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The Droplet in the Tube: A Case of Phase Transition in the Canonical Ensemble

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Dedicated to Roland Dobrushin

Abstract. We consider the 2-dimensional Ising model with ferromagnetic nearest neighbour interaction at inverse temperature β. Let $S_N = \sum \sigma_t$ be the total magnetization inside an $N \times N$ square box Λ, $\mu_\Lambda^{\mathrm{per}}$ be the Gibbs state in Λ with periodic b.c., and $m(\beta)$ be the spontaneous magnetization. We show the existence of the limit

$$\psi(\varrho) = \lim_{N\to\infty} \left(-\frac{1}{\beta N}\right) \ln \mu_\Lambda^{\mathrm{per}}(S_N = [N\varrho])$$

for $|\varrho| < m(\beta)$, provided β is large enough. It turns out that the quantity $\psi(\varrho)$ is closely related to the Wulf construction, and the dependence of the function $\psi(\varrho)$ on ϱ is singular.

I. Introduction and Statement of Results

1.1. Preliminaries

We consider the Ising model with nearest neighbour interaction on a d-dimensional lattice $\mathbb{Z}^d$. The spin at each point $t \in \mathbb{Z}^d$ takes the values $\sigma_t = \pm 1$, and the Hamiltonian of a spin configuration σ is given by:

$$H(\sigma) = -\sum_{\langle s,t\rangle} \sigma_s \sigma_t,$$

where the summation goes over nearest neighbours. Let $\Lambda \subset \mathbb{Z}^d$ be any finite region, and $\bar{\sigma}$ be some spin configuration on $\mathbb{Z}^d$. By $\mu_\Lambda^{\bar{\sigma}}$ we denote the Gibbs state in Λ, corresponding to inverse temperature β and boundary condition $\bar{\sigma}$. It is given by the formula

$$\mu_\Lambda^{\bar{\sigma}}(\sigma_\Lambda) = Z_\Lambda^{-1}(\beta, \bar{\sigma}) \exp\{-\beta H_\Lambda(\sigma_\Lambda|\bar{\sigma})\},$$

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where $\sigma_\Lambda \in \Omega_\Lambda$ is a spin configuration in Λ,

$$H_\Lambda(\sigma_\Lambda|\bar{\sigma}) = -\sum_{\langle s,t\rangle,\, s\in\Lambda} (\sigma_\Lambda \cup \bar{\sigma}_{\Lambda^c})_s (\sigma_\Lambda \cup \bar{\sigma}_{\Lambda^c})_t\,,$$

and

$$Z_\Lambda(\beta,\bar{\sigma}) = \sum_{\sigma_\Lambda \in \Omega_\Lambda} \exp\{-\beta H_\Lambda(\sigma_\Lambda|\bar{\sigma})\}\,.$$

We shall denote by the symbol $\emptyset$ the case of free boundary conditions, which one obtains by putting $\bar{\sigma}$ to be identically zero. For the special case when Λ is a d-dimensional cube in $\mathbb{Z}^d$, we shall also use periodic boundary conditions, which correspond to wrapping Λ into a discrete torus. The corresponding quantities then will be indexed by the symbol per. Let $\mu^+ = \lim_{\Lambda\to\infty} \mu_\Lambda^+$ be the limiting measure for the boundary condition $\bar{\sigma} \equiv +1$. Let $m(\beta) = \int \sigma_0 d\mu^+$. It is known that $m(\beta) > 0$ for $\beta > \beta_c(d)$ with $\beta_c(d) < \infty$, provided $d \geqq 2$.

