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Some Comments on Chern-Simons Gauge Theory

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Abstract. Following M. F. Atiyah and R. Bott [AB] and E. Witten [W], we consider the space of flat connections on the trivial $SU(2)$ bundle over a surface M, modulo the space of gauge transformations. We describe on this quotient space a natural hermitian line-bundle with connection and prove that if the surface M is now endowed with a complex structure, this line bundle is isomorphic to the determinant bundle. We show heuristically how path-integral quantisation of the Chern-Simons action yields holomorphic sections of this bundle.

1. Introduction

In [W], Witten studied a $2+1$ dimensional quantum Yang-Mills theory, with an action consisting purely of the Chern-Simons term,

$$CS(\mathbf{A}) = \frac{1}{4\pi} \int Tr\left(\mathbf{A}d\mathbf{A} + \frac{2}{3}\mathbf{AAA}\right).$$

He obtained the Jones polynomials of knots on S^3 and their extensions to other 3-manifolds as expectation values of Wilson loop functionals. A key point was the identification of the quantum state space as the space of holomorphic sections of a line bundle.

We first describe this line bundle from an algebraic point of view. Let M denote a compact 2-manifold without boundary (with genus $\mathbf{g} \geqq 3$ – the other cases can be treated with analogous results), $\mathscr{A}$ the space of connections on the trivial $SU(2)$ bundle on M, $\mathscr{A}_F$ the space of flat connections, $\mathscr{A}_F^s$ the space of irreducible flat connections, and $\mathscr{G}$ the group of gauge transformations. Then it is well-known that $\mathscr{A}_F^s/\mathscr{G}$ is in a natural way a symplectic manifold. A choice of conformal structure M_c on M endows $\mathscr{A}_F^s/\mathscr{G}$ with a compatible Kähler structure, and it can be

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identified with the moduli space $\mathcal{S}$ of stable vector bundles on M_c of rank 2 and trivial determinant [NS, AB]. The Kähler structure on $\mathcal{S}$, suitably normalised, comes from an ample class on $\bar{\mathcal{S}}$, the moduli space of *semistable* vector bundles, (which in turn can be identified with the quotient $\mathcal{A}_F/\mathcal{G}$). In fact, by recent results of Narasimhan and Drezet [ND] there is a unique line bundle $\mathcal{L}$ in this class.

The state-space of relevance for the quantum field theory is the space of global sections $H^0(\bar{\mathcal{S}}, \mathcal{L}^k)$, where the integer k multiplies the Chern-Simons action in functional integrals. Witten obtains this finite-dimensional space via holomorphic quantization of a classically constrained phase space. On physical grounds he concludes that the vector bundle $\mathcal{W}$ over Teichmuller space with fibre $H^0(\mathcal{S}, \mathcal{L}^k)$ is projectively flat, i.e., has a natural connection such that the curvature is a form with values in scalar endomorphisms. (Equivalently, parallel transport in the associated projective bundle has trivial holonomy.) Moreover, every 3-manifold N such that $\partial N = M$ determines via the functional integral

$$\int \mathcal{D}\mathbf{A} \exp ikCS(\mathbf{A})$$

a state vector $v_N \in H^0(\mathcal{S}, \mathcal{L}^k)$, and, given two such 3-manifolds N_1 and N_2 one has

$$(v_{N_1}, v_{N_2}) = \int \mathcal{D}\mathbf{A} \exp ikCS(\mathbf{A}),$$

where the integral is over connections $\mathbf{A}$ on the 3-manifold $\tilde{N}$ obtained by identifying N_1 and N_2 along M.[2]

This paper was motivated in part by trying to understand the existence of the projectively flat connection in terms of differential geometry. Some progress can be found in Sect. 2 where we prove

Theorem 1. *There exists a natural hermitian line bundle $\mathcal{L}$ on $\mathcal{A}_F/\mathcal{G}$. Restricted to $\mathcal{A}_F^s/\mathcal{G}$, this line bundle carries a natural connection whose curvature is (up to a factor of i) the standard symplectic form.*

This theorem does not require a choice of conformal structure on M. However given a conformal structure, we can consider the determinant line bundle of the family of elliptic operators $\{\bar{\partial}_{\vec{A}} | \vec{A} \in \mathcal{A}_F^s\}$. If further a choice of compatible Riemannian metric on M is made, the construction of Quillen [Q] yields a hermitian structure and connection on this bundle, and one can show that it descends to $\mathcal{S}$ as holomorphic hermitian line bundle $\mathcal{L}_D$ with connection. We then have

Theorem 2. *$\mathcal{L}$ is isomorphic to the determinant bundle $\mathcal{L}_D$, as a hermitian line bundle with connection on $\mathcal{S}$.*

Theorems 1 and 2 show that the fibres of the holomorphic bundle $\mathcal{W}$ imbed naturally into $C^\infty(\mathcal{S}, \mathcal{L})$.

In Sect. 3 we sketch a heuristic path-integral approach which yields the quantum state-space $H^0(\bar{\mathcal{S}}, \mathcal{L}^k)$ affirming Witten's geometric quantization. In Sect. 4 we point out some subtleties when the gauge group is $U(1)$ so that $\mathcal{A}_F/\mathcal{G}$ is the Jacobian. We conclude with some speculations concerning the correspondence $N \mapsto v_N$.

