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BRST Cohomology of the Super-Virasoro Algebras

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Abstract. We study the superextension of the semi-infinite cohomology theory of the Virasoro Algebra. In particular, we examine the BRST complex with coefficients in the Fock Space of the RNS superstring. We prove a theorem of vanishing cohomology, and establish the unitary equivalence between a positive definite transversal space, a physical subspace and the zeroth cohomology group. The cohomology of a subcomplex is identified as the covariant equivalent of the well-known GSO subspace. An exceptional case to the vanishing theorem is discussed.

0. Introduction

The BRST approach has long been known to be an effective method for studying quantization of string theories. It was first applied to the Virasoro algebra of the bosonic string by Kato and Ogawa [11]. Based on a vanishing theorem, unitary equivalence between the BRST cohomology groups and the physical spaces known to physicists was proven by Frenkel, Garland and Zuckerman (FGZ) [5, 17]. They have also provided a conceptual proof of the no-ghost theorem. Several authors have recently studied the BRST quantization of the Ramond–Neveu–Schwarz (RNS) model [13, 15]. In their work, a BRST differential operator was defined and shown to be nilpotent at the critical dimension of spacetime $D = 10$ together with an appropriate normal ordering. An extension of the GSO (Gilozzi, Scherk, Olive)-projection was also proposed.

In this paper, we apply some of the ideas introduced in [5] to the Super-Virasoro algebras. Using some standard techniques in homological algebra, we prove a vanishing theorem. Formal characters and signatures of the cohomology groups are expressed in terms of modular functions. We show that the canonical hermitian forms on the BRST complexes naturally lead to ones on the relative subcomplexes and induce an (positive definite) inner product on the physical spaces. We define

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a natural generalization of the GSO projection and show that it coincides with that proposed in [13, 15].

We now outline the organization of this paper. In Sect. I, we review a construction of the BRST complexes of the RNS model, mostly following the notations of [19]. In Sect. 2, we define the relative BRST complexes and examine several important consequences of the vanishing theorem and we study the hermitian structures on the complexes. We also discuss the correspondence between the zeroth relative cohomology classes and the physical states known to physicists. In Sect. 3, we discuss the proof of the vanishing theorem. The generalization of the GSO projection is defined and shown to have the required properties in Sect. 4. Finally, in Sect. 5, we return to the BRST complexes and examine their cohomology. An exceptional case to the vanishing theorem is discussed.

1. BRST Complex of the RNS Model

The Super-Virasoro algebras Vir_κ are super-extensions of the Virasoro algebra given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{8}(m^3 - 2\kappa m)\delta_{m+n} \quad (\delta_{m+n} \equiv \delta_{m+n,0}),$$

$$[L_m, G_{n+\kappa}] = (\tfrac{1}{2}m - n - \kappa)G_{m+n+\kappa},$$

$$\{G_{m+\kappa}, G_{n-\kappa}\} = 2L_{m+n} + \frac{c}{2}(m^2 + 2\kappa m)\delta_{m+n},$$

$$[L_m, c] = 0 = [c, G_{m+\kappa}],$$

where $m, n \in \mathbf{Z}, \kappa = 0, \frac{1}{2}$. Vir_0 is called the Ramond algebra and $\mathrm{Vir}_{1/2}$ is called the Neveu–Schwarz algebra. Vir_κ has a $\mathbf{Z}_2$-grading: respectively $\coprod_{n\in Z} \mathbf{C}L_n \oplus \mathbf{C}c$, $\coprod_{n\in Z} \mathbf{C}G_{n+\kappa}$ are the even and odd parts of Vir_κ. These algebras arise as the "covariant constraints" in the classical formulation of the RNS model [12, 14, 19].

Two important representations of each of these algebras are well-known. They are Fock spaces constructed from certain complex super Heisenberg algebras. We will briefly review them here. Since we will consider both Vir_κ, $\kappa = 0, \frac{1}{2}$, simultaneously, whenever κ appears unspecified, we will mean either case.

Consider the infinite dimensional super Heisenberg algebra with the Lie brackets

$$[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n} g^{\mu\nu}\,\mathrm{Id},$$

$$[\alpha_m^\mu, d_{n+\kappa}^\nu] = 0,$$

$$\{d_{m+\kappa}^\mu, d_{n-\kappa}^\nu\} = \delta_{m+n} g^{\mu\nu}\,\mathrm{Id},$$

where $m, n \in \mathbf{Z}$, $\mu, \nu = 1, \ldots, D$, $g^{\mu\nu}$ is the inverse of the Lorentz metric with signature $((D-1)+,\ 1-)$ and Id denotes the center. For each $p \in \mathbf{R}^{D-1,1}$, there corresponds an irreducible representation of this algebra. It is the linear space $V(p,\kappa) = R(p,\kappa) \otimes U(\kappa)$, where $U(\kappa)$ is the space of polynomials $\mathbf{C}[\alpha_{-n}^\mu, d_{-n+\kappa}^\mu]|p\rangle$[1]

[1] Unless otherwise stated, n in this notation is always ranging over $\mathbf{N} = \{1, 2, \ldots\}$ and μ over $\{1, \ldots, D\}$. The same notation will be used in the future without specifying these ranges of n and μ

which respect the above commutation relations, and $R(p,\frac{1}{2})=\mathbf{C}$, $R(p,0)$ is the spinor representation of $\mathrm{Spin}(D-1,1)$ in which d_0^μ acts. We will sometimes drop the factor $R(p,\frac{1}{2})$ when $\kappa=\frac{1}{2}$. Here $|p\rangle$ denotes the highest weight vector with $\alpha_n^\mu|p\rangle=d_{n-\kappa}^\mu|p\rangle=0$ for $n>0$, $\alpha_0^\mu|p\rangle=p^\mu|p\rangle=0$. In all subsequent uses, p will always be assumed non-zero until Sect. 5. The operator α_0^μ is interpreted as the "center of mass momentum," also denoted by p. Let

$$\pi(L_m)=\tfrac{1}{2}\sum_{n\in Z}:\alpha_{-n}\cdot\alpha_{n+m}:+\tfrac{1}{2}\sum_{n\in Z}(\tfrac{1}{2}m+n-\kappa):d_{-n+\kappa}\cdot d_{m+n-\kappa}:$$
$$\pi(G_{m+\kappa})=\sum_{n\in Z}\alpha_{-n}\cdot d_{n+m+\kappa},$$

where

$$:\alpha_n\cdot\alpha_m:=g_{\mu\nu}\alpha_n^\mu\alpha_m^\nu \quad \text{if} \quad m>0,$$
$$=g_{\mu\nu}\alpha_m^\nu\alpha_n^\mu \quad \text{otherwise},$$

and

$$:d_{n+\kappa}\cdot d_{m-\kappa}:=g_{\mu\nu}d_{n+\kappa}^\mu d_{m-\kappa}^\nu \quad \text{if} \quad m>0$$
$$=-g_{\mu\nu}d_{m-\kappa}^\nu d_{n+\kappa}^\mu \quad \text{otherwise}.$$

Note that we have summed over repeated Greek indices. In view of the commutation relations, the normal ordering : : above is well-defined. A direct calculation (see [19] for example) shows that

Proposition 1.1.

$$[\pi(L_m),\pi(L_n)]=(m-n)\pi(L_{m+n})+\frac{D\cdot\mathrm{Id}_V}{8}(m^2-2\kappa m)\delta_{m+n},$$
$$[\pi(L_m),\pi(G_{n+\kappa})]=(\tfrac{1}{2}m-n-\kappa)\pi(G_{m+n+\kappa}),$$
$$\{\pi(G_{m+\kappa},\pi(G_{n-\kappa})\}=2\pi(L_{m+n})+\frac{D\cdot\mathrm{Id}_V}{2}(m^2+2\kappa m)\delta_{m+n},$$

where $m,n\in\mathbf{Z}$.

Thus for $\pi(c)=D\cdot\mathrm{Id}_V$, $(V(p,\kappa),\pi)$ is a representation of Vir_κ. This representation has long been known in physics as the "matter sector" of the Super-Virasoro algebras. We will call this the Fock space.

The ghost sector is constructed as follows. Consider the infinite dimensional complex super Heisenberg algebra with Lie brackets:

$$[\gamma_{m+\kappa},\beta_{n-\kappa}]=\delta_{m+n}\mathrm{Id}, \tag{1}$$
$$\{c_m,b_n\}=\delta_{m+n}\mathrm{Id}, \tag{2}$$
$$[\gamma_{m+\kappa},c_n]=[\gamma_{m+\kappa}b_n]=[c_m,\beta_{n-\kappa}]=[b_m,\beta_{n-\kappa}]=0, \tag{3}$$
$$\{b_n,b_m\}=\{c_n,c_m\}=[\gamma_{m+\kappa},\gamma_{n+\kappa}]=[\beta_{m+\kappa},\beta_{n+\kappa}]=0,\quad m,n\in\mathbf{Z}. \tag{4}$$

Friedan and coworkers [6] have defined a class of irreducible representations of (1). Specify a vacuum vector $|q_B\rangle$, $q_B\in\mathbf{Z}+\kappa$ and let

$$\beta_{n+\kappa}|q_B\rangle=0 \quad \text{for} \quad n+\kappa>-q_B,$$
$$\gamma_{n+\kappa}|q_B\rangle=0 \quad \text{for} \quad n+\kappa\geqq q_B.$$

The representation space is then the linear space of monomials generated by the commuting operators $\{\beta_{n+\kappa}, \gamma_{m+\kappa}, n+\kappa \leqq -q_B, m+\kappa < q_B\}$. The parameter q_B is called the Bose-sea charge (Note: q_B here differs from that in [6] by a constant). Representations corresponding to distinct charges are inequivalent. We denote each of them by $\mathscr{T}_{q_B}$.

A similar class of irreducible representations of (2) is defined by specifying a vacuum $|q_F\rangle$, $q_F \in \mathbf{Z}$ and letting

$$b_n|q_F\rangle = 0 \quad \text{for} \quad n > -q_F,$$
$$c_n|q_F\rangle = 0 \quad \text{for} \quad n \geqq q_F.$$

The representation space is the linear space spanned by the monomials generated by the anti-commuting operators $\{b_n, c_m, n \leqq -q_F, m < q_F\}$. q_F is called the Fermi-sea charge. Each of these representations is equivalent to the linear space of semi-infinite forms constructed by Feigin [4] and FGZ [5]. To be definite, we fix $q_F = 0$ and denote the space by Λ_∞ and denote its vacuum vector by $|q_F = 0\rangle$. Note that under the above equivalence we can identify $|q_F\rangle$ with $b_{-n}b_{-n+1}\cdots b_0|0\rangle$ for $n = q_F - 1 \geqq 0$ and with $c_n c_{n+1}\cdots c_{-1}|0\rangle$ for $n = q_F < 0$.

Define

$$\rho(L_m) = \sum_{n\in Z}(n-m):c_{-n}b_{n+m}: + \sum_{n\in Z}(\tfrac{1}{2}m - n - \kappa):\gamma_{-n-\kappa}\beta_{n+m+\kappa}: - \kappa\delta_m,$$
$$\rho(G_{m+\kappa}) = -2\sum_{n\in Z} b_{-n}\gamma_{n+m+\kappa} + \sum_{n\in Z}(\tfrac{1}{2} - m - \kappa)c_{-n}\beta_{n+m+\kappa},$$

where

$$:c_n b_m: = c_n b_m \quad \text{if} \quad m > 0$$
$$= -b_m c_n \quad \text{otherwise}$$

and

$$:\gamma_{n+\kappa}\beta_{m-\kappa}: = \gamma_{n+\kappa}\beta_{m-\kappa} \quad \text{if} \quad m > 0$$
$$= \beta_{m-\kappa}\gamma_{n+\kappa} \quad \text{otherwise.}$$

Similar to Proposition 1.1, one has (see [19])

Proposition 1.2. *For* $q_B \in \mathbf{Z} + \kappa$, $(\Lambda_\infty \otimes \mathscr{T}_{q_B}, \rho)$ *is a representation of* Vir_κ *iff* $\rho(c) = -10\,\mathrm{Id}_{\Lambda_\infty \otimes \mathscr{T}_{q_B}}$.

It is sometimes convenient to write the above representations of Vir_κ in terms of generating functions or "quantum fields." Let (cf. [6])

$$b(z) = \sum_{n\in Z} b_n z^{-n-2},$$
$$c(z) = \sum_{n\in Z} c_n z^{-n+1},$$
$$\beta(z) = \sum_{n\in Z} \beta_{n+\kappa} z^{-n-\kappa-3/2},$$
$$\gamma(z) = \sum_{n\in Z} \gamma_{n+\kappa} z^{-n-\kappa+1/2},$$
$$T_V(z) = \sum_{n\in Z} \pi(L_n) z^{-n-2},$$

$$\mathscr{T}_V(z) = \sum_{n\in Z} \pi(G_{n+\kappa}) z^{-n-\kappa-3/2},$$

$$T_\Omega(z) = \sum_{n\in Z} \rho(L_n) z^{-n-2},$$

$$\mathscr{T}_\Omega(z) = \sum_{n\in Z} \rho(G_{n+\kappa}) z^{-n-\kappa-3/2}.$$

Then

$$T_\Omega(z) =: c(z)\frac{d}{dz}b(z) + 2\frac{d}{dz}c(z)b(z): - :\frac{1}{2}\gamma(z)\frac{d}{dz}\beta(z) - \frac{3}{2}\frac{d}{dz}\gamma(z)\beta(z):,$$

$$\mathscr{T}_\Omega(z) = -2b(z)\gamma(z) + c(z)\frac{d}{dz}\beta(z) + \frac{3}{2}\frac{d}{dz}c(z)\beta(z).$$

We now construct hermitian bilinear forms $(,)_R$, $(,)_U$, $(,)_\Lambda$, $(,)_{\mathscr{T}}$, on each of the spaces: $R(p,\kappa)$, $U(\kappa)$, $\Lambda_\infty, \mathscr{T}_{q_B}$. The notations $(,)$ $\langle,\rangle$ will be used interchangeably.

Definition 1.3.

(i) *For* $\kappa = \frac{1}{2}$, *(recall that* $R(p,\frac{1}{2}) = \mathbf{C}$) *let* $\langle 1,1\rangle_R = 1$.

(ii) *For* $\kappa = 0$, *will assume that* $D = 10$. *Let* $\Psi^i_{\leftarrow}$, $\Psi^i_{\rightarrow}$, $i = 1,\ldots,2^{D/2-1}$ *be a basis of* $R(p,0)$ *with definite chirality and satisfying* $p\cdot d_0\Psi^i_{\leftarrow} = \Psi^i_{\rightarrow}$. *Set* $\langle \Psi^i_{\leftarrow}, \Psi^j_{\rightarrow}\rangle_R = \sqrt{-1}\delta^{ij}$, $\langle \Psi^i_{\leftarrow}, \Psi^j_{\leftarrow}\rangle_R = 0 = \langle \Psi^i_{\rightarrow}, \Psi^j_{\rightarrow}\rangle_R$. *For* $p\cdot p < 0$, *we set* $\Gamma_0\Psi^i_{\rightarrow} = +\Psi^i_{\rightarrow}, \Gamma_0\Psi^i_{\leftarrow} = -\Psi^i_{\leftarrow}$, *where* Γ_0 *is the chirality operator of* $R(p,0)$, $\Gamma_0 = 2^5(d_0)_1,\ldots,(d_0)_{10}$. *For* $p\cdot p = 0$, *half of* $\Psi^i_{\rightarrow}$ *and half of* $\Psi^i_{\leftarrow}$ *are in* $\mathrm{Ker}(\Gamma_0 - 1)$ *and the other two halves are in* $\mathrm{Ker}(\Gamma_0 + 1)$.

Remark. We note that $(,)_R$ above is defined so that $(p\cdot d_0)^* = -p\cdot d_0$. We will see later that this indeed gives the correct signature for the physical space. For convenience, we will assume that p is always in the future half $\mathbf{R}^{D-1,1}_+$ of $\mathbf{R}^{D-1,1}$, i.e. $\mathbf{R}^{D-1,1}_+ = \{p\in\mathbf{R}^{D-1,1} : p^D > 0\}$. For if we allow $p^D < 0$, "time reversal" will reverse the signature of $(,)_R$. The construction of the "chiral" basis in the above definition is quite standard, and the reader is referred to [19] for details.

To define $(,)_U$, let $\alpha^{\mu *}_n = \alpha^\mu_{-n}$, $d^\mu_{n-\kappa}{}^* = d^\mu_{-n+\kappa}$, $n = 1,2,3,\ldots$ and $(|p\rangle, |p\rangle)_U = 1$. It is well known that these conditions define a unique non-degenerate hermitian form on $U(\kappa) = \mathbf{C}[\alpha^\mu_{-n}, d^\mu_{-n+\kappa}]|p\rangle$. Similarly a non-degenerate hermitian form $(,)_\Lambda$ on Λ_∞ is defined by $c^*_n = c_{-n}$, $b^*_n = b_{-n}$, $n\in\mathbf{Z}$, and $(|q_F\rangle, |-q_F+1\rangle)_\Lambda = 1$ (for any $q_F\in\mathbf{Z}$). Before we look at the hermitian forms on $\mathscr{T}_{q_B}$, we state the following lemma.

Lemma 1.4. *Let* $\mathscr{A}$ *be a complex Heisenberg Lie algebra with a canonical basis* $\{a_i, c_i, \mathrm{Id}, i\in\mathbf{Z}\}$ *such that* $[a_i,a_j] = 0 = [c_i,c_j]$ *and* $[a_i,c_j] = \delta_{ij}\mathrm{Id}$. *Let* $T = \mathrm{Sym}\coprod_{i\in Z}\mathbf{C}c_i)$ *be the representation of* $\mathscr{A}$ *in which* $a_i(c_j)$ *acts by derivation (multiplication), and* $\langle,\rangle_T$ *be a hermitian form on* T. *If* $c^*_k\in\mathscr{A}$ *for all* k *and* $a^*_i = a_j$ *for some* i,j *then* $\langle,\rangle_T \equiv 0$.