In this paper we shall consider the question of calculating explicitly the large deviation exponent for the total spin variable

$$S_\Lambda = S_\Lambda(\sigma) = \sum_{t\in\Lambda} \sigma_t\,.$$

The magnetization per site will be denoted by $M_\Lambda = S_\Lambda/|\Lambda|$. The first natural question is to look on the behaviour of the quantity

$$p_\Lambda(a,b) = \mu^+\{\sigma\colon M_\Lambda(\sigma) \in [a,b]\}\,.$$

The following is known about it (see the paper [S] and the references therein):

$$\lim_{\Lambda\to\infty} |\Lambda|^{-1} \log p_\Lambda(a,b) \begin{cases} =0 & -m(\beta) < a < b < m(\beta)\,, \\ <0 & m(\beta) < a < 1 \quad \text{or} \quad -1 < b < -m(\beta)\,. \end{cases}$$

The main result of [S] was to show that for the case $[a,b] \cap [-m(\beta), m(\beta)] \neq \emptyset$ the actual decay of the quantity $p_\Lambda(a,b)$ is of the order of the $\exp\{-c|\partial\Lambda|\}$. Namely, under some conditions, which hold for β large enough, it was shown in [S] that for some positive A_1, A_2, c_1, c_2,

$$A_1 e^{-c_1|\partial\Lambda|} \leqq p_\Lambda(a,b) \leqq A_2 e^{-c_2|\partial\Lambda|}\,. \tag{1}$$

The result was then extended in [CCS] by showing that in the dimension $d=2$ the same behaviour of $p_\Lambda(a,b)$ is valid for all $\beta > \beta_c(2)$.

1.2. The Results

The subject of the present paper is to obtain the precise value of the constant $c = c_1 = c_2$ in (1). The setting is the following. Suppose that for any Λ the number R_Λ is given, such that

$$R_\Lambda - |\Lambda| \equiv 0 \bmod 2\,, \tag{2}$$

$$R_\Lambda/|\Lambda| \to \varrho \quad \text{when} \quad \Lambda \to \mathbb{Z}^d\,. \tag{3}$$

Consider the probability

$$q_\Lambda^{\bar{\sigma}}(R_\Lambda) = \mu_\Lambda^{\bar{\sigma}}\{\sigma_\Lambda\colon S_\Lambda(\sigma_\Lambda) = R_\Lambda\}\,.$$

It turns out that the limit behaviour of the quantity $q_\Lambda^{\bar\sigma}(R_\Lambda)$ is very sensitive not only to the value of ϱ, but also to the shape of the box Λ as well as to the boundary condition $\bar\sigma$. We begin by considering the simplest case of a square box $\Lambda = N \times N$ in the dimension $d=2$, and with periodic boundary condition.

Theorem 1. *The limit:*

$$\psi^{\mathrm{per}}(\varrho) = \lim_{N\to\infty} -\frac{1}{\beta N} \ln q_\Lambda^{\mathrm{per}}(R_\Lambda) \tag{4}$$

exists and is positive, provided the inverse temperature β is large enough and the conditions (2), (3) are satisfied with $|\varrho| < m(\beta)$. Moreover, the function $\psi^{\mathrm{per}}(\varrho)$ is given by the following formula:

$$\psi^{\mathrm{per}}(\varrho) = \begin{cases} w\sqrt{m(\beta) - |\varrho|} & \text{for} \quad |\varrho| \geqq \varrho_0 > 0, \\ w\sqrt{m(\beta) - \varrho_0} & \text{for} \quad |\varrho| \leqq \varrho_0 . \end{cases} \tag{5}$$

The constants w, ϱ_0 can also be specified (see the Theorem 2 below). One sees clearly that at the point $\varrho = \pm\varrho_0$ the function $\psi^{\mathrm{per}}(\varrho)$ is singular (see Fig. 1).

The qualitative picture behind formula (5) is the following. The typical configuration σ_Λ of the system under the constraint $S_\Lambda(\sigma_\Lambda) = R_\Lambda > 0$ has the following structure: it contains one "large" droplet of $(-)$-phase of the size of order N, immersed in the sea of $(+)$-phase (see Fig. 2). For the values ϱ, which are near $m(\beta)$, the shape of the droplet is somewhat "round." This shape grows in size when ϱ decreases down to certain value ϱ_0, where the optimal shape of the droplet