[2] The functional integrals can be defined only after a choice of metric on $\tilde{N}$. A suitably modified integral depending on a choice of framing of $\tilde{N}$ turns out to only depend on the homotopy class of the framing. We will not consider the metric dependence in this paper

Witten considered gauge groups $SU(n)$, with $n \geqq 2$. For simplicity we have limited our attention to the case $n=2$.

For a recent study of Chern-Simons gauge theory from the Schrödinger viewpoint see [DJT].

One of us (T.R.R.) would like to thank M.S. Narasimhan for very useful conversations.

2. The Line Bundle

2.1. As in the introduction, let $\mathscr{A}$ be the space of $su(2)$-valued 1-forms on M, $\mathscr{A}_F$ the subspace of flat connections, $\mathscr{A}_F^s$ the submanifold of irreducible flat connections, $\mathscr{G}$ the space of gauge transformations. Also let N be a 3-manifold with $\partial N = M$; in fact we assume given a diffeomorphism of a neighbourhood of M in N with $M \times [0,1)$.

Consider the following $U(1)$-valued function on $\mathscr{A} \times \mathscr{G}$:

$$\Theta(\vec{A}, g) \equiv \exp i(CS(\mathbf{A}^{\tilde{g}}) - CS(\mathbf{A})),$$

where $\mathbf{A}$ and $\tilde{g}$ are extensions of $\vec{A}$ and g into N and $\mathbf{A}^{\tilde{g}}$ is the gauge transform of $\mathbf{A}$ by $\tilde{g}$. Such an extension of g into N is always possible in the case when the structure group is $SU(2)$ because $\pi_1(SU(2)) = \pi_2(SU(2)) = 0$. We shall choose the extensions such that on $M \times [0,1)$, $\mathbf{A}$ and $\tilde{g}$ are the pull-backs of $\vec{A}$ and g respectively by the projection to M. Now Θ is independent of N and the extensions $\mathbf{A}$ and $\tilde{g}$. In fact extensions $(\mathbf{A}_1, \tilde{g}_1)$ and $(\mathbf{A}_2, \tilde{g}_2)$ as above into N_1 and N_2 give a connection $\mathbf{B}$ and gauge transformation h on $N = N_1 \bigcup_M -N_2$ so that

$$\begin{aligned}\Theta(\mathbf{A}_1, \tilde{g}_1)\, \Theta(\mathbf{A}_2, \tilde{g}_2)^{-1} &= \exp i(CS(\mathbf{B}^{\tilde{h}}) - CS(\mathbf{B})) \\ &= 1\end{aligned}$$

because $CS(\mathbf{B}^{\tilde{h}}) - CS(\mathbf{B})$ is 2π times an integer.

The function Θ is a cocycle:

$$\Theta(\vec{A}, g)\, \Theta((\vec{A})^g, h) = \Theta(\vec{A}, gh).$$

We define

$$\mathscr{L} \equiv \mathscr{A}_F^s \times_\Theta \mathbf{C},$$

where on the right we mean the quotient by the equivalence relation:

$$(\vec{A}, z) \sim (\vec{A}^g, \Theta(\vec{A}, g) z).$$

It is clear that $\mathscr{L}$ is a complex line bundle on $\mathscr{A}_F^s/\mathscr{G}$, and since Θ is $U(1)$-valued, it is a *hermitian* line bundle. One can also define the corresponding principal $U(1)$ bundle $\mathscr{U}$ in a similar fashion.

One can easily check that Θ is C^∞ on $\mathscr{A} \times \mathscr{G}$ in the appropriate Sobolev norms. Integration by parts shows that its differential $d\Theta(\alpha, \phi)$ at $(\vec{A}, g)$ equals

$$\frac{i}{4\pi} \Theta \times \left\{ \int_M \operatorname{Tr}(g^{-1} dg g^{-1} \alpha g) - \int_M \operatorname{Tr}(\vec{A}^g \wedge d_{\vec{A}^g} \phi) + 2 \int_M \operatorname{Tr}(F_{\vec{A}^g} \phi) \right\},$$

where α is a tangent vector to $\mathscr{A}$ [i.e., an $su(2)$-valued 1-form on M] and ϕ is an infinitesimal gauge transformation [i.e., an $su(2)$-valued 0-form].

We recall next the definition of the symplectic structure on $\mathscr{A}_F^s/\mathscr{G}$. On the space $\mathscr{A}$ of all connections, the two-form

$$\hat{\Omega}(\alpha,\beta)\equiv\frac{i}{2\pi}\int_M \mathrm{Tr}(\alpha\wedge\beta)$$

(where α and β are tangent vectors at $A\in\mathscr{A}$) is closed. Restricted to $\mathscr{A}_F^s$ it is singular in the direction of gauge transformations:

$$\int_M \mathrm{Tr}(d_A\phi\wedge\beta)=-\int_M \mathrm{Tr}(\phi\wedge d_A\beta)=0$$

if $d_A\beta=0$ (the condition that β be tangent to $\mathscr{A}_F^s$). In fact the restricted form descends to a form Ω on $\mathscr{A}_F^s/\mathscr{G}$ which is symplectic (modulo a factor of i) [AB].