Proof. It is enough to show that $\langle\mu 1, 1\rangle_T = 0$ for an arbitrary monomial μ of the c's. Now $\langle\mu 1,1\rangle_T = \langle\mu 1,[a_i,c_i]1\rangle = \langle\mu 1, a_i c_i 1\rangle = \langle c^*_i a_j \mu 1, 1\rangle$. If $a_j\mu 1 = 0$, then $\langle\mu 1,1\rangle_T = 0$. If not, then $\mu = c^n_j\mu'$ for some $n > 0$ and some monomial μ' such that $[\mu', a_j] = 0$. Thus

$$0 = \langle c_j\mu 1, a_i 1\rangle = \langle a_j c_j \mu 1, 1\rangle = (n+1)\langle\mu 1, 1\rangle. \quad \square$$

The lemma states that the representation T does not admit a non-degenerate hermitian form for which an annihilator (derivation) a_j is conjugate to another annihilator a_i. We now search among the $\mathscr{T}_{q_B}$, $q_B \in \mathbf{Z}+\kappa$, for a representation that has a non-degenerate hermitian form such that

$$\gamma^*_{n+\kappa} = \gamma_{-n-\kappa} \quad \beta^*_{n+\kappa} = -\beta_{-n-\kappa}. \tag{5}$$

Thus the γ's the β's and their conjugates are creators or annihilators on each $\mathscr{T}_{q_B}$. Lemma 1.4 implies that in order for $\mathscr{T}_{q_B}$ to admit such a hermitian form, q_B must be chosen so that the conjugate of any annihilator is never an annihilator. We find that for $\kappa=\frac{1}{2}$, then $q_B=\frac{1}{2}$ is the only choice. For if $q_B>\frac{1}{2}(q_B<\frac{1}{2})$, then $\beta_{1/2}$ and $\beta^*_{1/2}(\gamma_{1/2}$ and $\gamma^*_{1/2})$ are both annihilators on $\mathscr{T}_{q_B}$. When $\kappa=0$, there is no such choice. For if $q_B>0$ $(q_B \leqq 0)$, then β_0 and $\beta^*_0(\gamma_0$ and $\gamma^*_0)$ are both annihilators on $\mathscr{T}_{q_B}$. However, consider the spaces $\mathscr{T}_{1/2}$ and $\mathscr{T}_{q_B} \oplus \mathscr{T}_{-q_B+1}$, $q_B \neq \frac{1}{2}$. Each of them actually admits a non-degenerate hermitian form with property (5). If we let (see [6])

$$(|q_B\rangle, |-q_B+1\rangle)_{\mathscr{T}} = 1, \quad q_B \in \mathbf{Z}+\kappa \tag{6}$$

then (5), (6) together defines a unique non-degenerate pairing between $\mathscr{T}_{q_B}$ and $\mathscr{T}_{-q_B+1}$. Note that with respect to $(,)_{\mathscr{T}}$, the conjugate of an annihilator is always a creator. Each of the spaces $\mathscr{T}_{1/2}$ or $\mathscr{T}_{q_B} \oplus \mathscr{T}_{-q_B+1}$, $q_B \neq \frac{1}{2}$ inherits a natural gradation from the graded operators, $\gamma_{n+\kappa}$ and $\beta_{n+\kappa}$, with $\deg \gamma_{n+\kappa} = n+\kappa = \deg \beta_{n+\kappa}$. A space in which deg (or "energy") is bounded neither above nor below will give rise to various difficulties. Fortunately, for each sector $\kappa=\frac{1}{2}, 0$, there is a unique such space in which deg is bounded above, namely

$$\begin{aligned} &\mathscr{T}_{1/2} \quad \text{for} \quad \kappa=\tfrac{1}{2} \\ &\mathscr{T}_0 \oplus \mathscr{T}_1 \quad \text{for} \quad \kappa=0. \end{aligned} \tag{7}$$

Therefore we will restrict to these spaces for our construction. Denote

$$\begin{aligned} C(p,\kappa) &= R(p,0) \otimes U(0) \otimes \Lambda_\infty \otimes (\mathscr{T}_0 \otimes \mathscr{T}_1) \quad \text{if} \quad \kappa=0 \\ &= R(p,\tfrac{1}{2}) \otimes U(\tfrac{1}{2}) \otimes \Lambda_\infty \otimes \mathscr{T}_{1/2} \qquad \text{if} \quad \kappa=\tfrac{1}{2}. \end{aligned} \tag{8}$$

Definition 1.4. *Define $\langle,\rangle_C$ on $C(p,\kappa)$ as the tensor product of the four hermitian forms $\langle,\rangle_R \langle,\rangle_U \langle,\rangle_\Lambda \langle,\rangle_{\mathscr{T}}$ defined above.*

We now define a $\mathbf{Z}_2$-gradation on $R, U, \Lambda_\infty, \mathscr{T}_{q_B}$ and hence on $\mathscr{C}$, as follows: Ψ is $\mathbf{Z}_2$-even ($\mathbf{Z}_2$-odd) if $\Gamma_0 \Psi = +\Psi (= -\Psi)$ for $\kappa=0$. Since $U, \Lambda_\infty, \mathscr{T}_{q_B}$ are all linear spaces of polynomials, it is enough to assign a $\mathbf{Z}_2$-grading to the generators and the vacuo. Let $\alpha^\mu_n, |p\rangle$ be $\mathbf{Z}_2$-even and $d^\mu_{n-\kappa}$ be $\mathbf{Z}_2$-odd. Let $b_n, c_n, |q_F=0\rangle$ be $\mathbf{Z}_2$-odd and $\beta_{n-\kappa}, \gamma_{n-\kappa}, |q_B\rangle$ be $\mathbf{Z}_2$-even.

We extend the action of b_n, c_n, $d^\mu_{n+\kappa}$, α^μ_n, $\beta_{n+\kappa}$, $\gamma_{n+\kappa}$, to $C(p,\kappa)$ by demanding that (indices omitted)

$$\begin{aligned} \{b,d\} &= [b,\alpha] = [b,\beta] = [b,\gamma] = 0, \\ \{c,d\} &= [c,\alpha] = [c,\beta] = [c,\gamma] = 0, \\ [d,\beta] &= [d,\gamma] = [\alpha,\beta] = [\alpha,\gamma] = 0. \end{aligned}$$

More precisely, we let the $b_n, c_n, d^\mu_{n+\kappa}, \alpha^\mu_n, \beta_{n+\kappa}, \gamma_{n+\kappa}$ act on $C(p,\kappa)$ as

$$(-1)^V \otimes b_n \otimes 1_{\mathscr{T}}, \quad (-1)^V \otimes c_n \otimes 1_{\mathscr{T}}, \quad d^\mu_{n+\kappa} \otimes 1_\Lambda \otimes 1_{\mathscr{T}},$$
$$\alpha^\mu_n \otimes 1_\Lambda \otimes 1_{\mathscr{T}}, \quad 1_V \otimes 1_\Lambda \otimes \beta_{n+\kappa}, \quad 1_V \otimes 1_\Lambda \otimes \gamma_{n+\kappa}$$

respectively. (Note: If T is a $\mathbf{Z}_2$-graded vector space $(-1)^T\omega = +\omega$ if ω is $\mathbf{Z}_2$-even, $-\omega$ if ω is $\mathbf{Z}_2$-odd.) Using these definitions of the actions, one can easily check the following

Proposition 1.5. *With respect to $\langle,\rangle_C$, the above operators on $C(p,\kappa)$ have the following hermiticity property:*

(i) *For* $\kappa = \frac{1}{2}$,

$$b_n^* = b_{-n}, \quad c_n^* = c_{-n}, \quad d^\mu_{n+1/2}{}^* = d^\mu_{-n-1/2},$$
$$\alpha_n^* = \alpha^\mu_{-n}, \quad \beta^*_{n+1/2} = -\beta_{-n-1/2}, \quad \gamma^*_{n+1/2} = \gamma_{-n-1/2}, \quad n \in \mathbf{Z};$$

(ii) *For* $\kappa = 0$,

$$b_n^* = -b_{-n}, \quad c_n^* = -c_{-n}, \quad d_n^{\mu *} = -d^\mu_{-n},$$
$$\alpha_n^* = \alpha^\mu_{-n}, \qquad \beta_n^* = -\beta_{-n}, \quad \gamma_n^* = \gamma_{-n}, \quad n \in \mathbf{Z}.$$

Definition 1.6 [19]. *Define the* BRST, *Kinetic, Dirac–Ramond, and the ghost-number operators:*

$$Q = \sum_{n,m \in Z} (\pi(L_{-n})c_n + \pi(G_{-n+\kappa})\gamma_{n-\kappa} - \tfrac{1}{2}(m-n){:}\, b_{m+n}c_{-m}c_{-n}{:}$$
$$+ (\tfrac{3}{2}n + m + \kappa){:}\, c_{-n}\beta_{-m-\kappa}\gamma_{m+n+\kappa}{:} - \gamma_{-m-\kappa}\gamma_{-n+\kappa}b_{m+n}) - \kappa c_0,$$
$$K(=\pi(L_0) + \rho(L_0)) = \sum_{n>0} (\alpha_{-n}\cdot\alpha_n + (n-\kappa)d_{-n+\kappa}\cdot d_{n-\kappa} + nc_{-n}b_n + nb_{-n}c_n$$
$$-(n-\kappa)\gamma_{-n+\kappa}\beta_{n-\kappa} + (n-\kappa)\beta_{-n+\kappa}\gamma_{n-\kappa}) + \tfrac{1}{2}p\cdot p - \kappa,$$
$$D_R(=\pi(G_0) + \rho(G_0)) = p\cdot d_0 + \sum_{n>0} (\alpha_{-n}\cdot d_n + d_{-n}\cdot\alpha_n$$
$$-2b_{-n}\gamma_n - 2\gamma_{-n}b_n + \tfrac{1}{2}nc_{-n}\beta_n - \tfrac{1}{2}n\beta_{-n}c_n) - 2b_0\gamma_0,$$
$$U = c_0b_0 + \sum_{n>0} (c_{-n}b_n - b_{-n}c_n - \gamma_{-n+\kappa}\beta_{n-\kappa} - \beta_{-n+\kappa}\gamma_{n-\kappa}) - (1-2\kappa)\gamma_{-\kappa}\beta_\kappa - \kappa.$$

We note that Q can be interpreted as the charge of some BRST current [6], i.e.

$$Q = \frac{1}{2\pi i}\int J(z)dz,$$

where

$$J(z) = {:}\, T_V c(z) + \mathscr{T}_V\gamma(z) + \tfrac{1}{2}T_\Omega c(z) + \tfrac{1}{2}\mathscr{T}_\Omega\gamma(z){:},$$

and the integral means taking $\mathrm{Res}_{z=0}(J(z))$.

Proposition 1.7.

(i) *All the operator sums above are well-defined on C, i.e. for each $v \in C$, only finitely many of the operator products in each sum act non-trivially on v.*

(ii) $Q^2 = 0$ iff $D = 10$.

(iii) $D_R^* = -D_R, K^* = K, Q^* = (-1)^{2\kappa+1}Q$ *with respect to* $\langle , \rangle_C$.
(iv) $[Q, K] = 0$.
(v) $\{Q, D_R\} = 0$.
(vi) $[U, Q] = Q$.
(vii) $[U, K] = 0$.
(viii) $[U, D_R] = 0$.
(ix) $D_R^2 = K$.
(x) *For* $\kappa = 0$, $Q = Kc_0 + Mb_0 + D_R\gamma_0 + N\beta_0 + Q_0 + b_0\gamma_0^2$, *where* M, N, Q_0 *are some operator sums of products of* $b_n, \beta_n, c_n, \gamma_n, n \in \mathbf{Z}\backslash 0$.
(xi) *For* $\kappa = \frac{1}{2}, Q = Kc_0 + Mb_0 + Q_0$, *where* M, Q_0 *are some operator sums of products of* $b_n, \beta_{m+1/2}, c_n, \gamma_{m+1/2}$, $n \in \mathbf{Z}\backslash 0$, $m \in \mathbf{Z}$.

Proof.
(i) is a direct consequence of the normal ordering in the operator sums.
(ii) is done in [13, 15].
(iii) follows from Proposition 1.5.
(iv) to (viii) are obtained by straightforward computations, some of which are done in [13, 15, 19].
(ix)

$$\begin{aligned} D_R^2 &= \tfrac{1}{2}\{\pi(G_0) + \rho(G_0), \pi(G) + \rho(G_0)\} \\ &= \pi(L_0) + \rho(L_0) = K. \end{aligned}$$

(x) and (xi) Calculations are done in [13]. □

It is obvious that K, U are diagonalizable in $C(p, \kappa)$ and their eigenvalues are respectively $[(1-\kappa)\mathbf{Z} + \frac{1}{2}p \cdot p]$- and $(\mathbf{Z} + \kappa)$-valued. Thus $C(p, \kappa)$ is turned into a doubly graded space. By convention, we let $C(p, \kappa)$ be graded by U and $-K$:

$$C = C(p, \kappa) = \coprod_{r \in Z + \kappa} C^r, \quad C^r = \coprod_{s \in (1-\kappa)Z - (1/2)p \cdot p} C^{r;s},$$

where $C^{r;s} = \{v \in C : Uv = rv, Kv = -sv\}$.

From now on, we will take $D = 10$. Then by Proposition 1.7 (ii), (iv), (vi), (C^*, Q) is a complex with $Q: C^{r,s} \to C^{r+1;s}$.

Definition 1.8. *The* $(\mathbf{Z} + \kappa)$*-graded complex* $(C^*(p, \kappa), Q)$ *is called the* BRST *complex. Its cohomology is denoted by* $H^*(p, \kappa)$.

We note here that since the eigenvalues of $-K$ are bounded above, for all sufficiently large positive s (independent of r) $C^{r;s} = 0$. We will call the $\mathbf{Z} + \kappa$-gradation, the ghost-number, and call the $(1-\kappa)\mathbf{Z} - \frac{1}{2}p \cdot p$-gradation, the degree. Finally, we state an important fact about $\langle , \rangle_C$ which follows from its definition.

Proposition 1.9. *For each* $r, s, \langle , \rangle_C$ *is non-degenerate when restricted to* $C^{r;s} \oplus C^{-r;s}$. *In particular it is non-degenerate in each eigenspace of* K *and in the whole* $C(p, \kappa)$.

2. Relative BRST Cohomology

Proposition 2.1. [13] $\{Q, b_0\} = K, [Q, \beta_0] = D_R$.

Definition 2.2. *Define the subspaces of* $C^r(p,\kappa), r\in\mathbf{Z}+\kappa$,

$$\begin{aligned}\mathscr{B}^r(p,\kappa) &= \operatorname{Ker} b_0 && \text{for } \kappa=\tfrac{1}{2}, \quad r\in\mathbf{Z}+\kappa\\ &= \operatorname{Ker} b_0 \cap \operatorname{Ker}\beta_0 && \text{for } \kappa=0, \quad r\in\mathbf{Z}\\ \mathscr{F}^n(p,\kappa) &= \mathscr{B}^{n-\kappa}(p,\kappa)\cap\operatorname{Ker} K && \text{for } n\in\mathbf{Z}\\ C^n_{\mathrm{rel}}(p,\kappa) &= \mathscr{F}^n(p,\kappa) && \text{for } \kappa=\tfrac{1}{2}\\ &= \mathscr{F}^n(p,\kappa)\cap\operatorname{Ker} D_R && \text{for } \kappa=0.\end{aligned}$$

We write $\mathscr{B}=\coprod_{\alpha\in Z+\kappa}\mathscr{B}^\alpha, \mathscr{F}=\coprod_{n\in Z}\mathscr{F}^n$ *and* $C_{\mathrm{rel}}=\coprod_{n\in Z}C^n_{\mathrm{rel}}$.

Remark. Note that $\mathscr{B}(p,\kappa)$ is the linear span of canonical basis vectors of the form $\Psi^k\otimes A|p\rangle\otimes B|q_F=1\rangle\otimes C|q_B=1-\kappa\rangle$, where A,B,C are monomials in $\mathbf{C}[\alpha^\mu_{-n}, d^\mu_{-n+\kappa}]$, $\mathbf{C}[b_{-n},c_{-n}]$ and $\mathbf{C}[\beta_{-n+\kappa},\gamma_{-n+\kappa}]$[2] respectively. Similarly, $\mathscr{F}(p,\kappa)$ is the (finite dimensional) linear span of those canonical basis vectors with $\deg A+\deg B+\deg C=p\cdot p/2-\kappa$.

By Proposition 2.1 and Definition 2.2, we see that Q leaves C_{rel} invariant and that C_{rel} is graded by $U+\kappa$:

$$C^n_{\mathrm{rel}}=\{v\in C_{\mathrm{rel}}:(U+\kappa)v=nv\},\quad n\in\mathbf{Z}.$$

Since $[U,Q]=Q, (C^*_{\mathrm{rel}}(p,\kappa),Q)$ is a $\mathbf{Z}$-graded complex.

Definition 2.3. $(C^*_{\mathrm{rel}}(p,\kappa),Q)$ *is called the relative* BRST *complex. Its cohomology is denoted by* $H^*_{\mathrm{rel}}(p,\kappa)$. *We will also call the* $\mathbf{Z}$*-gradation of the complex given by* $U+\kappa$*, the ghost-number.*

Theorem 2.4. *Suppose* $p\neq 0$: *then* $H^n_{\mathrm{rel}}(p,\kappa)=0$ *unless* $n=0$.

This is the analogue of the vanishing theorem proven by FGZ. We will see that as consequences, the zeroth relative cohomology group can be identified with the positive definite subspace of physical states and that its Euler characteristic and signature is closely related to various modular functions [10, 17]. We will first discuss these consequences and will return to the proof of this theorem in the next section.

In order for $H^0_{\mathrm{rel}}(p,\kappa)$ to be physical, there must be an inner product. Does there exist such a "natural" inner product? We need first a hermitian form on the complex $C_{\mathrm{rel}}(p,\kappa)$.

Proposition 2.5. $\langle,\rangle_C$ *is identically zero when restricted to* $\mathscr{B}(p,\kappa)$ *and hence to* $C_{\mathrm{rel}}(p,\kappa)$.