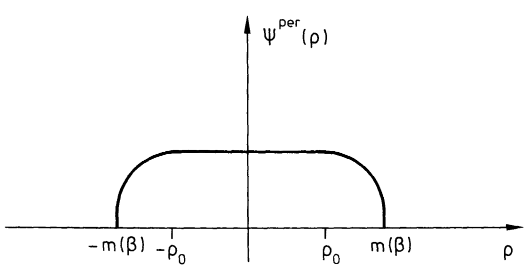

Fig. 1. The graph of the function $\psi^{\mathrm{per}}(\varrho)$

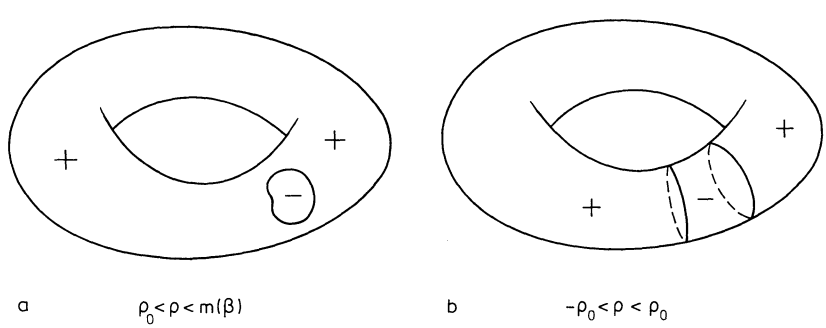

Fig. 2a, b. The shape of the droplet in two-dimensional torical volume for different densities ϱ

becomes that of a strip between two meridians of the torus Λ. Further decay of ϱ results in an increase of the volume of the droplet while the length of the boundary remains the same, and that explains the flat part of the function $\psi^{\mathrm{per}}(\varrho)$, see Fig. 2.

1.3. The Wulf Construction

To state the above results more explicitly and to obtain the values of the constants ϱ_0 and w, entering in (5), we shall use the technique and the results of the paper [DKS], where the famous Wulf construction [Wu] is put on a rigorous basis. Namely, it is shown there that the asymptotic shape of the droplet of one phase, floating in the opposite phase, is indeed given by this construction. To remind the reader of this construction we shall give the necessary definitions. In order to simplify the notations we shall restrict ourselves to the case $d=2$.

The Surface Tension. Let $D_N \subset \mathbb{Z}^2$ be the square box centred at the origin with the size $2N \times 2N$ and $n=(\cos\theta, \sin\theta) \in \mathbb{R}^2$ be the unit vector. Consider the boundary condition $\bar{\sigma}^n$, which is given by:

$$(\bar{\sigma}^n)_t = \begin{cases} -1 & (t,n)>0\,, \\ +1 & (t,n)\leqq 0\,. \end{cases}$$

The surface tension in the direction orthogonal to the vector n, is given by the formula

$$\tau_\beta(n) = -\lim_{N\to\infty} \frac{\cos\theta}{2N\beta} \ln \frac{Z_{D_N}(\beta, \bar{\sigma}^n)}{Z_{D_N}(\beta, \bar{\sigma}^+)}\,,$$

where $\bar{\sigma}^+ \equiv +1$, and we suppose that $|\theta| \leqq \frac{\pi}{4}$. In [DKS] one can find the detailed study of the quantity $\tau_\beta(n)$.

Let now $\gamma \subset \mathbb{R}^2$ be a piecewise smooth curve. Then we can define the Wulf functional $\mathscr{W}_\tau(\gamma)$ by

$$\mathscr{W}_\tau(\gamma) = \int_\gamma \tau_\beta(n_s)\,ds\,,$$

where n_s is a normal vector to γ at the point s. The Wulf shape $W = W(\tau_\beta) \subset \mathbb{R}^2$ is defined as

$$W = \{t \in \mathbb{R}^2 : \forall n,\ |(t,n)| \leqq \lambda \tau_\beta(n)\}\,,$$

where the constant λ is chosen in such a way that $|W|=1$.