Note that on $\mathscr{A}, \hat{\Omega}=d\hat{\omega}$, where the 1-form $\hat{\omega}$ is defined by

$$\hat{\omega}(\alpha)=\frac{i}{4\pi}\int_M \mathrm{Tr}(\vec{A}\wedge\alpha).$$

We now check that $\hat{\omega}$, restricted to $\mathscr{A}_F^s$ is the pull-back [via the map $\mathscr{A}_F^s\to\mathscr{A}_F^s\times 1 \hookrightarrow \mathscr{A}_F^s\times U(1)\to\mathscr{A}\times_\Theta U(1)\to\mathscr{U}$] of a unitary connection on $\mathscr{U}$ whose curvature is Ω. First, $\hat{\omega}$ defines a connection one-form $\hat{\omega}_P$ on the principal $U(1)$ bundle $\mathscr{A}_F\times U(1)$. We have a twisted action of $\mathscr{G}$ on this bundle, and $\mathscr{U}$ is the quotient. Let X be a vertical vector field for this action. Using the earlier formula for $d\Theta$ one verifies that $\hat{\omega}_P$ vanishes on X and

$$L_X\hat{\omega}_P=i_X d\hat{\omega}_P=i_X\hat{\Omega}=0,$$

where L_X is the Lie derivative. This completes the proof of Theorem 1.

Remarks. 1. In fact the above construction defines a continuous line bundle over $\mathscr{A}/\mathscr{G}$. One needs to check the following: if g fixes $\vec{A}\in\mathscr{A}$, then $\Theta(\vec{A},g)=1$. This is trivially true when $\vec{A}=0$ or when $g\equiv\pm$Identity. When that is not the case and g fixes $\vec{A}$, there exists $h\in\mathscr{G}$ such that

$$\vec{A}^h=\begin{pmatrix}\vec{a} & 0\\ 0 & -\vec{a}\end{pmatrix},$$

where $\vec{a}$ is a 1-form, and

$$h^{-1}gh=\begin{pmatrix}u & 0\\ 0 & \bar{u}\end{pmatrix},$$

for some const u with $|u|=1$. It is easy to check that $\Theta(\vec{A}^h,h^{-1}gh)=1$ and this implies $\Theta(\vec{A},g)=1$.

2. The cocycle Θ was first considered, in a slightly different form, by Jackiw [J]. A construction similar to the above, of a line bundle over a loop group, (using the Wess-Zumino-Novikov-Witten term) appears in Mickelsson [M]. We thank one of the referees for pointing out the latter reference.

3. It is worth pointing out the naturality of $\mathscr{L}$: Introducing subscripts in an obvious way, a map of surfaces $\sigma: M_1\to M_2$ induces a map $\tilde{\sigma}:\mathscr{A}_{F,2}/\mathscr{G}_2\to\mathscr{A}_{F,1}/\mathscr{G}_1$ such that $\tilde{\sigma}^*\Omega_1=(\text{degree}\,\sigma)\Omega_2$. We also have a morphism of hermitian line bundles $\mathscr{L}_2^{\text{degree}\,\tilde{\sigma}}\to\mathscr{L}_1$ "over" $\tilde{\sigma}$ which preserves connections.

To identify $\mathscr{L}$ with the determinant bundle we will need to know $\pi_j(\mathscr{A}_F^s/\mathscr{G})$, $j=0,1,2$. We will use:

$$\mathscr{A}_F/\mathscr{G} \equiv \{\text{conjugacy classes of representations of } \pi_1(M)\} \equiv \mathscr{R}(\pi_1).$$

Let $\{(a_i, b_i)|i=1,\ldots,\mathbf{g}\}$ be loops on M which form a standard set of generators of $\pi_1(M)$ so that we have $\prod_i a_i b_i a_i^{-1} b_i^{-1} = 1$. Then we can write $\mathscr{R}(\pi_1) \equiv Z_\pi/SU(2)$, where

$$Z_\pi = \left\{A_1, \ldots, A_{\mathbf{g}}, B_1, \ldots, B_{\mathbf{g}} | A_i, B_i \in SU(2), \text{ with } \prod_i A_i B_i A_i^{-1} B_i^{-1} = 1\right\}$$

and $SU(2)$ acts adjointly on Z_π. We will let $Z_\pi^s \subset Z_\pi$ denote the irreducible representations of $\pi_1(M)$. It is easy to see that $Z_\pi^s = Z_\pi \setminus Y$, where Y is the orbit of $T \times \ldots \times T$ under the adjoint action of $SU(2)$, T being the standard diagonal $U(1)$ subgroup of $SU(2)$.

We now prove the

Lemma. *$\mathscr{R}(\pi_1)$ is connected and simply connected;* $\pi_2(\mathscr{R}(\pi_1)) = \mathbf{Z}$.

Proof. The projection $Z_\pi^s \to \mathscr{R}(\pi_1)$ is a locally trivial fibration with fibre $SO(3)$. We show below that Z_π^s has $\pi_0 = 0$, $\pi_1 = 0$, and $\pi_2 = \mathbf{Z}$. Hence the same result holds for $\mathscr{R}(\pi_1)$.

Let Z denote the product of $2\mathbf{g}$ copies of $SU(2)$:

$$Z \equiv \{A_1, \ldots, A_{\mathbf{g}}, B_1, \ldots, B_{\mathbf{g}} | A_i, B_i \in SU(2)\}.$$

The set Y is a submanifold in Z of codimension $4\mathbf{g}-2$; since $\mathbf{g} > 2$, $Z \setminus Y$ satisfies $\pi_0 = 0$, $\pi_1 = 0$, and $\pi_2 = 0$. Consider the map $R: (Z \setminus Y) \to SU(2)$ given by

$$R((A_i, B_i)) = \prod_i A_i B_i A_i^{-1} B_i^{-1}.$$

Because this (proper) map has a differential of maximal rank at every point, it is a fibration. The space Z_π^s is the inverse image of the identity element under R, and the standard exact sequence in homotopy proves that $\pi_0(Z_\pi^s) = 0$, $\pi_1(Z_\pi^s) = 0$. To see that $\pi_2(Z_\pi^s) = \mathbf{Z}$ we observe that the map $\pi_3(Z \setminus Y) \to \pi_3(SU(2))$ induced by R is zero.