Proof. Since $C_{\mathrm{rel}}(p,\kappa)\subset\mathscr{B}(p,\kappa)$, it is enough to show that $\langle,\rangle_C\equiv 0$ when restricted to $\mathscr{B}(p,\kappa)$. Since $\{b_0,c_0\}=1$ and $b_0^2=0$, we have $\operatorname{Ker} b_0=\operatorname{Im} b_0$. Thus by Definition 2.2, $\mathscr{B}(p,\kappa)=b_0C(p,\kappa)$. Proposition 1.5 implies that $b_0^*=\pm b_0$. Thus $\langle,\rangle_C\equiv 0$ on $\mathscr{B}(p,\kappa)$. □

Definition 2.6. *For* $\kappa=\frac{1}{2}$, *define the hermitian form on* $\mathscr{B}(p,\frac{1}{2})$ *by* $\langle\cdot,\cdot\rangle=\langle\cdot,c_0\chi_{1/2}\cdot\rangle_C$, *where* $\chi_{1/2}=(-1)^C$ *is the* $\mathbf{Z}_2$*-grading on* $C(p,\frac{1}{2})$.

[2] See footnote on notation in Sect. 1

Note: $\mathscr{B}(p,\frac{1}{2}) = b_0 C(p,\frac{1}{2}), \{b_0, c_0\} = 1$, together with the second part of Proposition 1.9 implies that $\langle \cdot, \cdot \rangle_{\mathscr{B}}$ is non-degenerate on $\mathscr{B}(p,\kappa)$. Hermiticity can be easily checked.

The case $\kappa = 0$ requires special treatment. In fact, $\langle \cdot, c_0 \cdot \rangle_C \equiv 0$ when restricted to $C_{\text{rel}}(p,0) \subset \mathscr{B}(p,0) = V(p,0) \otimes b_0 \Lambda_\infty \otimes \mathscr{T}_1 \cap \operatorname{Ker} \beta_0$ because $\langle \mathscr{T}_1, \mathscr{T}_1 \rangle_{\mathscr{T}} = 0$. To define $\langle , \rangle_{\text{rel}}$ on $C_{\text{rel}}(p,0)$, we need to understand the structure of the space better. First note that $D_R^2 = K$, $\{D_R, b_0\} = 0$ and $[D_R, \beta_0] = -2b_0$. It follows that $\mathscr{F} \to \mathscr{F} \to \mathscr{F}$ is a complex ($\mathscr{F} = \operatorname{Ker} b_0 \cap \operatorname{Ker} \beta_0 \cap \operatorname{Ker} K$), with $D_R|_{\mathscr{F}}$ as its differential.

Theorem 2.7. *Assume* $p \neq 0$: *the above complex has zero cohomology. Thus* $C_{\text{rel}}(p,0) = D_R \mathscr{F}(p,0)$.

Proof. For $p \neq 0$, say $p^1 \neq 0$, we have $\{D_R|_{\mathscr{F}}, d_0^1/p^1\} = 1$. Applying this formula on $\Omega \in \operatorname{Ker} D_R|_{\mathscr{F}}$, one sees that $\Omega \in \operatorname{Im} D_R|_{\mathscr{F}}$. Thus, $\operatorname{Ker} D_R|_{\mathscr{F}} = \operatorname{Im} D_R|_{\mathscr{F}}$. □

Note that $\langle \mathscr{T}_1, \mathscr{T}_1 \rangle_{\mathscr{T}} = 0$ because $\langle , \rangle_{\mathscr{T}}$ is a non-degenerate pairing between $\mathscr{T}_0$ and $\mathscr{T}_1$. Also $\mathscr{T}_1 \cap \operatorname{Ker} \beta_0 = \mathbf{C}[\beta_{-n}, \gamma_{-n}] | q_B = 1 \rangle$.[3] Thus, following [6, 16], we define a "picture changing operator"

$$\chi_{\mathscr{T}} : \mathbf{C}[\beta_{-n}, \gamma_{-n}] | q_B = 1 \rangle \to \mathbf{C}[\beta_{-n}, \gamma_{-n}] | q_B = 0 \rangle$$

by

$$\chi_{\mathscr{T}} P | q_F = 1 \rangle = P | q_F = 0 \rangle$$

for any $P \in \mathbf{C}[\beta_{-n}, \gamma_{-n}]$.

Definition 2.8. *Define a hermitian form on* $\mathscr{B}(p,0)$ *by* $\langle \cdot, c_0 \chi_0 \cdot \rangle_C$, *where* $\chi_0 = (-1)^C 1_V \otimes 1_\Lambda \otimes \chi_{\mathscr{T}}$, *and* $(-1)^C$ *is the* $\mathbf{Z}_2$*-grading on* $C(p,0)$.

Again it is easy to check hermiticity and non-degeneracy. Furthermore, it follows from the commutation relations (Eqs. (1) to (4) in Sect. 1) and Proposition 1.5 that

Proposition 2.9.

(i) *With respect to* $\langle \cdot, c_0 \chi_0 \cdot \rangle_C$ *on* $\mathscr{B}(p,0)$,

$$b_n^* = -b_{-n} \quad c_n^* = -c_{-n} \quad d_n^{\mu *} = -d_{-n}^{\mu}$$
$$\beta_n^* = -\beta_{-n} \quad \gamma_n^* = \gamma_{-n} \quad \alpha_n^{\mu *} = \alpha_{-n}^{\mu}, \quad n \in \mathbf{Z}$$

(except for $b_0, c_0, \beta_0, \gamma_0$*).*

(ii) *With respect to* $\langle \cdot, c_0 \chi_{1/2} \cdot \rangle_C$ *on* $\mathscr{B}(p,\frac{1}{2})$,

$$b_n^* = b_{-n} \quad c_n^* = c_{-n} \quad d_{n-1/2}^{\mu *} = d_{-n+1/2}^{\mu}$$
$$\beta_{n-1/2}^* = -\beta_{-n+1/2} \quad \gamma_{n-1/2}^* = \gamma_{-n+1/2} \quad \alpha_n^{\mu *} = \alpha_{-n}^{\mu}, \quad n \in \mathbf{Z}$$

(except for b_0, c_0*).*

[3] See footnote on notation in Sect. 1

Proposition 2.10.

(i) $\langle \cdot, c_0\chi_0 \cdot \rangle_C$ *restricted to* $\mathscr{F}(p,0)$ *is non-degenerate.*
(ii) $D_R|_{\mathscr{F}}^* = -D_R|_{\mathscr{F}}$. *Thus* $\langle \cdot, c_0\chi_0 \cdot \rangle_C \equiv 0$ *when restricted to* $C_{\mathrm{rel}}(p,0) = \mathscr{F}(p,0) \cap \operatorname{Ker} D_R$.
(iii) $\langle \cdot, c_0\chi_{1/2} \cdot \rangle_C$ *restricted to* $C_{\mathrm{rel}}(p,\frac{1}{2}) = \mathscr{F}(p,\frac{1}{2})$ *is non-degenerate.*

Proof.

(i) Recall that $\langle \cdot, c_0\chi_0 \cdot \rangle_C$ is non-degenerate on $\mathscr{B}(p,0)$ and that $\mathscr{F}(p,0) = \mathscr{B}(p,0) \cap \operatorname{Ker} K$. Given $u \in \mathscr{F}(p,0) \backslash 0$, we can find $v \in \mathscr{B}(p,0)$ such that $\langle u, c_0\chi_0 v \rangle_C \neq 0$. Since $Ku = 0$ and $K^* = K$ with respect to $\langle , \rangle_C$, we can choose $v \in \operatorname{Ker} K$.
(ii) Since $\mathscr{F}(p,0) \subset \mathscr{B}(p,0)$, it is enough to check $D_R|_{\mathscr{B}}^* = -D_R|_{\mathscr{B}}$ with respect to $\langle \cdot, c_0\chi_0 \cdot \rangle_C$. But this follows from Proposition 2.9 and that $c_0\chi_0$ commutes with all the generators (except $b_0, c_0, \beta_0, \gamma_0$). Now $D_R|_{\mathscr{F}}^* = K$ acts like zero in $\mathscr{F}(p,0)$. Using Theorem 2.7 and $D_R|_{\mathscr{F}}^* = -D_R|_{\mathscr{F}}$, we have the result.
(iii) Since $\mathscr{F}(p,\frac{1}{2}) = \mathscr{B}(p,\frac{1}{2}) \cap \operatorname{Ker} K$, the argument for (i) applies here. □

Now using Theorem 2.7 and Proposition 2.10, we have the following well-defined non-degenerate hermitian form on $C_{\mathrm{rel}}(p,\kappa)$.

Definition 2.11. *Define the hermitian form* $\langle , \rangle_{\mathrm{rel}}$ *on* $C_{\mathrm{rel}}(p,\kappa)$ *as follows: for* $u, v \in C_{\mathrm{rel}}(p,\kappa)$, *let*

$$\begin{aligned} \langle u, v \rangle_{\mathrm{rel}} &= \langle u, v \rangle_{\mathscr{B}} \quad \text{if} \quad \kappa = \tfrac{1}{2} \\ &= \sqrt{-1}\langle u, \hat{v} \rangle_{\mathscr{B}} \quad \text{if} \quad \kappa = 0 \quad \text{where} \quad v = D_R\hat{v}. \end{aligned}$$

Remark. In the case $\kappa = 0$, although the way $\langle\ \rangle_{\mathrm{rel}}$ is defined seems rather peculiar, given any $u, v \in C_{\mathrm{rel}}(p,0)$ one can in principle compute $\langle u, v \rangle_{\mathrm{rel}}$. Note first that

$$\langle u, v \rangle_{\mathrm{rel}} = \sqrt{-1}\langle u, \hat{v} \rangle_{\mathscr{B}}, \quad v = D_R\hat{v}$$

is independent of the choice of $\hat{v}$. For if $v = D_R\hat{v} = D_R\tilde{v}$, then

$$\langle u, \hat{v} - \tilde{v} \rangle_{\mathscr{B}} = -\langle \hat{u}, D_R(\hat{v} - \tilde{v}) \rangle_{\mathscr{B}} = 0.$$

Recall that (Theorem 2.7) there is a contracting homotopy $\varepsilon: \mathscr{F} \to \mathscr{F}$ such that $\{D_R, \varepsilon\} = 1$. Then one finds that

$$\langle u, \varepsilon v \rangle_{\mathscr{B}} = \langle u, \varepsilon D\hat{v} \rangle_{\mathscr{B}} = \langle u, \hat{v} \rangle_{\mathscr{B}}.$$

Thus

$$\langle u, v \rangle_{\mathrm{rel}} = \sqrt{-1}\langle u, \varepsilon v \rangle_{\mathscr{B}}.$$

It is remarkable that the right-hand side of this equation is independent of the choice ε.

Definition 2.12.

(i) *If* T *is a* $\frac{1}{2}\mathbf{Z}$*-graded vector space,* $T = \coprod_{n \in 1/2\mathbf{Z}} T^n$, *with a non-degenerate hermitian form* $\langle , \rangle_T$, *then we call* T *hermitian. Suppose* $\dim T^n < +\infty$ *for all* n *and* $T^n = 0$ *for all large enough positive* n. *Then the character and signature of* T *are* *respectively*

$$\mathrm{ch}_q T = \sum_{n\in 1/2Z} \dim T^{-n} q^n,$$
$$\mathrm{sign}_q T = \sum_{n\in 1/2Z} \mathrm{sign}\, T^{-n} q^n, \quad \textit{where}$$
$$\mathrm{sign}\, T^n = \#\{+ \text{ signs of } \langle,\rangle_T \textit{ restricted to } T^n \textit{ in its diagonal form}\}$$
$$- \#\{- \text{ signs of } \langle,\rangle_T \textit{ restricted to } T^n \textit{ in its diagonal form}\}.$$

(ii) *If* (C^*, Q) *is a complex with finite support and* **Z**-*gradation, then we write* $\mathrm{char}\, C = \sum_{k\in Z} (-1)^k \dim C^k$.

Definition 2.13. [10] *Let* $\varphi(q) = \prod_{n>0} (1-q^n)$, $\varphi_0(q) = \varphi(q)^2 \varphi(q^2)^{-1}$, $\varphi_{1/2}(q) = \varphi(q^{1/2})\varphi(q^2)\varphi(q)^{-1}$, *and* $p_\kappa^{(N)}(n)$, $n\in(1-\kappa)\mathbf{Z}$, *be the coefficient of* q^n *in* $\varphi_\kappa(q)^{-N}$.

Proposition 2.14.
(i) $Q^* = Q$ *with respect to* $\langle,\rangle_{\mathrm{rel}}$.
(ii) $\mathrm{char}\, C_{\mathrm{rel}}(p,\kappa) = \frac{1}{2}(2\kappa+1)\dim R(p,\kappa) p_\kappa^{(8)}(\kappa - (p\cdot p/2)) = \mathrm{sign}\, C_{\mathrm{rel}}(p,\kappa)$.

Proof.
(i) Recall that $\mathscr{B}(p,\kappa) = b_0 C(p,\kappa)$. Similarly, $\mathscr{F}(p,\kappa) = b_0(C(p,\kappa)\cap \mathrm{Ker}\, K)$. For $\kappa = \frac{1}{2}$, $C_{\mathrm{rel}}(p,\frac{1}{2}) = \mathscr{F}(p,\frac{1}{2})$. Let $u, v\in C_{\mathrm{rel}}(p,\frac{1}{2})$. Then

$$\langle Qu, v\rangle_{\mathrm{rel}} = \langle Qu, c_0\chi_{1/2} v\rangle_C = \langle u, c_0 \chi_{1/2} Qv\rangle_C + \langle u, \{Q, c_0\}\chi_{1/2} v\rangle_C. \tag{1}$$

But $u = b_0\hat{u}, v = b_0\hat{v}$ for some $\hat{u},\hat{v}\in C(p,\kappa)\cap \mathrm{Ker}\, K$ and $[b_0, \{Q, c_0\}] = 0$. Thus, the second term of (1) vanishes because $b_0^* = b_0$ and $b_0^2 = 0$. Thus the Right-hand side of (1) is $\langle u, Qv\rangle_{\mathrm{rel}}$. By non-degeneracy we have $Q^* = Q$. For $\kappa = 0$, let $u, v\in C_{\mathrm{rel}}(p,0)$. By Theorem 2.7,

$$u = D_R\hat{u}, \quad v = D_R\hat{v} \quad \text{for some} \quad \hat{u},\hat{v}\in\mathscr{F}(p,0).$$

By definition,

$$\langle Qu, v\rangle_{\mathrm{rel}} = \sqrt{-1}\langle QD_R\hat{u}, \hat{v}\rangle_{\mathscr{B}} = \sqrt{-1}\langle QD_R\hat{u}, c_0\chi_0\hat{v}\rangle_C. \tag{2}$$

As in the case of $\kappa = \frac{1}{2}$, the right-hand side $= -\sqrt{-1}\langle D_R\hat{u}, c_0\chi_0 Q\hat{v}\rangle_C = -\sqrt{-1}\langle D_R\hat{u}, Q\hat{v}\rangle_{\mathscr{B}} = \langle u, Qv\rangle_{\mathrm{rel}}$ since $Qv = -D_R Q\hat{v}$. Thus $Q^* = Q$.

(ii) Recall that $\mathscr{B}(p,\kappa)$ is the linear span of the canonical basis vectors of the form

$$\begin{aligned}\Omega = N_\Omega \Psi^i &\otimes (\alpha^1_{-1})^{p_{11}}(\alpha^2_{-1})^{p_{12}}\cdots(d^1_{-1+\kappa})^{q_{11}}(d^2_{-1+\kappa})^{q_{12}}\cdots|p=0\rangle\\ &\otimes (c_{-1})^{m_1}(c_{-2})^{m_2}\cdots(b_{-1})^{n_1}(b_{-2})^{n_2}\cdots|q_F = 1\rangle\\ &\otimes (\gamma_{-1+\kappa})^{k_1}(\gamma_{-2+\kappa})^{k_2}\cdots(\beta_{-1+\kappa})^{l_1}(\beta_{-2+\kappa})^{l_2}\cdots|q_B = 1-\kappa\rangle,\end{aligned} \tag{3}$$

where the powers $m_i, n_i, q_{i\mu}\in\{0,1\}, k_i, l_i, p_{i\mu}\in\{0,1,2,\ldots\}$ such that all but finitely many of these powers are zero. $\{\Psi^i\}$ is a basis of $R(p,\kappa)$ and N_Ω is some fixed but yet undetermined positive constant. We will denote the basis by A. Recall also that $\mathscr{F}(p,\kappa) = \mathscr{B}(p,\kappa)\cap \mathrm{Ker}\, K$ (Definition 2.2). Thus $\mathscr{F}(p,\kappa)$ is the zeroth eigenspace of K. Note that this subspace is **Z**-graded by $U+\kappa$, i.e. $\mathscr{F}^n(p,\kappa)$ is the nth eigenspace of $U+\kappa$. Thus we have

$$\sum_{n\in Z} (-1)^n \dim \mathscr{F}^n(p,\kappa) = \text{const term}\, \mathrm{Tr}\langle (-1)^{U+\kappa} q^K(A)\rangle_{\mathrm{trivial}}. \tag{4}$$

Here $\langle(-1)^{U+\kappa}q^K(A)\rangle_{\text{trivial}}$ is the matrix of the operator $(-1)^{U+\kappa}q^K$ in the basis A, with respect to the bilinear form defined by $\langle\Omega,\Omega'\rangle_{\text{trivial}}=\delta_{\Omega,\Omega'},\Omega,\Omega'\in A$. Now using the explicit expression for the ghost number operator U and the kinetic operator K together with the above basis A, direct calculations give

$$\operatorname{Tr}\langle(-1)^{U+\kappa}q^K(A)\rangle_{\text{trivial}}=q^{(p\cdot p/2)-\kappa}\varphi_\kappa(q)^{-8}\dim R(p,\kappa). \tag{5}$$

For $\kappa=\frac{1}{2}$, $C^n_{\text{rel}}(p,\frac{1}{2})=\mathscr{F}^n(p,\frac{1}{2})$. Thus (4) and (5) give the desired result for $\operatorname{char}C_{\text{rel}}(p,\frac{1}{2})$. For $\kappa=0$, unfortunately the above formula for char does not hold. However, Theorem 2.7 implies that

$$\dim\mathscr{F}^n(p,0)=2\dim C^n_{\text{rel}}(p,0). \tag{6}$$

Thus (4), (5) and (6) give the desired result for $\operatorname{char}C_{\text{rel}}(p,0)$. □

To compute the signatures, we introduce the following notion, Let $A=\{a_1,a_2,\dots,a_N\}$ be a basis of a hermitian space W, and $L:W\to W$ be linear. Then $\langle L(A)\rangle_W$ denotes the matrix whose (i,j)-entry is $\langle a_i,La_j\rangle_W$. We say that A has the canonical pairing property with respect to $\langle,\rangle_W$, if the matrix $\langle 1(\mathrm{A})\rangle_W$ has exactly one non-zero entry in each column and $\langle a_i,a_j\rangle=0$ or ±1 for each (i,j).