Theorem [DKS]. *Suppose that γ is some closed curve in $\mathbb{R}^2$, and the area enclosed by γ is equal to one. Then*

$$\mathscr{W}_\tau(\gamma) \geqq \mathscr{W}_\tau(\partial W)\,,$$

and the equality holds only in case γ is a shift of ∂W. Moreover, the curve ∂W is a stable extremal point of the functional $\mathscr{W}_\tau$ in the following sense: if

$$\mathscr{W}_\tau(\gamma) - \mathscr{W}_\tau(\partial W) < \delta\,,$$

then for some vector $x \in \mathbb{R}^2$

$$\varrho_H(\gamma + x, \partial W) \leqq C\delta^{1/4}\,.$$

Here $C = C(\tau_\beta) < \infty$ *and* $\varrho_H(\ ,\ )$ *is the Hausdorff distance: for* $A, B \in \mathbb{R}^2$,

$$\varrho_H(A,B) = \max\left\{\sup_{x\in A} \operatorname{dist}(x,B),\ \sup_{y\in B} \operatorname{dist}(y,A)\right\}.$$

Partial results of that kind were obtained earlier in [T].

Now we can complete the statement of Theorem 1 by the following

Theorem 2. *The coefficient* w *in Theorem 1 is given by*

$$w = \mathscr{W}_\tau(\partial W).$$

The singularity point ϱ_0 *of the function* ψ^{per} *satisfies the equality*

$$w\sqrt{m(\beta) - \varrho_0} = 2\tau_\beta((1,0)), \tag{6}$$

where $(1,0) \in \mathbb{R}^2$ *is the unit vector.*

The right-hand side of (6) is just the value of the functional $\mathscr{W}$ on the curve γ in the (flat) two-dimensional torus of unit size, which is the union of two meridians. (The functional $\mathscr{W}$ extends to the functional on the torus in an obvious way.)

1.4. Generalizations

Now we state the results for the large deviation exponents for other boundary conditions. First we consider the case of empty boundary conditions.

Theorem 3. *The limit* $\psi^\emptyset(\varrho)$ *defined as in* (4), *with* $q_\Lambda^\emptyset$ *instead of* q_Λ^{per}, *exists and is positive under the same conditions as is Theorem 1. The following formula holds:*

$$\psi^\emptyset(\varrho) = \begin{cases} \frac{1}{4} w\sqrt{m(\beta) - |\varrho|} & |\varrho| \geqq \varrho_1, \\ \frac{1}{4} w\sqrt{m(\beta) - \varrho_1} & |\varrho| \leqq \varrho_1, \end{cases}$$

where w *is the same as in Theorem 2, and* ϱ_1 *satisfies the equation*

$$\tfrac{1}{4} w\sqrt{m(\beta) - \varrho_1} = \tau_\beta((1,0)).$$

The typical configurations of the canonical ensemble with empty boundary conditions and for different values of ϱ are shown in Fig. 3. The statement that the droplet with $|\varrho| \geqq \varrho_1$ has the form of one fourth of the Wulf shape W follows from the fact that the surface tension along the boundary with empty boundary conditions is zero.

The case of $(+)$-boundary conditions is more delicate. Qualitatively the picture is the following. As long as the difference $m(\beta) - \varrho$ is small enough – namely, the Wulf shape $[\frac{1}{2}(1 - \varrho/m(\beta))]^{1/2} W$ can be placed inside the unit square without

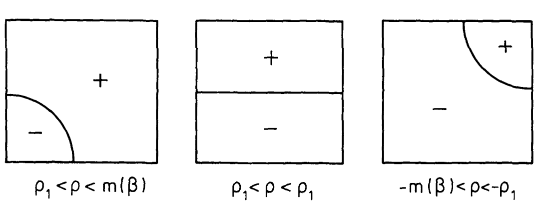

Fig. 3. The shape of the droplet in two-dimensional square volume with empty b.c. for different densities ϱ

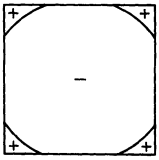

Fig. 4. The shape of the droplet in two-dimensional square volume with (+) b.c. and with high density of (−)-phase