2.2. Choose a conformal structure c on M. The space $\mathscr{A}^s/\mathscr{G}$ can then be identified [NS] with the moduli space $\mathscr{S}$ of stable vector bundles of rank 2 and trivial determinant on M_c. In fact, by [S] $\mathscr{A}_F/\mathscr{G}$ is a complete complex algebraic variety – the moduli space $\bar{\mathscr{S}}$ of (s-equivalence classes of) semistable vector bundles – in which $\mathscr{S} = \mathscr{A}^s/\mathscr{G}$ sits as the smooth part.

The complex structure on $\mathscr{A}^s/\mathscr{G}$ can also be obtained [AB] via the $\star$-operator on 1-forms defined by the conformal structure c. The form Ω is of type $(1,1)$ with respect to any one of these holomorphic structures; $\mathscr{L}$ has therefore a unique holomorphic structure such that the unitary connection ω is compatible with it.

We proceed to identify the line bundle constructed above with the determinant bundle $\mathscr{L}_D$ of the family of $\bar{\partial}$ operators parametrised by $\mathscr{S}$. We will first define $\mathscr{L}_D$ as a bundle on $\mathscr{S}$. In [Q] Quillen constructs a determinant line bundle $\tilde{\mathscr{L}}_D$ on $\mathscr{A}$. This bundle has a hermitian metric and connection invariant under a lift of the action of $\mathscr{G}$, and its curvature is $\hat{\Omega}$. Restricted to $\mathscr{A}_F^s$ it descends to give $\mathscr{L}_D$. The curvatures of $\mathscr{L}_D$ and $\mathscr{L}$ coincide, and using the above lemma one sees that $\mathscr{L}_D$ and $\mathscr{L}$ are isomorphic.

We now give the details. First, it is enough to consider $\tilde{\mathscr{L}}_D$ on the neighbourhood $\mathscr{A}^s$ of $\mathscr{A}_F^s$ consisting of $\vec{A}$'s which define stable bundles. On this set the bundle $\tilde{\mathscr{L}}_D$ is easy to describe. We have [NS] $\ker \bar{\partial}_{\vec{A}} = H^0(M_c, \bar{\partial}_{\vec{A}}) = 0$ and so the vector spaces $\operatorname{coker} \bar{\partial}_{\vec{A}} = H^1(M_c, \bar{\partial}_{\vec{A}})$ form a holomorphic vector bundle over $\mathscr{A}^s$ of rank $2(\mathbf{g}-1)$ and $\tilde{\mathscr{L}}_D$ is the determinant bundle of this vector bundle. We have, for $g \in \mathscr{G}$ the relation $\bar{\partial}_{(\vec{A})^g} = g^{-1} \circ \bar{\partial}_{\vec{A}} \circ g$ which shows that the action of $\mathscr{G}$ on $\mathscr{A}^s$ lifts to the bundle of cokernels. One can check, using the definition of holomorphic structure on this bundle, that the lifted action is by holomorphic isomorphisms. Thus the action of $\mathscr{G}$ on $\mathscr{A}^s$ lifts to an action on $\tilde{\mathscr{L}}_D$ by holomorphic transformations, which, further, preserves the Quillen metric in it. This implies that the action preserves the canonical connection on the bundle: denoting by $\tilde{g}$ the lift of $g \in \mathscr{G}$ we have

$$\nabla_X((\tilde{g})^{-1} \circ \mu \circ g) = (\tilde{g})^{-1} \circ (\nabla_{g_*(X)}\mu) \circ g$$

for any section μ. Let us now restrict $\tilde{\mathscr{L}}_D$, as a hermitian line bundle with connection, to $\mathscr{A}_F^s$, continuing to denote it as $\tilde{\mathscr{L}}_D$. We now define $\mathscr{L}_D$ as a line bundle over $\mathscr{S}$ by taking the quotient of $\tilde{\mathscr{L}}_D$ by the action of $\mathscr{G}$. [The isotopy subgroup of $\mathscr{G}$ at any $\vec{A} \in \mathscr{A}_F^s$ is $\pm$Identity. Since $H^1(M_c, \bar{\partial}_{\vec{A}})$ has even rank this subgroup acts trivially on $\tilde{\mathscr{L}}_D$ as well.]

We need to check that the connection descends. Let μ be any covariant section of $\tilde{\mathscr{L}}_D$: i.e., $(\tilde{g})^{-1} \circ \mu \circ g = \mu$, X a covariant vector field. The covariance of the connection shows that $\nabla_X \mu$ is covariant. We will now show that $\nabla_X \mu = 0$ if X is vertical. Let $\mathscr{O}$ be a $\mathscr{G}$ orbit in $\mathscr{A}_F^s$. On $\mathscr{O}$ we can write $\nabla_X \mu = \kappa(X)\mu$, where κ is a linear functional on {invariant vertical vector fields on $\mathscr{O}$} $= \operatorname{Lie}\mathscr{G}$, the Lie algebra of $\mathscr{G}$. Since the curvature $\hat{\Omega} = 0$ on $\mathscr{O}$, we have $\kappa([X_1, X_2]) = 0$. Since $\operatorname{Lie}\mathscr{G}$ is perfect, $\kappa = 0$.