Lemma 2.14.1 *If A has the canonical pairing property, then*

$$\operatorname{sign}W=\operatorname{Tr}\langle 1(A)\rangle_W.$$

Proof. Let $A=\{a_1\cdots a_N\}$. By reordering, we can assume that for each i, exactly one of $\langle a_i,a_{i-1}\rangle_W,\langle a_i,a_i\rangle_W,\langle a_i,a_{i+1}\rangle_W$ is nonzero ($a_0\equiv0\equiv a_{N+1}$). Thus $\langle 1(A)\rangle_W$ is block diagonal, each block being of the form $\pm[1]$ or

$$\pm\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Thus, its diagonal form $O\langle 1(A)\rangle_W O^T$ (where O is an orthogonal matrix) can be obtained from $\langle 1(A)\rangle_W$ by replacing

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

by

$$\begin{bmatrix}1/\sqrt{2}&0\\0&-1/\sqrt{2}\end{bmatrix}.$$

By definition, $\operatorname{sign}W=\sum_{i=1\cdots N}\operatorname{sign}(O\langle 1(A)\rangle_W O^T)_{ii}$. But the right-hand side $=\sum_{i=1\cdots N}\langle a_i,1(A)a_i\rangle_W=\operatorname{Tr}\langle 1(A)\rangle_W$. □

Remark. $\operatorname{sign}W=\operatorname{Tr}\langle 1(A)\rangle_W$ does not hold for an arbitrary basis A.

Counter Example. Let $W=\mathbf{R}^2, A=\{a_1,a_2\}$ be a basis,

$$\langle a_1,a_1\rangle_W=0,\langle a_1,a_2\rangle_W=\langle a_2,a_1\rangle_W=\frac{1}{\sqrt{2}},\langle a_2,a_2\rangle_W=1.$$

Then obviously $\operatorname{Tr}\langle 1(A)\rangle_W=1$. But $\operatorname{sign}W=0$.

We now compute $\operatorname{sign}C_{\text{rel}}(p,\frac{1}{2})$. Denote the canonical basis (given by (3)) of the subspace $\mathscr{B}^{;s}(p,\frac{1}{2}) \equiv \mathrm{Ker}(K-(p\cdot p/2)+\frac{1}{2}-s)\cap\mathscr{B}(p,\frac{1}{2})$ by A_s. Then $A=\bigcup_s A_s$ is the canonical basis of $\mathscr{B}(p,\frac{1}{2})=\bigoplus_{s\in 1/2Z_+}\mathscr{B}^{;s}(p,\frac{1}{2})$. Because $R(p,\frac{1}{2})=\mathbf{C}$, we will drop Ψ^i from (3) for $\kappa=\frac{1}{2}$.

Lemma 2.14.2. *For each s, the basis A_s of $\mathscr{B}^{;s}(p,\frac{1}{2})$ (with suitable normalization constant N_Ω; see (3)) has the canonical pairing property with respect to $\langle\cdot,\cdot\rangle_{\mathscr{B}}$.*

Proof. Let Ω,Ω' be two canonical basis vectors. Then

$$\langle\Omega,\Omega'\rangle_{\mathscr{B}}\equiv\langle\Omega,c_0\chi_{1/2}\Omega'\rangle_C\neq 0,$$

iff

$$m_i=n_i',\quad n_i=m_i',\quad k_i=l_i'.$$
$$l_i=k_i',\quad p_{i\mu}=p_{i\mu}',\quad q_{i\mu}=q_{i\mu}',$$

for all i,μ, where the powers with (without) primes correspond to the basis vector $\Omega'(\Omega)$. This implies that each $\Omega\in A_s$ pairs with exactly one other $\Omega'\in A_s$. Thus we can choose N_Ω so that each matrix entry of $\langle 1(A_s)\rangle_{\mathscr{B}}$ is either 0 or ± 1. □

Now $C_{\mathrm{rel}}(p,\frac{1}{2})=\mathscr{B}^{;s}(p,\frac{1}{2})$ for $s=-(p\cdot p/2)+\frac{1}{2}$. Thus by the above two lemmas:

$$\begin{aligned}\operatorname{sign} C_{\mathrm{rel}}(p,\tfrac{1}{2}) &= \mathrm{Tr}\langle 1(A_s)\rangle_{\mathscr{B}}\\ &= \text{const term}\left[\sum_{r\in 1/2Z_+} q^{r+(p\cdot p)/2-(1/2)}\,\mathrm{Tr}\langle 1(A_r)\rangle_{\mathscr{B}}\right]\\ &= \text{const term}\,\mathrm{Tr}\langle q^K(A)\rangle_{\mathscr{B}}. \end{aligned}\tag{7}$$

Using the above basis vectors and the commutation relations in Sect. 1 to do direct calculations, we get

$$\mathrm{Tr}\langle q^K(A)\rangle_{\mathscr{B}}=q^{(p\cdot p/2)-(1/2)}\varphi_{1/2}(q)^{-8}.\tag{8}$$

Combining (7) and (8) gives the desired result for $\operatorname{sign} C_{\mathrm{rel}}(p,\frac{1}{2})$.

We now turn to $\kappa=0$. This calculation is slightly less trivial because $C_{\mathrm{rel}}(p,0)=\mathscr{F}(p,0)\cap\mathrm{Ker}\,D_R$ and D_R is in general not diagonal in the basis defined by (3). Note also that $C_{\mathrm{rel}}(p,0)$ is zero unless $p\cdot p/2$ is a non-positive integer.

Case 1. $p\cdot p=0$. One can easily check that (Definition 1.3)

$$C_{\mathrm{rel}}(p,0)=\mathrm{Span}\{\Psi^i_{\rightarrow}\otimes|p\rangle\otimes|q_F=1\rangle\otimes|q_B=1\rangle,\ i=1,\dots,16\}.$$

Using Definition 1.3, 2.11, we have

$$\begin{aligned}\operatorname{sign} C_{\mathrm{rel}}(p,0) &= \sum_{i=1,\dots,16}\sqrt{-1}(\Psi^i_{\rightarrow}\otimes|p\rangle\otimes|q_F=1\rangle\otimes|q_B=1\rangle,\\ &\qquad \Psi^i_{\leftarrow}\otimes|p\rangle\otimes|q_F=1\rangle\otimes|q_B=1\rangle)_{\mathscr{B}}\\ &= \sum_{i=1,\dots,16}\sqrt{-1}(\Psi^i_{\rightarrow},\Psi^i_{\leftarrow})_R=16.\end{aligned}$$

Case 2. $p\cdot p/2$ is negative integer. There are a few facts we need to establish first.

Recall (Definition 2.2) that[4]

$$\mathscr{B}(p,0)=R(p,0)\otimes \mathbf{C}[\alpha^{\mu}_{-n},d^{\mu}_{-n}]|p\rangle \otimes \mathbf{C}[b_{-n},c_{-n}]|q_F=1\rangle \otimes \mathbf{C}[\beta_{-n},\gamma_{-n}]|q_B=1\rangle. \tag{9}$$

Then $\mathscr{B}(p,0)=R(p,0)\otimes\mathscr{S}$, where we denote the last three factors as $\mathscr{S}$. Let $\Psi^i_{\pm}=1/\sqrt{2}(1/m\Psi^i_{\rightarrow}\pm\sqrt{-1}\Psi^i_{\leftarrow})$ (cf. Definition 1.3). Then $p\cdot d_0\Psi^i_{\pm}=\pm\sqrt{-1}m\Psi^i_{\pm}$ and $m\langle\Psi^i_{\pm},\Psi^i_{\pm}\rangle_R=\pm\delta_{ij},\langle\Psi^i_{+},\Psi^i_{-}\rangle_R=0$, for $i,j=1,\dots,16$, and $m=\sqrt{-p\cdot p/2}$. Let $\{\varepsilon(s,k)\}_{k=1\dots M(s)}\cup\{\Theta(s,l)\}_{l=1\dots N(s)}$ be the canonical monomials in $\mathbf{C}[\alpha^{\mu}_{-n},d^{\mu}_{-n}]\otimes\mathbf{C}[b_{-n},c_{-n}]\otimes\mathbf{C}[\beta_{-n},\gamma_{-n}]$ such that for each $s=1,2,\dots,$

(i) $[K,\varepsilon(s,k)]=s\varepsilon(s,k),[K,\Theta(s,l)]=s\Theta(s,l)$ for all k,l.
(ii) $\varepsilon(s,k)(\Theta(s,l))$ are $\mathbf{Z}_2$-even ($\mathbf{Z}_2$-odd).

We abbreviate $|p\rangle\otimes|q_F=1\rangle\otimes|q_B=1\rangle$ as Ω_0. Let

$$A^0_s=\{\varepsilon(s,k)\Psi^i_{\pm}\otimes\Omega_0,k=1,\dots,M(s),i=1,\dots,16\},$$
$$A^1_s=\{\Theta(s,l)\Psi^i_{\pm}\otimes\Omega_0,l=1,\dots,N(s),i=1,\dots,16\}. \tag{10}$$

Note that $K\Psi^i_{\pm}\otimes\Omega_0=(p\cdot p/2)\Psi^i_{\pm}\otimes\Omega_0$. Thus by (i), $A^0_s\cup A^1_s$ is the canonical basis of $\mathscr{F}(p,0)=\mathscr{B}(p,0)\cap\operatorname{Ker}K$ whenever $s=-p\cdot p/2$. In this case $\dim\mathscr{F}(p,0)=32(N(s)+M(s))$.

Lemma 2.14.3. *$D_RA^0_s$, $D_RA^1_s$ are two bases of $C_{\rm rel}(p,0)=\mathscr{F}(p,0)\cap\operatorname{Ker}D_R$ when $s=-p\cdot p/2$.*

Proof. By Theorem 2.7, observe that $D_RA^0_s$, $D_RA^1_s$ are both in $C_{\rm rel}(p,0)$. Using the definition of D_R, we can write $D_R|_{\mathscr{F}}=p\cdot d_0+\tilde{D}$, where $\tilde{D}$ is a $\mathbf{Z}_2$-odd operator containing no $c_0,b_0,d^{\mu}_0,\alpha^{\mu}_0,\beta_0,\gamma_0$. Using this formula, one can easily show that $D_RA^0_s$, $D_RA^1_s$ are both linearly independent sets containing $32\,M(s)$, $32\,N(s)$ elements respectively. But Theorem 2.7 implies that

$$2\dim C_{\rm rel}(p,0)=\dim\mathscr{F}(p,0)=32(N(s)+M(s)).$$

Thus we must conclude that

$$\dim C_{\rm rel}(p,0)=32\,N(s)=32\,M(s).$$

This means that $D_RA^0_s,D_RA^1_s$ are both bases of $C_{\rm rel}(p,0)$. □

Recall that $\mathscr{B}(p,0)=R(p,0)\otimes\mathscr{S}$ (see (9)). When we computed $\operatorname{sign}C_{\rm rel}(p,\frac{1}{2})$, the hermitian structure of $\mathscr{B}(p,\frac{1}{2})$ and its canonical basis (cf. Lemma 2.14.2) played a crucial role. We will see that as $\mathscr{B}(p,\frac{1}{2})$, $\mathscr{S}$ has a similar structure which simplifies the computation of $\operatorname{sign}C_{\rm rel}(p,0)$.

Recall that

$$\mathscr{S}=\mathbf{C}[\alpha^{\mu}_{-n},d^{\mu}_{-n}]\otimes\mathbf{C}[b_{-n},c_{-n}]\otimes\mathbf{C}[\beta_{-n},\gamma_{-n}]\Omega_0,\quad\text{where}$$
$$\Omega_0=|\mathrm{p}\rangle\otimes|q_F=1\rangle\otimes|q_B=1\rangle. \tag{11}$$

[4] See footnote on notation in Sect. 1

Define a non-degenerate hermitian bilinear form $(,)_{\mathscr{S}}$ on $\mathscr{S}$ by

$$(\Omega_0, \Omega_0)_{\mathscr{S}} = 1, \tag{12}$$

$$b_n^* = b_{-n} \quad c_n^* = c_{-n} \qquad d_n^{\mu *} = d_{-n}^{\mu}$$

$$\alpha_n^{\mu *} = \alpha_{-n}^{\mu} \quad \beta_n^* = -\beta_{-n} \quad \gamma_n^* = \gamma_{-n}$$

where $n \in \mathbf{Z} \backslash 0$.

Then one can easily check that

$$(,)_{\mathscr{B}} = (,)_R \otimes (,)_{\mathscr{S}} \tag{13}$$

$$(\Theta(s,k)\Omega_0, \varepsilon(s,l)\Omega_0)_{\mathscr{S}} = 0 \tag{14}$$

for all $k, l = 1, \ldots, N(s)$. Let $A_s = \{\Theta(s,k)\Omega_0, \varepsilon(s,k)\Omega_0, k = 1, \ldots, N(s)\}$. Then $\bigcup_{s \geqq 1} A_s$ is a basis of $\mathscr{S}$. Note that $\mathscr{S}$ inherits a $\mathbf{N}$-graded structure from this basis, $s = \bigoplus_{s \geqq 1} s^{;s}$, where $s^{;s}$ is the $\mathbf{C}$-span of A_s.

Lemma 2.14.4. *With suitable normalization, the basis A_s of $\mathscr{S}^{;s}$ has the canonical pairing property with respect to $(\cdot,\cdot)_{\mathscr{S}}$.*

The proof is similar to that of Lemma 2.14.2.

Lemma 2.14.5. *With suitable normalization, $D_R A_s^0$ and $D_R A_s^1 (s = -p \cdot p/2)$ are two bases of $C_{\mathrm{rel}}(p, 0)$ with the canonical pairing property with respect to $\langle \cdot,\cdot \rangle_{\mathrm{rel}}$.*

Proof. We will show it for $D_R A_s^0$. The argument is similar for $D_R A_s^1$. From Eqs. (13) and (14), we have

$$(\Theta(s,k)\Psi_\sigma^i \otimes \Omega_0, \varepsilon(s,l)\Psi_\rho^i \otimes \Omega_0)_{\mathscr{B}} = 0 \tag{15}$$

for any $k, l = 1, \ldots, N(s)$ and $\sigma, \rho = \pm$. Write $D_R = p \cdot d_0 + \tilde{D}$ as before. Observe that $\tilde{D}\varepsilon(s,k)\Psi_{\pm}^i \otimes \Omega_0$ are linear combinations of $\Theta(s,l)\Psi_{\pm}^i \otimes \Omega_0$. Thus using these two facts, we have

$$\begin{aligned}
&(D_R \varepsilon(s,k)\Psi_\sigma^i \otimes \Omega_0 \; D_R \varepsilon(s,l)\Psi_\rho^i \otimes \Omega_0)_{\mathrm{rel}} \\
&= \sqrt{-1}(\varepsilon(s,k) p \cdot d_0 \Psi_\sigma^i \otimes \Omega_0, \varepsilon(s,l)\Psi_\rho^j \otimes \Omega_0)_{\mathscr{B}} \\
&= \sigma m (\varepsilon(s,k)\Psi_\sigma^i \otimes \Omega_0, \varepsilon(s,l)\Psi_\rho^j \otimes \Omega_0)_{\mathscr{B}} \\
&= (\varepsilon(s,k)\Omega_0, \varepsilon(s,l)\Omega_0)_{\mathscr{S}} \delta_{ij} \delta_{\sigma\rho}.
\end{aligned} \tag{16}$$

Here σ, ρ ranges over $\pm$, and $m = \sqrt{-p \cdot p/2} > 0$. Thus by Lemma 2.14.4, for each (k, i, σ) there is a unique (l, j, ρ) such that the last expression is non-zero. With suitable normalization of the vectors $D_R \varepsilon(s,k)\Psi_{\pm}^i \otimes \Omega_0 \in D_R A_s^0$, this basis has the canonical pairing property. □

We are now ready to compute $\operatorname{sign} C_{\mathrm{rel}}(p, 0)$. By Lemma 2.14.1, 2.14.5 and a short calculation, we have for $s = -p \cdot p/2$,

$$\begin{aligned}
2 \operatorname{sign} C_{\mathrm{rel}}(p, 0) &= \operatorname{Tr} \langle 1(D_R A_s^0) \rangle_{\mathrm{rel}} + \operatorname{Tr} \langle 1(D_R A_s^1) \rangle_{\mathrm{rel}} \\
&= 32 \sum_{k=1,\ldots,N(s)} (\varepsilon(s,k)\Omega_0, \varepsilon(s,k)\Omega_0)_{\mathscr{S}} \\
&\quad + 32 \sum_{k=1,\ldots,N(s)} (\Theta(s,k)\Omega_0, \Theta(s,k)\Omega_0)_{\mathscr{S}}.
\end{aligned} \tag{17}$$

Thus

$$\operatorname{sign} C_{\mathrm{rel}}(p,0) = 16 \,\text{const term}\left[\sum_{s\in Z} q^{s+(p\cdot p/2)} \operatorname{Tr}\langle 1(A_s)\rangle_{\mathscr{S}}\right]$$

$$= 16\,\text{const term} \operatorname{Tr}\langle q^K(A)\rangle_{\mathscr{S}}, \tag{18}$$

where $A = \bigcup_{s\geqq 1} A_s$. Now direct calculations can be carried out using (12) and the canonical basis A of $\mathscr{S}$ (with suitable normalization). Just as the case $\kappa = \frac{1}{2}$ (cf. (8)), we have here

$$\operatorname{Tr}\langle q^K(A)\rangle_{\mathscr{S}} = q^{p\cdot p/2}\varphi_0(q)^{-8}. \tag{19}$$

Combining (18) and (19) gives the desired result for $\operatorname{sign} C_{\mathrm{rel}}(p,0)$ □

Corollary 2.15.
(i) ("*Quantization Condition*")

$$\dim H^0_{\mathrm{rel}}(p,\kappa) = \tfrac{1}{2}(2\kappa+1)\dim R(p,\kappa)p^{(8)}_\kappa\left(\kappa - \frac{p\cdot p}{2}\right).$$

(ii) (*No-ghost Theorem*) *$H^0_{\mathrm{rel}}(p,\kappa)$ is a positive definite space.*

Proof. (i) We recall the Euler–Poincaré Principle for characteristics:

$$\sum_{n\in Z}(-1)^n \dim H^n_{\mathrm{rel}}(p,\kappa) = \sum_{n\in Z}(-1)^n \dim C^n_{\mathrm{rel}}(p,\kappa).$$

Thus by Theorem 2.4, the left-hand side gives $\dim H^0_{\mathrm{rel}}(p,\kappa)$ while the right-hand side is $\operatorname{char} C_{\mathrm{rel}}(p,\kappa)$. Hence (i) follows from Proposition 2.14 (ii).