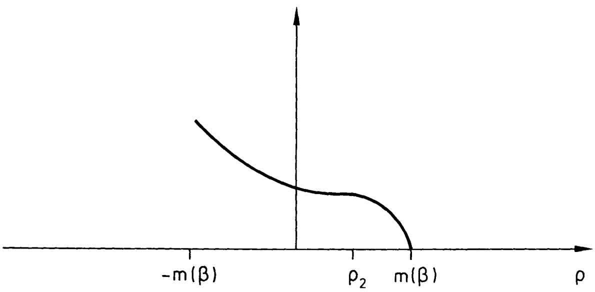

Fig. 5. The graph of the function $\psi^+(\varrho)$

touching its boundary – the corresponding exponent $\psi^+(\varrho)=w\sqrt{m(\beta)-\varrho}$ and the droplet of (−)-phase [of relative volume $(1/2)(1-\varrho/m(\beta))$] in a typical configuration with (+) b.c. does not touch the boundary. In other words, the (+)-boundary repels the (−)-droplet. The easiest way to see it is to apply the correlation inequalities obtained recently by Pfister [P]. To formulate exactly the statement about repulsion, we shall consider the surface tension along the boundary $\tau_\beta^b(n)$. It is defined to be the limit

$$\tau_\beta^b(n) = -\lim_{N\to\infty} \frac{\cos\theta}{2N\beta}\ln\frac{Z_{\bar{D}_N(\theta)}(\beta,\bar{\sigma}^n)}{Z_{\bar{D}_N(\theta)}(\beta,\bar{\sigma}^+)}$$

(compare with [FP]), where the volume $\bar{D}_N(\theta)$ is obtained by shifting the square D_N upward by the vector $(0,N)$ and applying a rotation of angle θ. Now it can be shown that $\tau_\beta^b(n)\geqq\tau_\beta(n)$, which is the reason why the droplet does not touch the boundary (if this is permitted by the volume constraint).

When the volume of the droplet becomes so large that the Wulf shape $[\frac{1}{2}(1-\varrho/m(\beta))]^{1/2}W$ cannot be placed inside a unit square, the shape of the droplet is given by the modified Wulf construction, which was given in [W], see also [ZAT]. In our case the result looks as follows: one has to cut off four equal pieces from W and then to magnify the resulting shape in such a way that it touches all four sides of our square volume by its flat pieces of the boundary, see Fig. 4.

The function $\psi^+(\varrho)$ looks as it is indicated in Fig. 5. The point ϱ_2 corresponds to the situation when the droplet touches the boundary of the square volume. For $\varrho\in[\varrho_2, m(\beta)]$, $\psi^+(\varrho)=w\sqrt{m(\beta)-\varrho}$, while for $\varrho<\varrho_2$ the formula is more complex, and we shall not give it.

II. Proofs

The main ingredient of the proofs is the technique of the paper [DKS]. Actually, the present paper has to be considered mainly as advertisement for [DKS]. We have to study the limit

$$\lim_{N\to\infty} -\frac{1}{\beta N}\ln\frac{\sum\limits_{\sigma\in\Omega_N,\,\Sigma\sigma_t=R_N}\exp\{-\beta H_N^{\mathrm{per}}(\sigma)\}}{\sum\limits_{\sigma\in\Omega_N}\exp\{-\beta H_N^{\mathrm{per}}(\sigma)\}}, \tag{7}$$

where the subscript N indicates the square box $\Lambda = N\times N$ with periodic b.c., and the numbers R_N satisfy the conditions (2), (3). The denominator in (7) is the partition function $Z_N^{\mathrm{per}}(\beta)$ of the grand canonical ensemble; we denote by $Z_N^{\mathrm{per}}(\beta, R_N)$ the numerator of (7). This is the partition function of the canonical ensemble in Λ with total spin R_N. The corresponding set of all configurations in Λ with total spin R_N will be denoted by $\Omega_N(R_N)$. In the usual way we define contours of the configuration σ as the collection of links of the dual lattice which separate n.n. points $s,t\in\Lambda$, for which $\sigma_s\sigma_t=-1$. The set of all contours of the configuration σ will be denoted by $G(\sigma)$. The contour $\Gamma\in G(\sigma)$ is called large, if its diameter $d(\Gamma)\geqq N^{1/2}(\ln N)^2$. The subset of large contours of $G(\sigma)$ will be denoted by $G_\ell(\sigma)$. For any configuration $\sigma\in\Omega_N$ with $G_\ell(\sigma)\neq\emptyset$, we define the configuration $\sigma_\ell\in\Omega_N$ by the following conditions:

1. $G_\ell(\sigma)=G(\sigma_\ell)$.
2. For any $t\in\Lambda$ such that $\mathrm{dist}(t,\Gamma)=1/2$ for some $\Gamma\in G_\ell(\sigma)$, $\sigma(t)=\sigma_\ell(t)$.