From the lemma one concludes easily that $H^2(\mathscr{S}, \mathbf{Z}) = \mathbf{Z}$. This, together with the fact that $\mathscr{S}$ is simply connected, implies that two hermitian line bundles with the same curvature 2-form are isomorphic as line bundles with connection, thus proving Theorem 2.

Remarks. 1. One can check that $H^1(M_c, \bar{\partial}_{\vec{A}}) \simeq H^0(M_c, K_\chi)^*$ when $\vec{A}$ is flat, χ is the associated representation of $\pi_1(M)$ in $SU(2)$ and K_χ the canonical line bundle twisted by χ. The latter space has a natural inner product $\langle \tau, \xi \rangle = \int_M (\tau, \wedge \bar{\xi})$. Hence the line $L_\chi \equiv \Lambda^{2g-2} H^0(M_c, K_\chi)$ has a natural inner product. The line bundle $\{L_\chi\}_{\chi \in Z_\pi^s}$ descends to $R(\pi_1)$ as a line bundle $\hat{\mathscr{L}}^*$ with hermitian structure. If one follows the path of defining a complex structure on $R(\pi_1)$ as in [NS], one could similarly define a holomorphic structure on $\hat{\mathscr{L}}^*$. We then have isomorphisms of holomorphic line bundles $\mathscr{L} = \mathscr{L}_D = \hat{\mathscr{L}}$.

However the Quillen metric on $\mathscr{L}_D$ differs from the metric on $\hat{\mathscr{L}}$ described above by the function $\det \Delta_\chi$, where Δ_χ is the Laplacian on functions equivariant under χ.

2. The determinant bundle is an algebraic object, and in fact exists over $\bar{\mathscr{S}}$. In fact since $\operatorname{Pic}\bar{\mathscr{S}} \sim \operatorname{Pic}\mathscr{S} \sim \mathbf{Z}$ [ND] every holomorphic line bundle on $\mathscr{S}$ extends uniquely as an invertible sheaf on $\bar{\mathscr{S}}$.

3. Holomorphic Quantisation *via* Formal Path-Integrals

Consider again the Chern-Simons action functional

$$CS(\mathbf{A}) = \frac{1}{4\pi} \int \mathrm{Tr}\left(\mathbf{A}d\mathbf{A} + \frac{2}{3}\mathbf{AAA}\right),$$

where $\mathbf{A}$ is a connection on the trivial $SU(2)$ bundle on a 3-manifold. When the 3-manifold is diffeomorphic to $M \times [0,1]$, with M a 2-manifold, we can write

$$\mathbf{A} = A(t) + A_0(t)\,dt,$$

where $A(t)$ and $A_0(t)$ are a one parameter family of $su(2)$-valued one- and zero-forms respectively on M parametrised by $t \in [0,1]$. The Chern-Simons action becomes then:

$$CS(A, A_0) = \frac{1}{4\pi} \int dt \int_M \mathrm{Tr}\left(2A_0 F_A - A\frac{dA}{dt}\right).$$

Standard constraint analysis à la Dirac leads to $\mathscr{A}_F^s/\mathscr{G}$, with its symplectic structure, as the reduced phase space. A conformal structure c on M determines (via the $\star$ operator) a compatible complex structure and Kähler metric on $\mathscr{A}^s/\mathscr{G}$ and this holomorphic polarisation determines (for $k=1$) the state-space of [W], described in Sect. 1.

Our purpose in this section is to obtain the state-space $H^0(\bar{\mathscr{S}}, \mathscr{L}^k)$ via the functional integral directly. We emphasize that the arguments in this section are *not* rigorous. Some of them are standard, but the one which suggests the holomorphy of the sections of $\mathscr{L}$ obtained by functional integrals is, to our knowledge, new.

Consider therefore the functional integral

$$\Phi(\vec{A}, A_{(0,M)}) = \int_{\mathbf{A}(0) = \vec{A} + A_{(0,M)}dt} \mathscr{D}_N\mathbf{A} \exp ikCS(\mathbf{A}) \prod_C W_{\mathbf{A}}(C),$$

where $\vec{A}$ is an $su(2)$-valued 1-form on M, $A_{(0,M)}$ a $su(2)$-valued 0-form, N is a 3-manifold such that $\partial N = M$, the subscript on $\mathscr{D}$ refers to the fact that the functional integral is over $\mathbf{A}$ on N and $\prod_C W_{\mathbf{A}}(C)$ is a product of Wilson loops in N not intersecting M. We will argue that Φ can be interpreted as a holomorphic section of $\mathscr{L}^k$.