(ii) Similarly, the Euler–Poincaré Principle for signature states that

$$\sum_{n\in Z}\operatorname{sign} H^n_{\mathrm{rel}}(p,\kappa) = \sum_{n\in Z}\operatorname{sign} C^n_{\mathrm{rel}}(p,\kappa).$$

Again Theorem 2.4 implies that the left-hand side $= \operatorname{sign} H^0_{\mathrm{rel}}(p,\kappa)$, while the right-hand side is $\operatorname{sign} C_{\mathrm{rel}}(p,\kappa)$. Thus (ii) follows from Proposition 2.14(ii) also. □

We now proceed to relating $H^0_{\mathrm{rel}}(p,\kappa)$ with the physical space well-known in the "old covariant" formalism. Thus we will recall some of the structure of $V(p,\kappa)$ (see [19]).

Definition 2.16. *Let $\mathscr{P}(p,\kappa) = \{v\in V(p,\kappa) : \pi(L_n)v = \delta_n\kappa v, \pi(G_{n+\kappa})v = 0,\ n\geqq 0\}$. It is called the space of physical states.*

When restricted to $\mathscr{P}(p,0)$, $\langle,\rangle_V = \langle\cdot\rangle_R \otimes \langle,\rangle_U$ is identically zero because $\pi(G_0)^* = -\pi(G_0)$ and $\pi(G_0)^2 = \pi(L_0)$ is zero on $\mathscr{P}(p,0)$. This is reminiscent of the fact that (cf. Proposition 2.10 (ii)) for $\kappa = 0$, $\langle,\rangle_{\mathscr{B}}$ is zero when restricted to $C_{\mathrm{rel}}(p,0)$ because $D_R^* = -D_R$ and $D_R^2 = K$ acts as zero on $C_{\mathrm{rel}}(p,0)$. Thus we have the analogue of Definition 2.11:

Definition 2.17. *Define the hermitian form $\langle,\rangle_{\mathscr{P}}$ on $\mathscr{P}(p,\kappa)$ as follows: for $u,v\in\mathscr{P}(p,\kappa)$, let*

$$\langle u,v\rangle_{\mathscr{P}} = \langle u,v\rangle_V \qquad \text{if } \kappa = \tfrac{1}{2}$$

$$= \sqrt{-1}\langle u,\hat{v}\rangle_V \quad \text{if } \kappa = 0 \quad \text{where} \quad v = \pi(G_0)\hat{v}.$$

We note that $\langle , \rangle_{\mathscr{P}}$ is a well-defined hermitian form on $\mathscr{P}(p,\kappa)$ in the very same way $\langle , \rangle_{\mathrm{rel}}$ is on $C_{\mathrm{rel}}(p,\kappa)$ (see Remark following Definition 2.11).

Proposition 2.18. *There exists a subspace $T(p,\kappa)$ of $V(p,\kappa)^{\mathrm{Vir}_\kappa^+}$ (i.e. elements annihilated by $\pi(L_n), \pi(G_{n-\kappa}), n > 0$) with*

$$\mathrm{ch}_q\, T(p,\kappa) = \dim R(p,\kappa) q^{p\cdot p/2} \varphi_\kappa(q)^{-8}.$$

$T(p,\frac{1}{2})$ is positive definite with respect to $\langle , \rangle_V$. $T(p,0)$ is a direct sum of two subspaces $T^\pm$ of equal characteristics, where T^+ (respectively T^-) is positive (respectively negative) definite with respect to $\langle , \rangle_V$.

Proof. Using the so-called spectrum generating algebra, physicists [1,2] have constructed these spaces. However, we will prove their existence the way FGZ did in the bosonic case.

It has been shown [7,9,10] that the Verma module of $M_\kappa(h,c)$ over Vir_κ is non-unitary for $h < 0, c = 1$ and unitary for $h > 0$, $c = 9$. By Kac's determinant [22] formula, for $h'' < 0$ and $h' > 0, M_\kappa(h'',1), M_\kappa(h',9)$ have no (non-zero) proper $\mathbf{Z}_2$-graded submodule. Note however that $M_0(h',9), h' > 0$, is the direct sum of two irreducible proper submodules $N_\pm(h',9)$, generated by the highest weight vectors (HWV)

$$w_\pm = |h',9\rangle \pm h'^{-1/2} G_0 |h',9\rangle \quad \text{with} \quad G_0 w_\pm = \pm h'^{-1/2} w_\pm, \tag{20}$$

where $|h',9\rangle$ is a $\mathbf{Z}_2$-homogeneous HWV of $M_0(h',9)$. Given $p \in \mathbf{R}^{9,1}\backslash 0$, we choose a decomposition $p = p'' \oplus p'$ with $p'' \in \mathbf{R}^{0,1}\backslash 0$, $p' \in \mathbf{R}^{9,0}\backslash 0$. Then

(i) $M_\kappa(p''\cdot p''/2, 1)$ has no proper $\mathbf{Z}_2$-graded submodule.
(ii) $M_{1/2}(p'\cdot p'/2 + r, 9)$, $N_\pm(p'\cdot p'/2 + n, 9)$ are unitary and irreducible for all $r \in \frac{1}{2}\mathbf{Z}_+$ and $n \in \mathbf{Z}_+$.

As in the bosonic case, there is a canonical isometry $V(p,\kappa) \equiv V(p'',\kappa) \otimes V(p',\kappa)$, where $V(p'',\kappa)(V(p',\kappa))$ is non-unitary (unitary). We will briefly describe their structures.

(iii) $V(p'',\kappa)$ and $V(p',\kappa)$ are given by

$$\begin{aligned}
V(p'',\kappa) &= R(p'',\kappa) \otimes \mathbf{C}[\alpha^{10}_{-n}, d^{10}_{-n+\kappa}, n \in \mathbf{N}] |p''\rangle, \quad \text{non-unitary}, \\
V(p',\kappa) &= R(p',\kappa) \otimes \mathbf{C}[\alpha^{i}_{-n}, d^{i}_{-n+\kappa}, n \in \mathbf{N}, i = 1,\dots,9] |p'\rangle, \quad \text{unitary}, \\
R(p'',\kappa) &= \mathbf{C}1 \quad \text{if} \quad \kappa = \tfrac{1}{2} \\
&= \text{2-dimensional representation of } \{d_0^{10}, d_0^{10}\} = -1 \quad \text{if} \quad \kappa = 0, \\
R(p',\kappa) &= \mathbf{C}1 \quad \text{if} \quad \kappa = \tfrac{1}{2} \\
&= \text{16-dimensional (positive definite) representation of} \\
&\qquad \{d_0^i, d_0^j\} = \delta^{ij} \quad \text{if} \quad \kappa = 0.
\end{aligned}$$

(iv) The characters and the c-values are:

$$\mathrm{ch}_q\, V(p'',\kappa) = (2 - 2\kappa) q^{p''\cdot p''/2} \varphi_\kappa(q)^{-1}, \tag{21}$$

$$\mathrm{ch}_q\, V(p',\kappa) = (16 - 30\kappa) q^{p'\cdot p'/2} \varphi_\kappa(q)^{-9}, \tag{22}$$

$$\pi''(c) = 1, \quad \pi'(c) = 9, \tag{23}$$

where $\pi''(x)(\pi'(x))$ is the action of $x \in \mathrm{Vir}_\kappa$ on $V(p'',\kappa)(V(p',\kappa))$. Thus (i) to (iv) imply that we can express these spaces in terms of irreducible modules:

$$V(p'',\kappa) \cong M_\kappa\left(\frac{p''\cdot p''}{2}, 1\right), \tag{24}$$

$$\begin{aligned} V(p',\kappa) &\cong \coprod_{n\in(1/2)Z_+} p_{1/2}^{(8)}(n) M_{1/2}\left(\frac{p'\cdot p'}{2}+n, 9\right) \quad \text{if} \quad \kappa = \tfrac{1}{2} \\ &\cong \coprod_{n\in Z_+} \left(a_n N_+\left(\frac{p'\cdot p'}{2}+n,9\right) \oplus b_n N_-\left(\frac{p'\cdot p'}{2}+n,9\right)\right) \quad \text{if} \quad \kappa=0, \end{aligned} \tag{25}$$

where $a_n, b_n \in \mathbf{Z}_+$ with $a_n + b_n = 16 p_0^{(8)}(n)$.

The eigenspace of $\pi''(L_0)$ corresponding to the eigenvalue $p''\cdot p''/2$ is given by

$$\begin{aligned} V_0(p'',\kappa) &= \mathbf{C}\cdot 1 \quad &\text{if} \quad \kappa = \tfrac{1}{2} \\ &= \mathbf{C}1_+ \oplus \mathbf{C}1_- \quad &\text{if} \quad \kappa = 0 \end{aligned} \tag{26}$$

where

$$\langle 1,1\rangle_{V(p'',1/2)} = 1, \quad \langle 1_\pm, 1_\pm\rangle_{V(p'',0)} = 0, \quad \langle 1_+, 1_-\rangle_{V(p'',0)} = \sqrt{-1} \tag{27}$$

$$d_0^{10} 1_\pm = \mp \frac{1}{\sqrt{2}} 1_\mp, 1_+ \text{ is } \mathbf{Z}_2\text{-even}. \tag{28}$$

Equation (24) implies that

$$V(p'',\kappa)^{\mathrm{Vir}_\kappa^+} = V_0(p'',\kappa). \tag{29}$$

Equation (25) says that each eigenspace of $\pi'(L_0)$,

$$V_n(p',\kappa) = \left\{ v \in V(p',\kappa) : \pi'(L_0)v = \left(n + \frac{p'\cdot p'}{2}\right)v\right\}, \quad n \in (1-\kappa)\mathbf{Z}_+$$

is such that

$$\dim V_n(p',\kappa)^{\mathrm{Vir}_\kappa^+} = (16-30\kappa)p_\kappa^{(8)}(n). \tag{30}$$

Since $V'(p',\kappa)$ is unitary, we can choose an orthonormal basis

$$\{w_l(n) : l = 1,\dots,(16-30\kappa)p_\kappa^{(8)}(n)\} \text{ of } V_n(p',\kappa)^{\mathrm{Vir}_\kappa^+} : \langle w_k(n), w_l(n)\rangle_{V(p',\kappa)} = \delta_{kl}. \tag{31}$$

Now following FGZ, we let

$$T(p,\kappa) = V(p'',\kappa)^{\mathrm{Vir}_\kappa^+} \otimes V(p',\kappa)^{\mathrm{Vir}_\kappa^+}. \tag{32}$$

It follows from (26), (29), (30) that

$$\begin{aligned} \mathrm{ch}_q T(p,\kappa) &= (2-2\kappa)(16-30\kappa) q^{p\cdot p/2} \varphi_\kappa(q)^{-8} \\ &= \dim R(p,\kappa) q^{p\cdot p/2}\varphi_\kappa(q)^{-8}. \end{aligned} \tag{33}$$

Equations (27), (31) imply that $T(p,\frac{1}{2})$ is positive definite with respect to $(,)_{V(p,1/2)} = (,)_{V(p'',1/2)} \otimes (,)_{V(p',1/2)}$, while

$$\begin{aligned} T(p,0) &= T^+ \oplus T^- \quad \text{with} \\ T^\pm &= \mathbf{C}(1_+ \mp \sqrt{-1}\,1_-) \otimes V(p',\kappa)^{\mathrm{Vir}_\kappa^+}, \end{aligned} \tag{34}$$

where $T^+(T^-)$ is positive (negative) definite with respect to $(,)_{V(p,0)}$. □

Remark. $V(p,\kappa)$ is in general not a direct sum of Verma modules of Vir_κ even though it is Vir_κ^--free. This was also pointed out in [5] in the bosonic case. To see a counter-example, we consider the Neveu–Schwarz case. Suppose $V=V(p,\frac{1}{2})$ is a direct sum of Verma modules. Then computing ch_q gives

$$V\cong\bigoplus_n a_n M\left(\frac{p\cdot p}{2}+n,10\right)\tag{35}$$

for some positive integers a_n, n ranges over $0,\frac{1}{2},1,\ldots$. We wish to derive a contradiction. Recall that V is hermitian. Thus

$$\bigoplus a_n M\left(\frac{p\cdot p}{2}+n,10\right)\cong\bigoplus a_n M\left(\frac{p\cdot p}{2}+n,10\right)^*.\tag{36}$$

By Kac's determinant formula, $M(h,10)$ is irreducible for $h>0$ and hence hermitian. Thus (36) implies that

$$\bigoplus_{0\leqq n\leqq |p\cdot p/2|} a_n M\left(\frac{p\cdot p}{2}+n,10\right)\cong\bigoplus_{0\leqq n\leqq |p\cdot p/2|} a_n M\left(\frac{p\cdot p}{2}+n,10\right)^*.\tag{37}$$

Since Verma modules are strongly indecomposable, (37) implies that (see [20], Sect. 3.4)

$$M\left(\frac{p\cdot p}{2}+n,10\right)\cong M\left(\frac{p\cdot p}{2}+n,10\right)^*\quad\text{for}\quad n\leqq\left|\frac{p\cdot p}{2}\right|.\tag{38}$$

But by Kac's determinant formula, there are some negative half integers h such that $M(h,10)$ is reducible. If $p\cdot p/2+n$ is one such number, (38) contradicts reducibility.

Corollary 2.19.

(i) $\dim T\cap\mathscr{P}=\frac{1}{2}(2\kappa+1)\dim R(p,\kappa)p_\kappa^{(8)}(\kappa-p\cdot p/2)$.
(ii) *$T\cap\mathscr{P}$ is positive definite with respect to $\langle,\rangle_{\mathscr{P}}$.*
(iii) *$V(p,\kappa)$ is a free* Vir_κ^-*-module.*

Proof.
(i) The case $\kappa=\frac{1}{2}$ is an immediate result of Proposition 2.18. Consider the case $\kappa=0$. Clearly $T(p,0)$ is left invariant by $\pi''(G_0)\otimes 1$. But

$$\{\pi(G_0),\pi''(G_0)\otimes 1\}=2\pi''(G_0)^2\otimes 1=p''\cdot p''\cdot\mathrm{Id}_{T(p,0)}.\tag{39}$$

This implies that in $T(p,0)\cap\mathrm{Ker}\,\pi(L_0)$,

$$T\cap\mathrm{Ker}\,\pi(G_0)=T\cap\mathrm{Im}\,\pi(G_0).\tag{40}$$

The left-hand side is $T\cap\mathscr{P}$. Thus by (40) and Proposition 2.18, we have

$$\dim T\cap\mathscr{P}=\tfrac{1}{2}\dim R(p,0)p_0^{(8)}\left(-\frac{p\cdot p}{2}\right).\tag{41}$$

(ii) Again the $\kappa=\frac{1}{2}$ case follows from Definition 2.17 and Proposition 2.18. Let $\kappa=0, n=-p\cdot p/2$. Then one can check that $\{\pi(G_0)1_+\otimes w_l(n)\}_{l=1,\ldots,\dim T\cap\mathscr{P}}$ is basis of $T \cap \mathscr{P}$. By using Definition 2.17, and Eqs. (27), (28), (31), we obtain

$$\langle \pi(G_0)1_+ \otimes w_k(n), \pi(G_0)1_+ \otimes w_l(n) \rangle_{\mathscr{P}} = \delta_{k,l}. \tag{42}$$

(iii) Equations (24), (25) imply that $V(p,\kappa)$ is a direct sum of tensor products of free modules over Vir_κ^-. One can easily check that each of the tensor products is also Vir_κ^--free. □

Theorem 2.20. *There are unitary isomorphisms* $T \cap \mathscr{P} \cong \mathscr{P}/\mathrm{rad}\,\mathscr{P} \cong H^0_{\mathrm{rel}}$.

Proof. Consider the map $\lambda: \mathscr{P} \to C^0_{\mathrm{rel}}$ defined by

$$\lambda(v) = v \otimes |q_F = 1\rangle \otimes |q_B = 1 - \kappa\rangle, \quad v \in \mathscr{P}.$$

We note that λ maps $\mathscr{P}$ into the cocycles Z^0_{rel} and thus induces an isometric inclusion $\mathscr{P}/\lambda^{-1}B^0_{\mathrm{rel}} \to H^0_{\mathrm{rel}}$, where B^0_{rel} is the (zeroth) coboundaries. But $\lambda^{-1}B^0_{\mathrm{rel}}$ is a subset of $\mathrm{rad}\,\mathscr{P}$. By Corollary 2.19(ii), $T \cap \mathscr{P} \to \mathscr{P}/\mathrm{rad}\,\mathscr{P}$ is also an isometric inclusion. Thus we have

$$\begin{array}{ccc} & & \mathscr{P}/\lambda^{-1}B^0_{\mathrm{rel}} \xrightarrow{1-1} H^0_{\mathrm{rel}} \\ & & \Big\downarrow \text{onto} \\ T \cap \mathscr{P} & \xrightarrow{1-1} & \mathscr{P}/\mathrm{rad}\,\mathscr{P}. \end{array}$$

Using Corollary 2.15(i) and Corollary 2.19(i), we have the desired result. □

Having discussed its consequences, we now return to the vanishing theorem (2.4).