We denote by $\Lambda_\pm(\sigma)$, the subsets of Λ, where $\sigma_\ell=\pm1$.
We will make several reductions of the ratio (7). Let

$$Z_N^{\mathrm{per}}(\beta,R_N,\ell)=\sum_{\sigma\in\Omega_N(R_N):\ G_\ell(\sigma)\neq\emptyset}\exp\{-\beta H_N^{\mathrm{per}}(\sigma)\}. \tag{8}$$

Then

$$\frac{Z_N^{\mathrm{per}}(\beta,R_N)}{Z_N^{\mathrm{per}}(\beta)}=\frac{Z_N^{\mathrm{per}}(\beta,R_N,\ell)}{Z_N^{\mathrm{per}}(\beta)}\cdot\left(\frac{Z_N^{\mathrm{per}}(\beta,R_N,\ell)}{Z_N^{\mathrm{per}}(\beta,R_N)}\right)^{-1}.$$

It is an easy corollary of the large deviation estimate for the variable $S_\Lambda(\sigma_\Lambda)$ conditioned by the requirement that $G_\ell(\sigma_\Lambda)=\emptyset$ that

$$\mathscr{P}\!r\{\sigma\colon G_\ell(\sigma)=\emptyset|\Omega_N(R_N)\}\equiv 1-\frac{Z_N^{\mathrm{per}}(\beta,R_N,\ell)}{Z_N^{\mathrm{per}}(\beta,R_N)}\to 0$$

as $N\to\infty$ (see [DKS, Sect. 3]). Hence

$$\frac{Z_N^{\mathrm{per}}(\beta,R_N)}{Z_N^{\mathrm{per}}(\beta)}\sim\frac{Z_N^{\mathrm{per}}(\beta,R_N,\ell)}{Z_N^{\mathrm{per}}(\beta)}.$$

The same estimate, applied to subvolumes of Λ, which one obtains by taking the complement $\Lambda\setminus\{t\in\Lambda,\ \mathrm{dist}(t,\Gamma)=1/2$ for some $\Gamma\in G_\ell(\sigma)\}$ for some $\sigma\in\Omega_N$ actually tells us that

$$\frac{Z_N^{\mathrm{per}}(\beta,R_N,\ell)}{Z_N^{\mathrm{per}}(\beta)}\sim\frac{Z_N^{\mathrm{per}}(\beta,R_N,\ell,\mathrm{vol})}{Z_N^{\mathrm{per}}(\beta)},$$

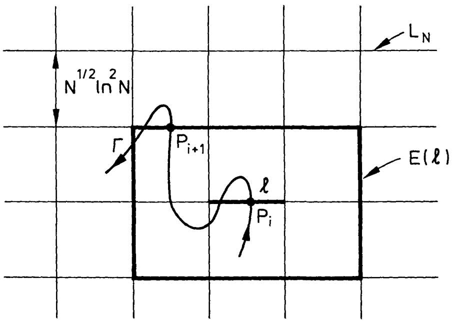

Fig. 6. The construction of the skeleton

where the partition function $Z_N^{\text{per}}(\beta, R_N, \ell, \text{vol})$ is obtained from $Z_N^{\text{per}}(\beta, R_N, \ell)$ by restricting the range of summation in (8) to those $\sigma \in \Omega_N(R_N)$ for which $G_\ell(\sigma) \neq \emptyset$ and

$$|(|\Lambda_+(\sigma)| - |\Lambda_-(\sigma)|)m(\beta) - R_N| \leqq N^{3/2+\varepsilon} \tag{9}$$

for any $\varepsilon > 0$, provided $N \geqq N(\varepsilon)$ [DKS, Sect. 3].