Choose a neighbourhood (not intersecting any of the loops C) of M in N diffeomorphic to $M \times [0,1)$ and in fact fix such a diffeomorphism, letting t denote as before the co-ordinate along $[0,1)$. The functional integral Φ above can be expressed as

$$\begin{aligned}
\Phi(\vec{A}, A_{(0,M)}) &= \int_{\mathbf{A}(0) = \vec{A} + A_{(0,M)}dt} \mathscr{D}_N\mathbf{A} \exp ikCS(\mathbf{A}) \prod_C W_{\mathbf{A}}(C) \\
&= \int_{A(0) = \vec{A},\, A_0(0) = A_{(0,M)}} \mathscr{D}A(t)\,\mathscr{D}A_0(t) \exp ikCS(A, A_0) \\
&\quad \times \int_{\mathbf{A}(1) = A(1) + A_0(1)dt} \mathscr{D}_{N \setminus M \times [0,1)}\mathbf{A} \exp ikCS(\mathbf{A}) \prod_C W_{\mathbf{A}}(C) \\
&= \int_{A(0) = \vec{A},\, A_0(0) = A_{(0,M)}} \mathscr{D}A(t)\,\mathscr{D}A_0(t) \exp ikCS(A, A_0)\ \Psi(A(1), A_0(1)),
\end{aligned}$$

where the effects of the integral over the fields on the rest of the manifold are subsumed in the functional Ψ. We will see below that the functional Φ is independent of $A_{(0,M)}$ and Ψ of $A_0(1)$, so we can write

$$\Phi(\vec{A}) = \int_{(A(0)=\vec{A})} \mathscr{D}A(t)\,\mathscr{D}A_0(t) \exp ikCS(A, A_0)\,\Psi(A(1)).$$

The expression

$$CS(A, A_0) = \frac{1}{4\pi}\int dt \int_M \mathrm{Tr}\left(2A_0F_A - A\frac{dA}{dt}\right)$$

shows that the above functional integral is nonzero only for paths $A(t)$ in $\mathscr{A}_F$. More precisely,

$$\int \mathscr{D}A_0(t) \exp\frac{ik}{4\pi}\int_M \mathrm{Tr}(2A_0F_A) = \frac{\delta(F_A(t)=0)}{2k\mathscr{D}_1(t)},$$

where $\mathscr{D}_1(t) = \sqrt{\det L(t)^* L(t)}$, and $L(t) = d_{A(t)}|_{\ker d^*_{A(t)}}$ is the differential (in the normal direction to $\{F_A(t)=0\}$) of the map $A \mapsto F_A$ of 1-forms to 2-forms. We have our first result: *Φ is supported on connections $\vec{A}$ which are flat*, i.e., $\Phi(\vec{A})=0$ unless $F_{\vec{A}}=0$.

We next demonstrate that the functional Φ is a section of $\mathscr{L}^k$, i.e., $\vec{A} \mapsto (\vec{A}, \Phi(\vec{A}))$ is a section of $\mathscr{L}^k$. Because of the equivalence relation defining $\mathscr{L}$, we need to check $\Phi(\vec{A}^g) = \Theta(\vec{A}, g)^k\,\Phi(\vec{A})$. Since $\mathscr{G}$ is connected, it suffices to verify this at the Lie algebra level: if ϕ is an infinitesimal gauge transformation,

$$d/dt|_{t=0}\,\Phi(\vec{A}^{e^{t\phi}}) = d/dt|_{t=0}\,\Theta(\vec{A}, e^{t\phi})^k\,\Phi(\vec{A}) = \frac{ik}{4\pi}\left(\int_M \mathrm{Tr}(\vec{A} \wedge d_{\vec{A}}\phi)\right)\Phi(\vec{A}),$$

where in the second equation we have used the expression for the differential of Θ from Sect. 2.

Extend the function ϕ into N so that it is supported within $M \times [0, \delta)$ and denote the extension by $\tilde{\phi}$. We have

$$\Phi(\vec{A}^{e^{t\phi}}) = \int_{\mathbf{A}(0)|_M = \vec{A}^{e^{t\phi}}} \mathscr{D}_N\mathbf{A} \exp ikCS(\mathbf{A}) \prod_C W_{\mathbf{A}}(C).$$

Make a change of variables $\mathbf{A} \mapsto \mathbf{A}^{e^{t\tilde{\phi}}}$. Assuming the integral is gauge invariant, we conclude because the Wilson loop functionals are gauge invariant that

$$\Phi(\vec{A}^{e^{t\phi}}) = \int_{\mathbf{A}(0)|_M = \vec{A}} \mathscr{D}_N\mathbf{A} \exp ikCS(\mathbf{A}^{e^{t\tilde{\phi}}}) \prod_C W_{\mathbf{A}}(C).$$

We note now that

$$\left.\frac{d}{dt}\right|_{t=0} CS(\mathbf{A}^{e^{t\tilde{\phi}}}) = \frac{1}{4\pi}\int_M \mathrm{Tr}(\vec{A} \wedge d_{\vec{A}}\phi)\,CS(\mathbf{A}),$$

which proves the required covariance. (Note that the above argument also shows that Φ is independent of $A_{(0,M)}$, since we can change $A_{(0,M)}$ arbitrarily by a gauge-transformation that is the identity on the boundary.)

We now give a heuristic argument to show that Φ is a holomorphic section of $\mathscr{L}^k$. It pretends that $\bar{\mathscr{S}}$ is a smooth Kähler manifold or that $\mathscr{S}$ is compact. Of course the holomorphic structure comes from a choice of metric on N needed to define the path integral – the metric on N induces a metric in M and hence a complex structure.

We will use the formal path integral expression for the heat kernel [FH] $e^{-T\Delta_{\omega_k}}$, where ω_k is the connection on $\mathscr{L}^k$ and $\Delta_{\omega_k}=d^*_{\omega_k}d_{\omega_k}$. If ψ is a section of $\mathscr{L}^k$ we have

$$e^{-T\Delta_{\omega_k}}\psi(x)=\int_{x:[0,T]\to\mathscr{S},\,x(0)=x}\mathscr{D}x(t)\exp\left(-\int_0^T dt|dx/dt|^2-\int_{x(t)}\omega_k\right)\psi(x(T)),$$

and $\exp\int\omega_k$ is the parallel transport operator from the fibre at $x(0)$ to the fibre at $x(T)$.