3. The Vanishing Theorem

Lemma 3.1 (Poincaré Duality). $H^m_{\mathrm{rel}}(p,\kappa)$ *is isomorphic to the anti-dual of* $H^{-m}_{\mathrm{rel}}(p,\kappa)$.

Proof. For each m, $C^m_{\mathrm{rel}}(p,\kappa)$ is a finite dimensional vector space over $\mathbf{C}$. Thus we can write

$$C^m_{\mathrm{rel}}(p,\kappa) = \mathrm{Ker}\,Q_m \oplus \mathrm{Ker}\,Q_m^\perp,$$

where $Q_m = Q$ restricted to $C^m_{\mathrm{rel}}(p,\kappa)$, and $\mathrm{Ker}\,Q_m^\perp$ is a subspace of $C^m_{\mathrm{rel}}(p,\kappa)$ complementary to $\mathrm{Ker}\,Q_m$. Similarly we can write $\mathrm{Ker}\,Q_m = \mathrm{Im}\,Q_{m-1} \oplus \mathrm{Im}\,Q^\perp_{m-1}$, where $\mathrm{Im}\,Q^\perp_{m-1}$ is a subspace of $\mathrm{Ker}\,Q_m$ complementary to $\mathrm{Im}\,Q_{m-1}$. Now $\langle , \rangle_C$ defines a non-degenerate pairing between $C^m_{\mathrm{rel}}(p,\kappa)$ and $C^{-m}_{\mathrm{rel}}(p,\kappa)$, i.e. restricted to $C^m_{\mathrm{rel}} \oplus C^{-m}_{\mathrm{rel}}$, $\langle , \rangle_{\mathrm{rel}}$ is non-degenerate. Using $Q^* = \pm Q$ (Proposition 2.14 (i)) and the non-degeneracy of $\langle , \rangle_{\mathrm{rel}}$, we have $\langle \mathrm{Im}\,Q_{m-1}, \mathrm{Ker}\,Q_{-m} \rangle_{\mathrm{rel}} = 0$ and that $\mathrm{Im}\,Q_{m-1}$ pairs with $\mathrm{Ker}\,Q^\perp_{-m}$. Similarly $\langle \mathrm{Im}\,Q_{-m-1}, \mathrm{Ker}\,Q_m \rangle_{\mathrm{rel}} = 0$ and that $\mathrm{Im}\,Q_{-m-1}$ pairs with $\mathrm{Ker}\,Q^\perp_m$. Note also that $\langle C^m_{\mathrm{rel}}, C^m_{\mathrm{rel}} \rangle_{\mathrm{rel}} = 0$ for all $m \neq 0$. Recall that

$$C^m_{\mathrm{rel}} = \mathrm{Im}\,Q_{m-1} \oplus \mathrm{Im}\,Q^\perp_{m-1} \oplus \mathrm{Ker}\,Q^\perp_m,$$
$$C^{-m}_{\mathrm{rel}} = \mathrm{Im}\,Q_{-m-1} \oplus \mathrm{Im}\,Q^\perp_{-m-1} \oplus \mathrm{Ker}\,Q^\perp_{-m}.$$

Assume $m \neq 0$. The case $m = 0$ is similar. Pick a basis for each of the six spaces: $\mathrm{Im}\,Q_{m-1}$, $\mathrm{Im}\,Q^\perp_{m-1}$, $\mathrm{Ker}\,Q^\perp_m$, $\mathrm{Im}\,Q_{-m-1}$, $\mathrm{Im}\,Q^\perp_{-m-1}$, $\mathrm{Ker}\,Q^\perp_{-m}$. Label these bases according to the same order: $I_1, \ldots, I_6$. Then, the matrix M of $\langle , \rangle_{\mathrm{rel}}$ in these bases must have the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 & \square & X \\ 0 & 0 & 0 & X & X & X \\ 0 & 0 & X & 0 & 0 & 0 \\ 0 & \square & X & 0 & 0 & 0 \\ X & X & X & 0 & 0 & 0 \end{pmatrix}$$

where the (i, j) block is the evaluation of $\langle I_i, I_j\rangle_{\text{rel}}$, and the X's denote some nonzero blocks. That M is non-degenerate implies that the square blocks $\square$ are both non-degenerate, i.e. $\operatorname{Im} Q^{\perp}_{m-1}$ pairs with $\operatorname{Im} Q^{\perp}_{-m-1}$. Thus

$$\operatorname{Im} Q^{\perp}_{m-1} \cong (\operatorname{Im} Q^{\perp}_{-m-1})^* \quad \text{(antidual)}.$$

This induces an isomorphism

$$H^m_{\text{rel}} \cong (H^{-m}_{\text{rel}})^*. \quad \square$$

To compute the cohomology, we will use some standard techniques in homological algebra. The same techniques were applied to the calculation of the BRST cohomology of the bosonic string by FGZ. Here we need only a slight modification. The strategy is to define a filtration of the complex such that the induced differential operator $\tilde{Q}$ simplifies considerably. One can then relate the new cohomology groups with the ones we want to compute, by some long exact sequence. As we will see, it is then enough to compute the new cohomology groups. In fact they are essentially the classical homology of a (super) Lie algebra (see [23]) which will be dealt with using the standard techniques we mentioned earlier.

Recall that $C_{\text{rel}}(p, \frac{1}{2})$ (respectively $C_{\text{rel}}(p, 0)$) is spanned by the canonical basis vectors of the form

$$\Omega = p_\alpha p_d |p\rangle \otimes p_b p_c |q_F = 1\rangle \otimes p_\beta p_\gamma |q_B = \tfrac{1}{2}\rangle,$$
$$(\Omega = D_R \Psi^i \otimes p_\alpha p_d |p\rangle \otimes p_b p_c |q_F = 1\rangle \otimes p_\beta p_\gamma |q_B = 1\rangle)$$

(see Definition 2.2 and Theorem 2.7), where $p_\alpha, p_d, p_b, p_c, p_\beta, p_\gamma$ are monomials of creation operators. Define on $C_{\text{rel}}(p, \kappa)$ the filtration degree

$$f \deg \Omega = -\frac{p \cdot p}{2} + \deg p_\alpha + \deg p_d + \deg p_b - \deg p_c + \deg p_\beta - \deg p_\gamma.$$

For example,

$$f \deg (\alpha^1_{-1} d^2_{-3/2} |p\rangle \otimes b_{-3} c_{-4} |1\rangle \otimes \beta_{-5/2} \gamma_{-7/2} |\tfrac{1}{2}\rangle)$$
$$= -\frac{p \cdot p}{2} - 1 - \tfrac{3}{2} - 3 + 4 - \tfrac{5}{2} + \tfrac{7}{2},$$

$$f \deg (D_R \Psi^i(p) \otimes \alpha_1^{-1} d^2_{-2} |p\rangle \otimes b_{-3} c_{-4} |1\rangle \otimes \beta_{-5} \gamma_{-6} |1\rangle)$$
$$= -\frac{p \cdot p}{2} - 1 - 2 - 3 + 4 - 5 + 6.$$

Here we emphasize that for $\kappa = 0$, Theorem 2.7 tells us that

$$C_{\mathrm{rel}}(p,0) = (\mathrm{Im}\, D_R : \mathscr{F}(p,0) \to \mathscr{F}(p,0));$$

also, $f \deg D_R = 0$ is essential for $f \deg$ to be well-defined on $C_{\mathrm{rel}}(p,0)$. Let $I_\kappa = (1-\kappa)\mathbf{Z} - p \cdot p/2$. Then $f \deg$ is I_κ-valued on $C_{\mathrm{rel}}(p,\kappa)$. Define

$$B^q = \{\Omega \in C_{\mathrm{rel}}(p,\kappa) : f \deg \Omega \geqq q\}, \quad q \in I_\kappa. \tag{1}$$

Then one can easily check that

Proposition 3.2. *$\{B^q\}_{q \in I_\kappa}$ is a finite filtration of the complex $(C_{\mathrm{rel}}(p,\kappa), Q)$, i.e. $QB^q \subset B^q, B^{q+1-\kappa} \subset B^q$ for all $q \in I_\kappa$, and there exists q_0, q_1 such that for $q \leqq q_0$, $B^q = C_{\mathrm{rel}}(p,\kappa)$ and for $q \geqq q_1, B^q = 0$.*

This means that for each $q, (B^q, Q)$ is a complex with differential Q, graded by the $\mathbf{Z}$-valued ghost number, $B^q = \bigoplus_{m \in Z} (B^q)^m$. Furthermore, $(B^q/B^{q+1-\kappa}, \tilde{Q})$ is a complex with the differential $\tilde{Q}$ induced by the map $B^q \to B^q/B^{q+1-\kappa}$. $\bigoplus_{q \in I_\kappa} B^q/B^{q+1-\kappa}$ is the associated graded space of C_{rel}.

Proposition 3.3. *For $p \neq 0$, the cohomology of the associated graded space is such that $H^m(B^q/B^{q+1-\kappa}) = 0$ for $m < 0$, $q \in I_\kappa$.*

First, its consequences

Corollary 3.4 (Theorem 2.4). *$H^m_{\mathrm{rel}}(p,\kappa) = 0$ for $m \neq 0$ and $p \neq 0$.*

Proof. From the short exact sequence of cochain complexes $0 \to B^{q+1-\kappa} \to B^q \to B^q/B^{q+1-\kappa} \to 0$, one obtains the long exact sequence

$$\cdots \to H^{m-1}(B^q/B^{q+1-\kappa}) \to H^m(B^{q+1-\kappa}) \to H^m(B^q) \to H^m(B^q/B^{q+1-\kappa}) \to \cdots.$$

Then by Proposition 3.3 we have $H^m(B^{q+1-\kappa}) \cong H^m(B^q)$ for all $q \in I_\kappa$ and $m < 0$. From the finiteness of the filtration (Proposition 3.2), it follows that $H^m_{\mathrm{rel}}(p,\kappa) = 0$ for $m < 0$. By Lemma 3.1, we have the desired result. □

Proof (*Proposition 3.3*). The proof consists of 3 steps

Step 1. Define $D^q = \{\Omega \in C_{\mathrm{rel}}(p,\kappa) : f \deg \Omega = q\}$. Thus $B^q = B^{q+1-\kappa} \oplus D^q$ and we can identify

$$D^q = B^q/B^{q+1-\kappa}. \tag{2}$$

Thus the induced differential $\tilde{Q}$ is now acting in $D^q, \tilde{Q} : D^q \to D^q$. Explicitly, $\tilde{Q} = Q_1 + Q_2$, where

$$\begin{aligned} Q_1 = {} & \sum_{n>0} \pi(L_{-n})\varepsilon(L_{-n}) + \sum_{n>0} \pi(G_{-n+\kappa})\varepsilon(G_{-n+\kappa}) \\ & - \frac{1}{2} \sum_{m,n>0} \iota([L_{-m}, L_{-n}])\varepsilon(L_{-m})\varepsilon(L_{-n}) \\ & + \sum_{m,n>0} \iota([L_{-m}, G_{-n+\kappa}])\varepsilon(L_{-m})\varepsilon(G_{-n+\kappa}) \\ & - \frac{1}{2} \sum_{m,n>0} \iota(\{G_{-m+\kappa}, G_{-n+\kappa}\})\varepsilon(G_{-m+\kappa})\varepsilon(G_{-n+\kappa}), \end{aligned} \tag{3}$$

$$Q_2 = -\frac{1}{2}\sum_{m,n>0} \varepsilon(L_m)\varepsilon(L_n)\iota([L_m, L_n]) + \sum_{m,n>0} \varepsilon(L_m)\varepsilon(G_{n-\kappa})\iota([L_m, G_{n-\kappa}])$$
$$-\frac{1}{2}\sum_{m,n>0} \iota(\{G_{m-\kappa}, G_{n-\kappa}\})\varepsilon(G_{m-\kappa})\varepsilon(G_{n-\kappa}), \tag{4}$$

where ε, ι are linear maps from Vir_κ onto the super Heisenberg algebra (Eqs. (1) to (4), Sect. 1) defined by

$$\varepsilon(L_n) = c_{-n} \quad \varepsilon(G_{-n+\kappa}) = \gamma_{n-\kappa},$$
$$\iota(L_n) = b_n \quad \iota(G_{-n+\kappa}) = \beta_{-n+\kappa}.$$

By definition of D^q, it is clear that

$$C^n_{\mathrm{rel}}(p,\kappa) = \bigoplus_{q\in I_\kappa} (D^q)^n, \tag{5}$$

where $(D^q)^n$ is the elements of D^q of ghost-number n. Then $(C^*_{\mathrm{rel}}, \tilde{Q})$ is a cochain complex with differential $\tilde{Q}$. Recall (Definition 2.2) that

$$\mathscr{B}(p,\kappa) = V(p,\kappa)\otimes \mathbf{C}[b_{-n}, c_{-n}]|q_F = 1\rangle \otimes \mathbf{C}[\beta_{-n+\kappa}, \gamma_{-n+\kappa}]|q_B = 1-\kappa\rangle.$$

Thus there is a canonical isomorphism

$$\mathscr{B}(p,\kappa) \cong C(\mathscr{G}_-, V)\otimes C(\mathscr{G}_+, \mathbf{C}), \tag{6}$$

where

$$C(\mathscr{G}_-, V) = V(p,\kappa)\otimes \mathbf{C}[b_{-n}, \beta_{-n+\kappa}], \tag{7}$$
$$C(\mathscr{G}_+, \mathbf{C}) = \mathbf{C}[c_{-n}, \gamma_{-n+\kappa}]. \tag{8}$$

Here, n ranges over $1, 2, \ldots$. Throughout this proof, we will denote the tensor product in the right-hand side of (6) by F. Here $\mathscr{G}_\pm$ denotes the subalgebras of $\mathscr{G} = \mathrm{Vir}_\kappa$ given by $\mathscr{G}_\pm = \mathrm{span}\,\{x\in\mathscr{G}: \pm \deg x > 0\}$. Note that $C(\mathscr{G}_-, V), C(\mathscr{G}_+, \mathbf{C}), F$ are also graded by the ghost number and that

$$C^n(\mathscr{G}_-, V) = C^{-n}(\mathscr{G}_+, \mathbf{C}) = 0 \quad \text{for} \quad n > 0, \tag{9}$$
$$F^n = \bigoplus_{n=b-a;\, a,b\geqq 0} C^{-a}(\mathscr{G}_-, V)\otimes C^b(\mathscr{G}_+, \mathbf{C}). \tag{10}$$

Furthermore (cf. (3), (4))

$$Q_1: C^n(\mathscr{G}_-, V) \to C^{n+1}(\mathscr{G}_-, V), \tag{11}$$
$$Q_2: C^n(\mathscr{G}_+, \mathbf{C}) \to C^{n+1}(\mathscr{G}_+, \mathbf{C}), \tag{12}$$
$$\tilde{Q}: F^n \to F^{n+1}, \tag{13}$$

and $Q_1^2 = Q_2^2 = \tilde{Q}^2 = 0$. Thus $C^*(\mathscr{G}_-, V)$, $C^*(\mathscr{G}_+, \mathbf{C}), F^*$ are complexes with the above differentials.

Recall that

$$C_{\mathrm{rel}}(p,\kappa) = \mathscr{B}(p,\tfrac{1}{2})\cap \mathrm{Ker}\, K \quad \text{for} \quad \kappa = \tfrac{1}{2}$$
$$= \mathscr{B}(p,0)\cap \mathrm{Ker}\, D_R \quad \text{for} \quad \kappa = 0. \tag{14}$$

By (6), $K(K$ and $D_R)$ has a canonically induced action on F when $\kappa=\frac{1}{2}$ (when $\kappa=0$). Thus by (5), (6) and (14), we have the canonical isomorphisms of complexes

$$\bigoplus_{q\in I_\kappa} D^q \cong F\cap \operatorname{Ker} K \quad \text{for} \quad \kappa=\tfrac{1}{2}$$
$$\cong F\cap \operatorname{Ker} D_R \quad \text{for} \quad \kappa=0. \tag{15}$$

Following FGZ, we denote the right-hand side of (15) in either case by $F^{\mathscr{G}_0}$. Here $\mathscr{G}_0=\mathbf{C}K(\mathscr{G}_0=\mathbf{C}K\oplus\mathbf{C}D_R)$ when $\kappa=\frac{1}{2}(\kappa=0)$. Combining (2) and (15), we have for each n,

$$\bigoplus_{q\in I_\kappa} H^n(B^q/B^{q+1-\kappa})\cong H^n(F^{\mathscr{G}_0}), \tag{16}$$

where H^n denotes the n^{th} cohomology group.

Step 2. It is easy to check that, as acting on F,

$$[K,\tilde{Q}]=0=\{D_R,\tilde{Q}\}. \tag{17}$$

In particular, K has a naturally induced action on $H^n(F)$. Let

$$H^n(F)^K=H^n(F)\cap \operatorname{Ker} K. \tag{18}$$

By the Künneth formula and (10) we have

$$H^n(F)\cong \bigoplus_{n=b-a;a,b\geqq 0} H^{-a}(C(\mathscr{G}_-,V))\otimes H^b(C(\mathscr{G}_+,\mathbf{C})). \tag{19}$$

As in [5], $H^{-a}(C(\mathscr{G}_-,V))$ will be computed in Step 3 using the standard technique we mentioned earlier. Suppose that

$$H^{-a}(C(\mathscr{G}_-,V))=0 \quad \text{for all} \quad a>0. \tag{20}$$

We will establish that for $\kappa=0,\frac{1}{2}$,

$$H^n(F^{\mathscr{G}_0})=0 \quad \text{for all} \quad n<0. \tag{21}$$

Proposition 3.3 then follows from (16) and (21).

Let $\mathscr{B}^n$ and $\mathscr{Z}^n$ be the spaces of n^{th} coboundaries and cocycles in F^n.