In other words, the partition function $Z_N^{\text{per}}(\beta, R_N, \ell, \text{vol})$ is obtained by summation over all configurations $\sigma \in \Omega_N(R_N)$, which have the large contours with "right" areas.

To make the third reduction we have to introduce the notion of the skeleton of a large contour. To do this we consider the sublattice L_N of Λ_N with spacing $N^{1/2}(\ln N)^2$. For each contour Γ we fix somehow its orientation and its initial point $p_0(\Gamma)$ (belonging to some bond of L_N), and define inductively the sequence $p_0(\Gamma)$, $p_1(\Gamma), \ldots$ of points on bonds of L_N: if $p_{i-1}(\Gamma)$ belongs to the bond $\ell \in L_N$, then $p_i(\Gamma)$ is the first hitting point of Γ in the set $E(\ell) \subset L_N$, which is the union of 10 bonds of L_N, see Fig. 6. The sequence terminates at the point $p_n(\Gamma)$, such that the diameter of the segment of Γ between $p_n(\Gamma)$ and $p_0(\Gamma)$ is between $2N^{1/2}(\ln N)^2$ and $\frac{1}{2}N^{1/2}(\ln N)^2$. The polygon $\Pi(\Gamma)$ with sites $p_0(\Gamma), p_1(\Gamma), \ldots$ is called the skeleton of Γ.

Now consider the Wulf shape W magnified $\sqrt{\frac{1}{2}(N^2 - R_N/m(\beta))}$ times, and let $\bar{W} = \bar{W}(R_N, N)$ be the resulting shape. We define the polygon Π_W to be:

the skeleton of $\partial\bar{W}(R_N, N)$ if $\varrho_0 \leqq \varrho < m(\beta)$;

the skeleton of $\partial\bar{W}(N^2 - R_N, N)$ if $-m(\beta) < \varrho \leqq -\varrho_0$;

the skeleton of the union of two parallel meridians, separated by the distance $\frac{1}{2}(N - R_N/(Nm(\beta)))$, if $-\varrho_0 < \varrho < \varrho_0$.

For Π to be a union of several polygons with sites on the bonds of L_N we introduce the partition function

$$Z_N^{\text{per}}(\beta, R_N, \Pi),$$

which is obtained by summation over all configurations $\sigma \in \Omega_N(R_N)$, such that the set of skeletons of their large contours is exactly the family Π. We define as above the subvolumes $\Lambda_\pm(\Pi) \subset \Lambda$, and we denote by $v_d(\Pi)$ the volume difference

$$v_d(\Pi) = |(|\Lambda_+(\Pi)| - |\Lambda_-(\Pi)|)m(\beta) - R_N|$$

[compare with (9)]. It is easy to see now that

$$Z_N^{\mathrm{per}}(\beta, R_N, \Pi_W) \leqq Z_N^{\mathrm{per}}(\beta, R_N, \ell, \mathrm{vol}) \leqq \sum_{\Pi:\, v_d(\Pi) \leqq N^{3/2+\varepsilon}} Z_N^{\mathrm{per}}(\beta, R_N, \Pi). \tag{10}$$

Hence, it is enough to show that

$$\lim_{N\to\infty} -\frac{1}{\beta N} \ln[Z_N^{\mathrm{per}}(\beta, R_N, \Pi_W)/Z_N^{\mathrm{per}}(\beta)] \leqq \psi^{\mathrm{per}}(\varrho), \tag{11}$$

$$\lim_{N\to\infty} -\frac{1}{\beta N} \ln\left[\sum_{\Pi:\, v_d(\Pi) \leqq N^{3/2+\varepsilon}} Z_N^{\mathrm{per}}(\beta, R_N, \Pi)/Z_N^{\mathrm{per}}(\beta)\right] \geqq \psi^{\mathrm{per}}(\varrho). \tag{12}$$