When $T\to\infty$, $e^{T\lambda_0}e^{-T\Delta_{\omega_k}}$ projects ψ onto the eigenspace corresponding to the smallest eigenvalue λ_0. In our case we are working on a Kähler manifold and the curvature of the line bundle is a constant multiple of the Kähler form; hence $\Delta_{\omega_k}=2\bar\partial^*_{\omega_k}\bar\partial_{\omega_k}+k\times(\dim\mathscr{S})$ so that the above eigenspace is the space of holomorphic sections. That is,

$$\lim_{T\to\infty}e^{T\lambda_0}\int_{x:[0,T]\to\mathscr{S},\,x(0)=x}\mathscr{D}x(t)\exp\left(-\int_0^T dt|dx/dt|^2-\int_{x(t)}\omega_k\right)\psi(x(T))$$

is a holomorphic section. We will now regulate our path integral to be exactly this.

Let us rewrite the expression for Φ, taking into account our observation that only paths in $\mathscr{A}_F$ contribute:

$$\begin{aligned}\Phi(\vec{A})&=\int_{A(0)=\vec{A},\,A(t)\in\mathscr{A}_F}\mathscr{D}A(t)\prod_t\mathscr{D}_1(t)\exp\left(-\frac{ik}{4\pi}\int dt\,\mathrm{Tr}\left(A\frac{dA}{dt}\right)\right)\Psi(A(1))\\&=\int_{A(0)=\vec{A},\,A(t)\in\mathscr{A}_F}\mathscr{D}A(t)\prod_t\mathscr{D}_1(t)\exp\left(-k\int_{A(t)}\hat\omega\right)\Psi(A(1)).\end{aligned}$$

Note that now A_0 is out of the picture and yet the above functional integral has the gauge freedom $A(t)\mapsto g^{-1}(t)A(t)g(t)+g^{-t}dg(t)$. We now argue that the above integral descends to an integral over the space of paths on $\mathscr{S}$,

$$\phi(x)=\int_{x:[0,1]\to\mathscr{S},\,x(0)=x}\mathscr{D}x(t)\exp\left(-\int_{x(t)}\omega_k\right)\psi(x(1)),$$

where ϕ and ψ are sections of $\mathscr{L}$ determined respectively by Φ and Ψ and ω is the connection one-form – the only point to note being that the volume of $\mathscr{G}$ cancels the determinant factor:

$$\sqrt{\det(d^*_{A(t)}d_{A(t)})_0}=\sqrt{\det(d^*_{A(t)}d_{A(t)})_1}=\mathscr{D}_1(t).$$

(The operator $\star d_A$ gives an isomorphism between the eigenvectors with nonzero eigenvalue.)

The last functional integral can be regulated by adding a kinetic energy term to the action:

$$\phi_T(x)=\int_{x:[0,1]\to\mathscr{S},\,x(0)=x}\mathscr{D}x(t)\exp\left(-\frac{1}{T}\int_0^1 dt|dx/dt|^2-\int_{x(t)}\omega_k\right)\psi(x(1)).$$

If we now make a change of variables $s=tT$ we get

$$\phi_T(x)=\int_{x:[0,T]\to\mathscr{S},\,x(0)=x}\mathscr{D}x(s)\exp\left(-\int_0^T ds|dx/ds|^2-\int_{x(s)}\omega_k\right)\psi(x(T)).$$

As $T\to\infty$, we get the projection of ψ on the subspace of holomorphic sections, provided we renormalise by multiplying by $e^{T\lambda_0}$.

4. The $U(1)$ Case; Concluding Remarks

4.1. We consider the case of $U(1)$ bundles; two subtleties become more evident from our point of view.

First, the space $\mathscr{G}$ of maps from M to G is connected in the case when $G=SU(2)$, but when $G=U(1)$ the group of connected component of $\mathscr{G}$ is precisely $H^1(M,\mathbf{Z})$. The argument in 2.1 that the functionals Φ descend to sections of a line bundle on $\mathscr{A}_F/\mathscr{G}$ fails when $SU(2)$ is replaced by $U(1)$ because the gauge transformation g need not extend into M. Also, there is no *à priori* reason now for the number k to be an integer since the Chern-Simons action (on a manifold without boundary) is single-valued modulo all gauge transformations. Let us, nevertheless see if the procedure *does* define a hermitian line bundle $\mathscr{L}$ with connection on $\mathscr{A}_F/\mathscr{G}=H^1(M,\mathbf{R})/H^1(M,\mathbf{Z})$ [where we have identified $\mathscr{A}_F$ with $H^1(M,\mathbf{R})$] by sending $\vec{A}$ to the class of $\frac{1}{2\pi i}\vec{A}$. To define the factor of automorphy for a given gauge transformation g one must now choose a three manifold N such that g extends into N. We can now evaluate Θ (we include the factor k in the definition)

$$\Theta_k(\vec{A},g)=\exp\left(\frac{ik}{4\pi}\int_M(g^{-1}dg\wedge\vec{A})\right).$$

We see that for this to be a factor of automorphy for all gauge transformations [i.e., $\Theta_k(\vec{A},gh)=\Theta_k(\vec{A},g)\,\Theta_k((\vec{A})^g,h)$] k must be an even integer. We shall assume this to be the case. We shall denote the line bundle defined by taking $k=2$ by l.