Case 1. $\kappa=\frac{1}{2}$. Equation (17) implies that there is a natural homomorphism

$$\Psi: H^n(F^K)\to H^n(F)^K,$$
$$w+(\mathscr{B}^n)^K\mapsto w+\mathscr{B}^n. \tag{22}$$

We will show that Ψ is bijective. Let $w\in(\mathscr{Z}^n)^K$ and $w+\mathscr{B}^n=0$. Then

$$w=\tilde{Q}w' \quad \text{for some} \quad w'\in F^{n-1}. \tag{23}$$

Since $[K,\tilde{Q}]=0, \tilde{Q}$ leaves each eigenspace of K invariant. Thus we can choose

$$w'\in(F^{n-1})^K. \tag{24}$$

By (23), (24), we have

$$w+(\mathscr{B}^n)^K=0. \tag{25}$$

Thus Ψ is injective.

Note. To prove surjectivity, it is tempting to use $\{Q, b_0\} = K$ to argue that every cocycle of F is annihilated by K. But remember that $Q \neq \tilde{Q}$ and that b_0 is not define in F!

However, let

$$w + \mathscr{B}^n \in H^n(F)^K. \tag{26}$$

Again, we can choose w to be an eigenvector of K,

$$Kw = \lambda w \quad \text{for some } \lambda. \tag{27}$$

But (26) implies that either

$$w \in \mathscr{B}^n \quad \text{or} \quad \lambda = 0. \tag{28}$$

In either case, $w + \mathscr{B}^n \in \mathrm{Im}\ \Psi$. This means that Ψ is surjective. Thus for $\kappa = \frac{1}{2}$, (22) is a natural isomorphism, i.e.

$$H^n(F^K) \cong H^n(F)^K. \tag{29}$$

Thus (19) and (20) imply (21).

Case 2. $\kappa = 0$. Recall that $D_R^2 = K, \mathscr{G}_0 = \mathbf{C}K \oplus \mathbf{C}D_R$. They imply that

$$H^n(F^{\mathscr{G}_0}) = H^n(F^{D_R}). \tag{30}$$

Repeating the argument used in Case 1, we have (cf. (29))

$$H^n(F^K) \cong H^n(F)^K. \tag{31}$$

By (17) and Theorem 2.7, we have the short exact sequence of cochain complexes:

$$0 \to F^{D_R} \xrightarrow{i} F^K \xrightarrow{\Psi} F^{D_R} \to 0, \tag{32}$$

where i is the inclusion map and $\Psi = D_R(-1)^{F^K}$. Thus there is a long exact sequence

$$\cdots \to H^{n-1}(F^{D_R}) \to H^n(F^{D_R}) \to H^n(F^K) \to H^n(F^{D_R}) \to \cdots. \tag{33}$$

Equations (19), (20) and (31) imply that

$$H^n(F^K) = 0 \quad \text{for all} \quad n < 0. \tag{34}$$

Since F^K is finite dimensional, the sequence (33) terminates on both ends. But (34) implies that

$$H^{n-1}(F^{D_R}) \cong H^n(F^{D_R}) \quad \text{for all} \quad n < 0. \tag{35}$$

This in turn implies (21).

We have now established that for $\kappa = 0, \frac{1}{2}$, if

$$H^{-a}(C(\mathscr{G}_-, V)) = 0 \quad \text{for all} \quad a > 0, \tag{36}$$

then

$$H^n(B^q / B^{q+1-\kappa}) = 0 \quad \text{for all} \quad n < 0. \tag{37}$$

Thus to complete the proof of Proposition 3.3, we need only to show Eq. (36).

Step 3. We first state a theorem.

Theorem 3.3.1. *Let $\mathscr{L}=\mathscr{L}^0\oplus\mathscr{L}^1$ be a super Lie algebra where $\mathscr{L}^0,\mathscr{L}^1$ are the even, odd parts. Let $C_n=\bigoplus_{n=p+q}U\mathscr{L}\otimes\wedge^p\mathscr{L}^0\otimes\vee^q\mathscr{L}^1$, where $\wedge^p\mathscr{L}^0(\vee^q\mathscr{L}^1)$ denotes the p^{th} exterior space generated by $\mathscr{L}^0$(q^{th} symmetric space generated by $\mathscr{L}^1$) and $U\mathscr{L}$ is the ($\mathbf{Z}_2$-graded) universal enveloping algebra of $\mathscr{L}$. Define $d:C_n\to C_{n-1}$ by*

$$
\begin{aligned}
&d(u\otimes x_1\wedge\cdots\wedge x_p\otimes y_1\vee\cdots\vee y_q)\\
&=(-1)^u\sum_{1\le i\le p}(-1)^{i+1}x_i\cdot u\otimes\wedge\cdots\hat{x}_i\cdots\wedge x_p\otimes y_1\vee\cdots\vee y_q\\
&\quad+\sum_{1\le i\le q}y_i\cdot u\otimes x_1\wedge\cdots\wedge x_p\otimes y_1\vee\cdots\hat{y}_i\cdots\vee y_q\\
&\quad-(-1)^u\sum_{1\le i<j\le p}(-1)^{i+j}u\otimes[x_i,x_j]\wedge x_1\wedge\cdots\hat{x}_i\cdots\hat{x}_j\cdots\wedge x_p\otimes y_1\vee\cdots\vee y_q\\
&\quad+(-1)^u\sum_{1\le i\le p,1\le j\le q}(-1)^i\cdot u\otimes x_1\wedge\cdots\hat{x}_i\cdots\wedge x_p\otimes[x_i,y_i]\vee y_1\vee\cdots\hat{y}_j\cdots\vee y_q\\
&\quad-(-1)^u\sum_{1\le i<j\le q}u\otimes\{y_i,y_j\}\wedge x_1\wedge\cdots\wedge x_p\otimes y_1\vee\cdots\hat{y}_i\cdots\hat{y}_j\cdots\vee y_q,
\end{aligned}
\tag{38}
$$

where $(-1)^u$ is the $\mathbf{Z}_2$-grading of $u\in U\mathscr{L}$. Let ε: $C_0\to\mathbf{C}$ be the map defined by linearly extending

$$
\varepsilon(u\otimes 1\otimes 1)=\begin{cases}1 & \text{if } u=1\\ 0 & \text{if } u\neq 1 \text{ and } u \text{ is a canonical basis vector of } U\mathscr{L}.\end{cases}
\tag{39}
$$

Then the sequence

$$
C:\cdots C_n\xrightarrow{d}C_{n-1}\to\cdots C_0\xrightarrow{\varepsilon}\mathbf{C}\to 0
\tag{40}
$$

is exact. In particular, the homology groups $H_n(C)=0$ for $n>0$.

This is a generalization of a theorem in the theory of classical Lie algebra homology (a good exposition is given in [18]; see also [23]). The proof of this super case requires only a slight modification of the ordinary case. Note that the theorem holds if we replace $U\mathscr{L}$ by any free $\mathscr{L}$-module A, for then $A\cong\bigoplus_{s\in S}(U\mathscr{L})_s$ where A is free on S.

We now apply this theorem to the subalgebra Vir_κ^- of Vir_κ. Let $\mathscr{L}=\mathrm{Vir}_\kappa^-$. Then it is clear that canonically, $\wedge\mathscr{L}^0\simeq\mathbf{C}[b_{-n}]$ and $\vee\mathscr{L}^1\simeq\mathbf{C}[\beta_{-n+\kappa}]$. By Corollary 2.19(iii), $V(p,\kappa)$ is a free $\mathscr{L}$-module. Now let $d=Q_1$ and $C_n=C^{-n}(\mathscr{G}_-,V)$. Then (C_*,d) satisfies the conditions of the above theorem. Thus, $0=H_n(C)=H^{-n}(C(\mathscr{G}_-,V))$ for $n>0$. This completes the proof of Proposition 3.3. □

4. The GSO Projection

It is clear from Corollary 2.15(i) that in the case $\kappa=\frac{1}{2}$, a "tachyonic" state (i.e. a state with $p\cdot p>0$) exists. In the early days of superstring theory, the RNS model was meant to be a "spacetime" supersymmetric extension of the bosonic string in which the Neveu–Schwarz sector ($\kappa=\frac{1}{2}$) consists of bosons and the Ramond sector ($\kappa = 0$) fermions. However, Corollary 2.15 (i) implies that $\dim H^0_{\mathrm{rel}}(p,0) \neq \dim H^0_{\mathrm{rel}}(p,\frac{1}{2})$. Thus one of the basic requirements of a supersymmetric theory $-$ # bosons = # fermions at each mass level—is violated. It turns out that these two undesirable features can be removed by certain natural truncations on the spectrum. They were first proposed by Gliozzi, Scherk and Olive [8] (GSO) in the "old covariant" formalism. It turns out that if one projects out the $\mathbf{Z}_2$-odd states of $\mathscr{P}(p,\kappa)$ in both sectors, the above two features will be eliminated. We will discuss these truncations below.

GSO defined the G-parity on the Fock space $V(p,\kappa)$ by

$$\tilde{G}_\kappa = \Gamma_\kappa(-1)^{\sum_{n\geq 1} d_{-n+\kappa} d_{n-\kappa}},$$

where $\Gamma_{1/2} = -1$ and Γ_0 is the chirality operator on $R(p,0)$ with $\Gamma_0^2 = 1$, $\{\Gamma_0, d_0^\mu\} = 0$. Note that $\tilde{G}_\kappa$ is essentially the $\mathbf{Z}_2$-gradation defined on $V(p,\kappa)$. Thus a natural extension to $C_{\mathrm{rel}}(p,\kappa)$ would be

Definition 4.1. *Let* $\bar{G}_\kappa = \Gamma_\kappa(-1)^{\sum_{n\geq 1}(d_{-n+\kappa}\cdot d_{n-\kappa} + b_{-n}c_n + c_{-n}b_n)}$. Then one has

$$Q: \tfrac{1}{2}(1 \pm \bar{G}_\kappa)C^n_{\mathrm{rel}}(p,\kappa) \to \tfrac{1}{2}(1 \mp \bar{G}_\kappa)C^{n+1}_{\mathrm{rel}}(p,\kappa). \tag{1}$$

Let

$$\bar{P}_\pm = \tfrac{1}{2}(1 \pm (-1)^{U+\kappa}\bar{G}_\kappa). \tag{2}$$

It follows that $(\bar{P}_\pm C^*_{\mathrm{rel}}(p,\kappa), Q)$ are two subcomplexes of $(C^*_{\mathrm{rel}}(p,\kappa), Q)$. We will denote their cohomology groups by $H^*(\bar{P}_\pm C_{\mathrm{rel}})$ and call $(\bar{P}_+ C^*_{\mathrm{rel}}, Q)$ the GSO subcomplex.

Proposition 4.2. $\mathrm{char}\,\bar{P}_+ C_{\mathrm{rel}}(p,k) = 8p_0^{(8)}(-p\cdot p/2)$.

Proof. Let $\kappa = \frac{1}{2}$. Then

$$\begin{aligned}\mathrm{char}\,\bar{P}_+ C_{\mathrm{rel}} &= \sum (-1)^n \dim \bar{P}_+ C^n_{\mathrm{rel}} \\ &= \sum_{\Omega\in A} \langle \Omega, \tfrac{1}{2}(1 + (-1)^{U+1/2}\bar{G}_{1/2})(-1)^{U+1/2}\Omega \rangle_{\mathrm{trivial}},\end{aligned} \tag{3}$$

where $\langle,\rangle_{\mathrm{trivial}}$ is defined by $\langle \Omega, \Omega' \rangle = \delta_{\Omega,\Omega'}$, where $\Omega, \Omega' \in A$ and A is the canonical basis of C_{rel}.

Recall that $C_{\mathrm{rel}} = \mathscr{B}(p,\frac{1}{2}) \cap \mathrm{Ker}\, K$. Thus (1) becomes

$$\mathrm{char}\,\bar{P}_+ C_{\mathrm{rel}} = \text{const. term}\,[\mathrm{Tr}\langle \tfrac{1}{2}(-1)^{U+1/2}q^K \rangle_{\mathrm{trivial}} + \mathrm{Tr}\langle \tfrac{1}{2}\bar{G}_{1/2}q^K \rangle_{\mathrm{trivial}}]. \tag{4}$$

Calculations as in Proposition 2.14 give

$$\mathrm{Tr}\langle \tfrac{1}{2}(-1)^{U+\kappa}q^K \rangle_{\mathrm{trivial}} = \tfrac{1}{2}q^{p\cdot p/2 - 1/2}\prod_{n>0}\left(\frac{1+q^{n-1/2}}{1-q^n}\right)^8, \tag{5}$$

$$\mathrm{Tr}\langle \tfrac{1}{2}\bar{G}_{1/2}q^K \rangle_{\mathrm{trivial}} = -\tfrac{1}{2}q^{p\cdot p/2 - 1/2}\prod_{n>0}\left(\frac{1-q^{n-1/2}}{1-q^n}\right)^8. \tag{6}$$

Combining (4), (5), (6) and using Jacobi formula (see [19], Chap. 4), we have

$$\operatorname{char} \bar{P}_+ C_{\text{rel}} = \text{const term}\left[8q^{p\cdot p/2} \prod_{n>0} \left(\frac{1+q^n}{1-q^n}\right)^8 \right]. \tag{7}$$

The reader should compare this with a calculation in [19] using the "transverse coordinates."

Now consider, $\kappa = 0$. We will do only the nontrivial case where $p\cdot p/2$ is negative integer. As before,

$$\operatorname{char} \bar{P}_+ C_{\text{rel}} = \tfrac{1}{2} \sum_{\Omega\in A} \langle \Omega, (-1)^U \Omega \rangle_{\text{trivial}} + \tfrac{1}{2} \sum_{\Omega\in A} \langle \Omega, \bar{G}_0 \Omega \rangle_{\text{trivial}}. \tag{8}$$

The first term of (8) is $\frac{1}{2}\operatorname{char} C_{\text{rel}} = 8p_0^{(8)}(-p\cdot p/2)$, by Proposition 2.14(ii). Thus it is enough to show that the second term of (8) is zero. Recall the $\mathbf{Z}_2$-odd canonical monomials $\{\Theta(s,k)\}_{k=1\ldots N(s)}$ defined in the proof of 2.14(ii) with $s = -p\cdot p/2 > 0$. Recall also the basis vectors of $R(p,0)$ (Definition 1.3(ii)) $\Psi^i_{\rightarrow}$, $\Psi^i_{\leftarrow}$, which satisfy $\Gamma_0 \Psi^i_{\rightarrow} = +\Psi^i_{\rightarrow}, \Gamma_0 \Psi^i_{\leftarrow} = -\Psi^i_{\leftarrow}$.

Lemma 4.2.1. *The set*

$$A = \{D_R \Theta(s,k) \Psi^i_{\rightleftharpoons} \otimes |p\rangle \otimes |q_F = 1\rangle \otimes |q_B = 1\rangle, i = 1\ldots 16, k = 1,\ldots, N(s)\}$$

is a basis of $C_{\text{rel}}(p,0)$.

The proof is exactly the same as that of Lemma 2.14.3. Now it is clear from the definition of $\bar{G}_0$ that precisely half of the above basis is in $\operatorname{Ker}(\bar{G}_0 - 1)$ and the other half in $\operatorname{Ker}(\bar{G}_0 + 1)$. Thus $\sum_{\Omega\in A} \langle \Omega, \bar{G}_0 \Omega \rangle_{\text{trivial}} = 0$. □

Remark. We note that each of the factors

$$(-1)^{\sum_{n\geqq 1} b_{-n}c_n}, \qquad (-1)^{\sum_{n\geqq 1} c_{-c}b_n},$$

$$(-1)^{\sum_{n\geqq 1} \beta_{-n+\kappa}\gamma_{n-\kappa}}, \quad (-1)^{\sum_{n\geqq 1} \gamma_{-n+\kappa}\beta_{n-\kappa}}$$

in $(-1)^{U+\kappa}$ is an involution. Thus we can write

$$(-1)^{U+\kappa}\bar{G}_\kappa = \Gamma_\kappa (-1)^{\sum_{n\geqq 1}(d_{-n+\kappa}\cdot d_{n-\kappa} + \beta_{-n+\kappa}\gamma_{n-\kappa} - \gamma_{n+\kappa}\beta_{n-\kappa})}.$$

The right-hand side is precisely the "G-parity" proposed by Terao, Uehara [15], and Ohta [13], as a generalization of the GSO projection.

Proposition 4.3. *For* $\kappa = 0, \frac{1}{2}$,

(*i*) *there is an isometry*

$$H^*_{\text{rel}} \cong H^*(\bar{P}_+ C_{\text{rel}}) \oplus H^*(\bar{P}_- C_{\text{rel}})$$

such that the right-hand side is an orthogonal decomposition;

(*ii*) *the cohomology of the* GSO *subcomplex has*

$$\dim H^n(\bar{P}_+ C_{\text{rel}}) = 8p_0^{(8)}\left(-\frac{p\cdot p}{2}\right) \quad \text{for} \quad n = 0$$

$$= 0 \quad \text{otherwise.}$$

Proof.
(i) It is easy to check that $\bar{P}_\pm$ are orthogonal projectors with respect to $\langle , \rangle_{\mathrm{rel}}$. i.e.

$$\bar{P}_\pm^2 = \bar{P}_\pm, \quad \bar{P}_\pm{}^* = \bar{P}_\pm,$$
$$\bar{P}_+\bar{P}_- = \bar{P}_-\bar{P}_+ = 0, \quad \bar{P}_+ + \bar{P}_- = 1.$$

Furthermore, $[Q, \bar{P}_\pm] = 0$. These equations imply that the map

$$H^*_{\mathrm{rel}} \to H^*(\bar{P}_+ C_{\mathrm{rel}}) + H^*(\bar{P}_- C_{\mathrm{rel}}),$$
$$[\omega] \mapsto [\bar{P}_+\omega] + [\bar{P}_-\omega]$$

is an isometry.
(ii) Combining (i) and Theorem 2.4, we have $H^n(\bar{P}_+ C_{\mathrm{rel}}) = 0$ unless $n = 0$. By the Euler-Poincaré Principle, we have

$$\sum_{n\in Z} (-1)^n \dim \bar{P}_+ C_{\mathrm{rel}} = \dim H^0(\bar{P}_+ C_{\mathrm{rel}}).$$

The left-hand side is char $\bar{P}_+ C_{\mathrm{rel}}$ which is given by Proposition 4.2. □

We note that the cohomology of the GSO subcomplexes $H^0(\bar{P}_+ C_{\mathrm{rel}}(p, \kappa))$, $\kappa = 0, \frac{1}{2}$ no longer have the two undesirable features—i.e. $H^0(\bar{P}_+ C_{\mathrm{rel}}(p, \frac{1}{2}))$ has no tachyonic states and $\dim H^0(\bar{P}_+ C_{\mathrm{rel}}(p, \frac{1}{2})) = \dim H^0(\bar{P}_+ C_{\mathrm{rel}}(p, 0))$. Thus we have achieved the "covariant" equivalent of the GSO projection.