To show (11) one can use the statement from [DSK, Sect. 5], that

$$\ln \frac{Z_N^{\mathrm{per}}(\beta, R_N, \Pi_W)}{Z_N^{\mathrm{per}}(\beta)} \geqq -\beta N \psi^{\mathrm{per}}(\varrho) - \frac{C\sqrt{N}}{\ln N},$$

where $C > 0$ is some constant. To show (12) one first can use the obvious fact that

$$Z_N^{\mathrm{per}}(\beta, R_N, \Pi) \leqq Z_N^{\mathrm{per}}(\beta, \Pi),$$

where the last partition function is calculated over the set of configurations such that the skeleton of their large contours is Π. Now, if Π satisfies the volume restriction

$$|(|\Lambda_+(\Pi)| - |\Lambda_-(\Pi)|) m(\beta) - R_N| \leqq N^{3/2+\varepsilon},$$

then

$$\ln \frac{Z_N^{\mathrm{per}}(\beta, \Pi)}{Z_N^{\mathrm{per}}(\beta) \exp\{-\beta N \psi^{\mathrm{per}}(\varrho)\}} \leqq C\sqrt{N^{1/2+\varepsilon}},$$

which follows from the upper bound in [DKS, Sect. 5], and the extremal properties of the Wulf functional. The last piece of information from [DKS], which is needed, is the statement that the contribution to the right-hand side of (10) of the skeletons Π which are too long or are disconnected, can be neglected. Namely, let us denote by K the set of those skeletons Π, which have two properties:

i) $\mathscr{W}(\Pi) \leqq N\psi^{\mathrm{per}}(\varrho) + \sqrt{N}/\ln N$,

ii) Π is connected.

Then

$$\frac{\sum_{\Pi:\, v_d(\Pi) \leqq N^{3/2+\varepsilon},\ \Pi \notin K} Z_N^{\mathrm{per}}(\beta, R_N, \Pi)}{\sum_{\Pi:\, v_d(\Pi) \leqq N^{3/2+\varepsilon}} Z_N^{\mathrm{per}}(\beta, R_N, \Pi)} \to 0$$

as $N \to \infty$. That means that

$$\lim_{N\to\infty} -\frac{1}{\beta N} \ln\left(\sum_{\Pi:\, v_d(\Pi) \leqq N^{3/2+\varepsilon}} Z_N^{\mathrm{per}}(\beta, R_N, \Pi)/Z_N^{\mathrm{per}}(\beta)\right)$$
$$= \lim_{N\to\infty} -\frac{1}{\beta N} \ln\left(\sum_{\Pi:\, v_d(\Pi) \leqq N^{3/2+\varepsilon},\ \Pi \in K} Z_N^{\mathrm{per}}(\beta, R_N, \Pi)/Z_N^{\mathrm{per}}(\beta)\right).$$

It remains only to estimate the number of terms in the last sum. To do this we observe first that there is $\leqq N^2$ possibilities for the first point of Π. Then, for any subsequent point there is, by definition, less than $11\sqrt{N}\ln N^2$ possibilities, see Fig. 6. Because $\mathscr{W}(\Pi)\leqq N\psi^{\mathrm{per}}(\varrho)+\sqrt{N}/\ln N$, the length of Π is less than CN for some constant C. So the number of different skeletons Π is less than

$$(11\sqrt{N}\ln N^2)^{CN/\sqrt{N}\ln N^2}\sim e^{C\sqrt{N}/\ln N}.$$

From that and the estimate (13) the inequality (12) follows.

Actually, in [DKS] only the case $|\varrho|>\varrho_0$ was considered, but the general case is studied exactly in the same way.

The proofs for the cases of free or $(+)$ b.c. as well as other boundary conditions go again in the same way as given above, modulo some additional technical points concerning the interaction of the droplet with the boundary of the volume. Because the additional details are rather lengthy, the proofs will appear elsewhere.

There is no doubt that the same kind of results hold true in higher dimensions. However the proof of the fact that the Wulf construction gives the right asymptotic shape of droplet in dimensions $d\geqq 3$ does not yet exist. This is because, in particular, the structure of the low temperature Gibbs states in dimensions $d\geqq 3$ is not completely known. Nevertheless, there are some partial results in this direction [DPS].

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