Choosing a complex structure on M yields one on $H^1(M,\mathbf{R})$; we can then identify $\mathscr{A}_F/\mathscr{G}=H^1(M,\mathbf{R})/H^1(M,\mathbf{Z})$ with the Jacobian J.

Note that even if k is an integer, different three-manifolds N define vectors in different infinite-dimensional vector spaces, namely, sections of $l^{k/2}$ on the (noncompact) space $H^1(M,\mathbf{R})/\operatorname{Im}H^1(N,\mathbf{Z})$ with no further automorphic property to ensure that they descend to J; the obvious guess that these are all θ-functions on the Jacobian may not be correct. [By $\operatorname{Im}H^1(N,\mathbf{Z})$ we mean the image of $H^1(N,\mathbf{Z})$ in $H^1(M,\mathbf{Z})$. We have the projection $H^1(M,\mathbf{R})/\operatorname{Im}H^1(N,\mathbf{Z})\to H^1(M,\mathbf{R})/H^1(M,\mathbf{Z})$ and can pull back $l^{k/2}$ by this map.]

Since $\pi_1(J)\neq 0$ it is interesting that path integral considerations have enabled us to choose a *particular* line bundle with connection whose curvature is the symplectic form out of a whole family of such bundles. Conventional geometric quantization does not do this.

4.2. When N is a handle-body, it determines [A] a Lagrangian manifold $\mathscr{E}_N$ of $\mathscr{A}_F^s/\mathscr{G}$. The line bundle $\mathscr{L}$ is flat along $\mathscr{E}_N$; we will show that $\mathscr{L}|_{\mathscr{E}_N}$ *has a nonzero flat section.*

Let $\{(a_i, b_i) | i=1, \ldots, \mathbf{g}\}$ be loops on M which form a standard set of generators of $\pi_1(M)$ so that we have $\prod_i a_i b_i a_i^{-1} b_i^{-1} = 1$. Let N be such that under the inclusion of fundamental groups $\pi_1(M) \to \pi_1(N)$ the a_i are annihilated. The subspace $\mathscr{E}_N$ of $\mathscr{A}_F/\mathscr{G}$ corresponding to connections which have flat extensions into the interior of N is a Lagrangian submanifold, i.e., a maximal submanifold on which Ω vanishes [A]. We give the brief argument: First, given two vectors α, β tangent to $\mathscr{E}_N$ at a point A,

$$\begin{aligned}\int_M \operatorname{Tr}(\alpha \wedge \beta) &= \int_N d \operatorname{Tr}(\alpha \wedge \beta) \\ &= \int_N (\operatorname{Tr}(d_{\mathbf{A}}\alpha \wedge \beta) - \operatorname{Tr}(\alpha \wedge d_{\mathbf{A}}\beta)) \\ &= 0,\end{aligned}$$

where in the second and third line we have extended A to a flat connection $\mathbf{A}$ and extended α, β to one-forms satisfying $d_{\mathbf{A}}(\cdot) = 0$. This shows that $\mathscr{E}_N$ is isotropic. We next show that $\dim \mathscr{E}_N = \frac{1}{2} \dim(\mathscr{A}_F/\mathscr{G})$. To see this, note that $\mathscr{E}_N \underset{\text{diffeo}}{=} X/SU(2)$, where

$$X = \underbrace{SU(2) \times \ldots \times SU(2)}_{\mathbf{g}\,\text{times}} \backslash SU(2) \{T \times \ldots \times T\} SU(2)^{-1}.$$

T is a maximal torus of $SU(2)$ and $SU(2)$ acts adjointly on X.

One can now also check that $\mathscr{E}$ is simply connected, and this implies that there is a nonzero flat section of of $\mathscr{L} | \mathscr{E}$, unique up to a nonzero scalar.

In fact this flat section exists even when the gauge group is $U(1)$. In that case we have to consider the bundle l, defined on $J \equiv \dfrac{H^1(M, \mathbf{R})}{H^1(M, \mathbf{Z})}$ by the factor of automorphy,

$$\Theta_1(x, u) = \exp(-2\pi i \textstyle\int x \wedge u).$$

The one-form

$$\hat{\omega}(x)[y] \equiv -(2\pi i \textstyle\int x \wedge y)$$

defines a hermitian connection ω on this bundle. The curvature is

$$\Omega[x, y] = -4\pi i \textstyle\int x \wedge y.$$

Let W be a subgroup of $H^1(M, \mathbf{Z})$ such that the subspace $\mathbf{R}W$ generated by it is of dimension $\mathbf{g}$ and isotropic for the intersection form. Then the expression for $\hat{\omega}$ shows: the image of $\mathbf{R}W$ in J is a Lagrangian submanifold for Ω, not simply connected, such that however *the holonomy of* ω *is trivial on it.*

4.3. One would expect a close relationship between a nonzero flat section (say s_N), along $\mathscr{E}_N$ and v_N. Computations in genus 1 show that s_N does not extend to a holomorphic section of $\mathscr{L}$ (although if it did, such an extension would be unique). A saddle-point approximation to the functional integral shows that the sections v_N are concentrated along $\mathscr{E}_N$, and this is sharper as k increases. Again, computations in genus 1 show that v_N is not the projection of s_N (thought of as a distribution with support along $\mathscr{E}_N$) onto $H^0(\bar{\mathscr{S}}, \mathscr{L}^k)$. The relationship, if any, between s_N and v_N is subtler than the above.

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