5. Cohomology of the BRST Complex and the Exceptional Case

We now return to the BRST complex defined in Sect. 1. First the Neveu–Schwarz case ($\kappa = \frac{1}{2}$). Recall that $(C^*_{\mathrm{rel}}(p, \frac{1}{2}), Q)$, $(C^*(p, \frac{1}{2}), Q)$ are $\mathbf{Z}$- and $\mathbf{Z} + \frac{1}{2}$-graded respectively (Definition 1.8, 2.3).

Theorem 5.1. *There are isomorphisms* $H^0_{\mathrm{rel}}(p, \frac{1}{2}) \cong H^{\pm 1/2}(p, \frac{1}{2})$ *and* $H^n(p, \frac{1}{2}) = 0$ *for* $n \neq \pm\frac{1}{2}$.

Proof. We observe that $\{Q, b_0\} = K$. Thus, the cohomology of $C^*(p, \frac{1}{2})$ is the same as that of $C^*(p, \frac{1}{2})^K = C^*(p, \frac{1}{2}) \cap \operatorname{Ker} K$. Define the map

$$\Psi: C^{n+1/2}(p, \tfrac{1}{2})^K \to C^n_{\mathrm{rel}}(p, \kappa)$$

by $\Psi(w) = (-1)^w b_0 w$, where $(-1)^w = +1$ if w is $\mathbf{Z}_2$-even, -1 if $\mathbf{Z}_2$-odd. Then we have a short exact sequence of cochain complexes:

$$0 \to C^{n+1}_{\mathrm{rel}}(p, \tfrac{1}{2}) \xrightarrow{i} C^{n+1/2}(p, \tfrac{1}{2})^K \xrightarrow{\Psi} C^n_{\mathrm{rel}}(p, \tfrac{1}{2}) \to 0, \tag{1}$$

where i is the inclusion map. This implies the long exact sequence

$$\cdots \to H^{n-1}_{\mathrm{rel}}(p, \tfrac{1}{2}) \to H^{n+1}_{\mathrm{rel}}(p, \tfrac{1}{2}) \to H^{n+1/2}(p, \tfrac{1}{2}) \to H^n_{\mathrm{rel}}(p, \tfrac{1}{2}) \to \cdots. \tag{2}$$

Theorem 2.4 then implies the desired result. □

In the Ramond case ($\kappa = 0$), again

$$H^*(p,0) \cong H^*(C(p,0)^K) \quad \text{canonically.} \tag{3}$$

Unfortunately, because $C(p,0)^K$ is infinite dimensional, a short exact sequence similar to (1) does not exist. Recall that (Sect. 1)

$$C(p,0)^K = C_0 \oplus C_1, \tag{4}$$

where $C(p,0)$ has a doubly graded structure (cf. Proposition 1.9) and

$$C_{q_B} = (V \otimes \Lambda_\infty \otimes \mathscr{T}_{q_B})^K, \quad q_B = 0, 1. \tag{5}$$

Let

$$C_{q_B}^{n+1/2} = \{v \in C_{q_B} : Uv = (n + q_B)v\}, \quad n \in \mathbf{Z}. \tag{6}$$

Remark. Note that although C_{q_B} is infinite dimensional, each $C_{q_B}^{n+1/2}$ is finite dimensional. This follows from the fact that for fixed $q_B = 0, 1$, and $n \in \mathbf{Z}$, there are only finitely many monomials v in $V \otimes \Lambda_\infty \otimes \mathscr{T}_{q_B}$ which satisfy both

$$Kv = 0 \quad \text{and} \quad Uv = (n + q_B)v.$$

Note that $QC_{q_B}^{n-1/2} \subseteq C_{q_B}^{n+1/2}$. Thus by (3), (4), we have

$$H^n(p,0) \cong H^{n+1/2}(C_0) \oplus H^{n-1/2}(C_1). \tag{7}$$

Proposition 5.2. *$H^{n+1/2}(C_0) \cong H^{-n-1/2}(C_1)^*$, for all $n \in \mathbf{Z}$.*

Proof. By Proposition 1.9, $(,)_C$ defines a non-degenerate pairing between $C^{n;0}$ and $C^{-n;0} \equiv (C^{-n})^K$. In particular, there is a similar pairing between the two finite dimensional spaces, $C_0^{n+1/2}$ and $C_1^{-n-1/2}$ (cf. (6) of Sect. 1). Since $Q^* = -Q$ with respect to $(,)_C$, the pairing induces the isomorphism we want. The detailed argument is very similar to that Lemma 3.1. □

Thus, it is enough to compute one of $H^*(C_{q_B}), q_B = 0, 1$. Observe that $C_{\mathrm{rel}}(p,0)$ is a subspace of C_1. As we did in the case $\kappa = \frac{1}{2}$, we will try to relate $H^*_{\mathrm{rel}}(p,0)$ and $H^*(C_1)$. Since both c_0, b_0 act in C_1 and $\{b_0, c_0\} = 1$, $b_0^2 = 0$, we have

$$\operatorname{Ker} b_0 = \operatorname{Im} b_0. \tag{8}$$

Let

$$D_1^n = \{v \in C_1^{n-1/2} : b_0 v = 0\}. \tag{9}$$

Then we have a short exact sequence similar to (1):

$$0 \to D_1^{n+1} \xrightarrow{i} C_1^{n+1/2} \xrightarrow{\Psi} D_1^n \to 0, \tag{10}$$

where $\Psi(w) = (-1)^w b_0 w$ as before. This implies the long exact sequence:

$$\cdots \to H^{n-1}(D_1) \to H^{n+1}(D_1) \to H^{n+1/2}(C_1) \to H^n(D_1) \to \cdots. \tag{11}$$

Proposition 5.3. *If $p \neq 0$, then $H^n(D_1) \cong H^n_{\mathrm{rel}}(p,0)$ for all $n \in \mathbf{Z}$.*

First its consequence:

Theorem 5.4. *For $\kappa = 0$, the cohomology of the* BRST *complex is given by*

$$H^n(p,0) \cong H^0_{\mathrm{rel}}(p,0) \qquad \text{for} \quad n = 1$$

$$\begin{aligned} &\cong H^0_{\rm rel}(p,0)^* \oplus H^0_{\rm rel}(p,0) && \textit{for} \quad n=0 \\ &\cong H^0_{\rm rel}(p,0)^* && \textit{for} \quad n=-1 \\ &=0 && \textit{otherwise}. \end{aligned}$$

Proof. By Proposition 5.3, Theorem 2.4 and the long exact sequence (11), we have

$$\begin{aligned} H^r(C_1) &\cong H^0_{\rm rel}(p,0) && \text{for} \quad r=\pm\tfrac{1}{2} \\ &=0 && \text{otherwise}. \end{aligned} \tag{12}$$

The theorem then follows from Proposition 5.2 and Eq. (7). □

We now return to the proof of Proposition 5.3.

Proof (*Proposition 5.3*). Recall that $D_1 = \bigoplus_{m\in Z} D_1^m$ has the structure, by (5), (8) and (9),

$$D_1 = (V \otimes \Lambda_1 \otimes \mathscr{T}_1)^K, \tag{13}$$

where

$$\mathscr{T}_1 = \mathbf{C}[\beta_n, \gamma_m, n \leqq -1, m < 1]|q_B = 1\rangle, \tag{14}$$

$$\begin{aligned} \Lambda_1 &= b_0 \mathbf{C}[b_n, c_m, n \leqq 0, m < 0]|q_F = 0\rangle \\ &= \mathbf{C}[b_n, c_n, n < 0]|q_F = 1\rangle. \end{aligned} \tag{15}$$

Thus, D_1 can be written as

$$D_1 = \bigoplus_{l \geqq 0} E_l, \quad E_l = \gamma_0^l \mathscr{F}(p,0), \tag{16}$$

where (Definition 2.2)

$$\begin{aligned} \mathscr{F}(p,0) = (V \otimes \mathbf{C}[b_n, c_n, n < 0]|q_F = 1\rangle \\ \otimes \mathbf{C}[\beta_n, \gamma_n, n < 0]|q_B = 1\rangle)^K. \end{aligned} \tag{17}$$

From Proposition 1.7 (x), we see that Q acts on D_1 like

$$Q = A\gamma_0 + B + C\beta_0, \tag{18}$$

where

$$A = D_R + 2b_0\gamma_0, \quad B = Q_0, \quad C = N. \tag{19}$$

One can easily check that A, B, C contain no zero modes $(b_0, c_0, \beta_0, \gamma_0)$ and they satisfy

$$A^2 = C^2 = 0, \quad B^2 = CA, \tag{20}$$

$$\{B, A\} = \{B, C\} = \{A, C\} = 0, \tag{21}$$

$$A, B, C \text{ commute with } \beta_0, \gamma_0, \tag{22}$$

$$\operatorname{Ker} A = \operatorname{Im} A. \tag{23}$$

Note that (23) is essentially Theorem 2.7 since b_0 acts as zero. A similar structure ((18) to (23)) was first realized by Dixon and Taylor [21] in their study of gauge theories. Kato and Ogawa [11] then apply it in a different context. It turns out that such a structure allows one to define a projection map from the cocycles of D_1^* to those of $C^*_{\rm rel}$.

Lemma 5.3.1 (Dixon–Taylor). *There is a projection map* $P_0: Z^m(D_1) \to Z^m_{\mathrm{rel}}$ *such that for each* $w \in Z^m(D_1)$, $w = P_0 w + Qw'$ *for some* $w' \in D_1^{m-1}$.

Proof. For each m, choose a fixed complementary subspace $(\mathrm{Ker}\, A)^{\perp} \subset D_1^m$. Then (20) and (23) implies that

$$A: (\mathrm{Ker}\, A)^{\perp} \to \mathrm{Ker}\, A \text{ is invertible.} \tag{24}$$

Now

$$Qw = 0 \tag{25}$$

and

$$w = P_0 w + Qw' \tag{26}$$

are equivalent to

$$A\gamma_0 w_{n-1} + Bw_n + C\beta_0 w_{n+1} = 0, \quad n \geqq 1, \tag{27}$$

$$w_0 = P_0 w + Bw'_0 + C\beta_0 w'_1 \tag{28}$$

and

$$w_n = A\gamma_0 w'_{n-1} + Bw'_n + C\beta_0 w'_{n+1}, \quad n \geqq 1, \tag{29}$$

where $w_n, w'_n \in E_n$ are components of w, w'. Note that $w = \sum_{n \geqq 0} w_n$ terminates. Thus using (20), (21), (24) and (27), one can inductively solve for a unique

$$w' = \sum_{n \geqq 0} w'_n \in (\mathrm{Ker}\, A)^{\perp} \tag{30}$$

such that (29) is satisfied. The solution and Eq. (28) then uniquely defines P_0. One can verify that P_0 has all the required properties. The reader can consult [11] for details. □

Corollary 5.3.2. *There is an isomorphism* $H^*(D_1) \to H^*_{\mathrm{rel}}(p, 0)$ *given by* $[w] \mapsto [P_0 w]$. This completes the proof of Proposition 5.3. □

Thus far, we have assumed $p \neq 0$. For $p = 0$, the vanishing theorem does not hold because the Fock space $V(p, \kappa)$ is not Vir_κ^--free, as in the case of the bosonic string (see [5]). In fact, $\pi(L_{-1})R(0, \kappa) \otimes \mathbf{C}|p = 0\rangle = 0$. Fortunately, the cohomology groups in this case are easy enough to compute explicitly.

Proposition 5.5. *Let* $p = 0$:

(i) *For* $\kappa = 0$

$$H^n_{\mathrm{rel}} = 0 \quad \textit{for} \quad n \neq 0$$

$$\cong C^0_{\mathrm{rel}} = \coprod_{i=1,\ldots,32} \mathbf{C}\Psi^i \otimes |p = 0\rangle \otimes |q_F = 1\rangle \otimes |q_B = 1\rangle \quad \textit{for} \quad n = 0,$$

where Ψ^i *are basis vectors of* $R(0, 0)$.

(ii) *For* $\kappa = \frac{1}{2}$,

$$H^n_{\mathrm{rel}} = 0 \quad \textit{for} \quad n \neq 0$$

$$\cong C^0_{\mathrm{rel}} = \coprod_{\mu=1,\ldots,10} \mathbf{C}d^{\mu}_{-1/2}|p = 0\rangle \otimes |q_F = 1\rangle \otimes |q_B = \tfrac{1}{2}\rangle \quad \textit{for} \quad n = 0$$

$$\cong \mathbf{C} \quad \textit{for} \quad n = \pm 1.$$

(iii) *For* $\kappa=0$,

$$
\begin{aligned}
H^n &= 0 \quad \textit{for} \quad n \neq 0, \pm 1\\
&\cong C^0 = \coprod_{i=1,\ldots,32} [\mathbf{C}\Psi^i \otimes |p=0\rangle \otimes |q_F=1\rangle \otimes |q_B=1\rangle\\
&\qquad \oplus \mathbf{C}\Psi^i \otimes |p=0\rangle \otimes |q_F=0\rangle \otimes |q_B=0\rangle] \qquad \textit{for} \quad n=0\\
&\cong C^{+1} = \coprod_{i=1,\ldots,32} \mathbf{C}\Psi^i \otimes |p=0\rangle \otimes |q_F=1\rangle \otimes \gamma_0 |q_B=1\rangle \quad \textit{for} \quad n=+1\\
&\cong C^{-1} = \coprod_{i=1,\ldots,32} \mathbf{C}\Psi^i \otimes |p=0\rangle \otimes |q_F=0\rangle \otimes \beta_0 |q_B=0\rangle \quad \textit{for} \quad n=-1.
\end{aligned}
$$

(iv) *For* $\kappa=\frac{1}{2}$,

$$
\begin{aligned}
H^n &= 0 \quad \textit{for} \quad n \neq \pm\tfrac{1}{2}, \pm\tfrac{3}{2}\\
&\cong C^{-1/2} = \coprod_{\mu=1,\ldots,10} \mathbf{C} d^{\mu}_{-1/2} |p=0\rangle \otimes |q_F=1\rangle \otimes |q_B=\tfrac{1}{2}\rangle \quad \textit{for} \quad n=-\tfrac{1}{2}\\
&\cong C^{1/2} = \coprod_{\mu=1,\ldots,10} \mathbf{C} d^{\mu}_{-1/2} |p=0\rangle \otimes |q_F=0\rangle \otimes |q_B=\tfrac{1}{2}\rangle \quad \textit{for} \quad n=+\tfrac{1}{2}\\
&\cong \mathbf{C} \quad \textit{for} \quad n=\pm\tfrac{3}{2}.
\end{aligned}
$$

Proof.

(i) For $\kappa=0$, the only non-trivial C^n_{rel} is C^0_{rel} as given above. One can also check that each element of C^0_{rel} is a cocycle.
(ii) For $\kappa=\frac{1}{2}$, the only non-trivial C^n_{rel} are

$$
\begin{aligned}
C^{-1}_{\mathrm{rel}} &= \mathbf{C}|p=0\rangle \otimes |q_F=1\rangle \otimes \beta_{-1/2} |q_B=\tfrac{1}{2}\rangle,\\
C^0_{\mathrm{rel}} &= \coprod_{\mu=1,\ldots,10} \mathbf{C} d^{\mu}_{-1/2} |p=0\rangle \otimes |q_F=1\rangle \otimes |q_B=\tfrac{1}{2}\rangle,\\
C^1_{\mathrm{rel}} &= \mathbf{C}|p=0\rangle \otimes |q_F=1\rangle \otimes \gamma_{-1/2} |q_B=\tfrac{1}{2}\rangle,
\end{aligned}
$$

all of which are cocycles that are not exact.
(iii) Let

$$
\begin{aligned}
A^i_m &= \Psi^i \otimes |p=0\rangle \otimes |q_F=1\rangle \otimes \gamma_0^m |q_B=1\rangle,\\
B^i_m &= \Psi^i \otimes |p=0\rangle \otimes |q_F=1\rangle \otimes \beta_0^m |q_B=0\rangle,\\
C^i_m &= \Psi^i \otimes |p=0\rangle \otimes |q_F=0\rangle \otimes \gamma_0^m |q_B=1\rangle,\\
D^i_m &= \Psi^i \otimes |p=0\rangle \otimes |q_F=0\rangle \otimes \beta_0^m |q_B=0\rangle.
\end{aligned}
$$

As in Proposition 5.1, we need only to consider $C(0,0)^K \equiv C(0,0) \cap \operatorname{Ker} K$. Then, the complex $C(0,0)^K$ is given by

$$
\begin{aligned}
(C^0)^K &= \coprod_{i=1,\ldots,32} \mathbf{C}A^i_0 \oplus \mathbf{C}D^i_0,\\
(C^m)^K &= \coprod_{i=1,\ldots,32} \mathbf{C}A^i_m \oplus \mathbf{C}C^i_{m-1} \quad m>0,\\
(C^{-m})^K &= \coprod_{i=1,\ldots,32} \mathbf{C}B^i_{m-1} \oplus \mathbf{C}D^i_m \quad m>0.
\end{aligned}
$$

Using these bases the calculations of $H^n(0,0)$ becomes trivial.
(iv) This is similar to part (ii). □

6. Conclusion

We have constructed a complete super analogue of the BRST cohomology theory for bosonic strings using the methods introduced by FGZ. We have shown that such a construction indeed leads to a theory equivalent to the "old covariant" quantization method.

As in [5], the construction here of the super-extension of the BRST complex can be generalized to an arbitrary super graded Lie algebra. The only mathematically non-trivial result is the vanishing theorem (2.4). Work in this direction is under way.

Note added in proof. After the completion of this work, we received preprints from J. Figueroa-O'Farrill and T. Kimura, whose work appears to be similar to ours.

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