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p-Adic Quantum Mechanics

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Abstract. An extension of the formalism of quantum mechanics to the case where the canonical variables are valued in a field of p-adic numbers is considered. In particular the free particle and the harmonic oscillator are considered. In classical p-adic mechanics we consider time as a p-adic variable and coordinates and momentum or p-adic or real. For the case of p-adic coordinates and momentum quantum mechanics with complex amplitudes is constructed. It is shown that the Weyl representation is an adequate formulation in this case. For harmonic oscillator the evolution operator is constructed in an explicit form. For primes p of the form $4l+1$ generalized vacuum states are constructed. The spectra of the evolution operator have been investigated. The p-adic quantum mechanics is also formulated by means of probability measures over the space of generalized functions. This theory obeys an unusual property: the propagator of a massive particle has power decay at infinity, but no exponential one.

Contents

1. Introduction

In modern theoretical and mathematical physics up to now only real and complex numbers were used. The reason is that the space-time coordinates have a good description in terms of real numbers. However there exists a more general point of view. In the paper [I] which was devoted to superanalysis we proposed to consider the superanalysis and corresponding supersymmetric field theories not only over the field of real numbers but also over the field of p-adic numbers and other locally-compact fields. In [2–4] the hypothesis on the non-Archimedean geometry of the space-time on very small distances, the so-called Planck length (10^{-33} cm), was suggested and the corresponding string theory over the field of p-adic numbers and the Galois field was initiated. Then a number of papers appeared in which possible physical applications of p-adic numbers approach has been discussed [5–17].

We are only in the beginning of p-adic mathematical physics. To answer many physical questions it is necessary to conduct a detailed investigation of suitable mathematical theory. It seems that the p-adic numbers will find applications not only in mathematical physics, in particular in string theory and field theory, but also in other natural sciences in which there are complicated fractal behaviours and hierarchical structures, for example in turbulence theory, dynamical systems, statistical physics, biology etc. Perhaps p-adic numbers, in particular 2-adic numbers, will be useful for computer construction.

Usual string theory is closely connected with quantum field theory and is its generalization. In the same way p-adic string theory has to be connected with p-adic quantum field theory. The aim of this paper is to start construction of quantum mechanics and further quantum field theory over the field of p-adic numbers. It seems to us that an extension of the formalism of quantum theory to the field of p-adic numbers is of great interest even independent of possible new physical applications because it can lead to better understanding of the formalism of usual quantum theory. We hope also that investigation of p-adic quantum mechanics and field theory will be useful in pure mathematical researches in number theory, representation theory and p-adic analysis. Let us recall here that the quantum-mechanical Weyl representation (see below) has wide applications in number theory and representation theory. However from the point of view of field theory it corresponds only to the simplest model of the free noninteracting system. No doubt investigations of p-adic nonlinear interacting systems will provide new deep pure mathematical results.

Let K be an arbitrary field, in particular the field of p-adic numbers or the Galois field. In considering mathematical and physical theories the following two natural possibilities appear: either to consider functions $f: K \to K$ or functions $f: K \to \mathbb{R}$ (or $\mathbb{C}$).[1] Both these possibilities were suggested in [2–4], in particular p-adic valued string amplitudes and complex-valued ones (convolution of characters) were proposed. Then the case of complex-valued amplitudes was discussed in [6, 7, 13]. In this paper we propose various most natural formulations of p-adic quantum and classical mechanics. The question of which formulation is

[1] It is interesting also to consider functions $f: \mathbb{R} \to K$

the most acceptable from the physical point of view will be solved after suitable development of the formalism, see also the conclusion.

The usual Schrödinger representation cannot be used for construction of p-adic quantum mechanics with complex wave functions. We propose to use a generalization of the Weyl representation and we develop the corresponding formalism of p-adic quantum mechanics with complex wave functions in the Hilbert space $L_2(Q_p)$. It is quite remarkable that the Weyl representation can be used not only in usual quantum mechanics but also in the p-adic case. p-adic quantum mechanics does not possess a Hamiltonian and we propose to work directly with a unitary group of time translations. Such a group is constructed for the free particle and for the harmonic oscillator. The existence of the vacuum state is nontrivial even for the harmonic oscillator. We introduce a notion of a generalized vacuum state and we show the existence of a vacuum state for prime numbers p of the form $4l+1$. It is very interesting to note that in contrast to the usual quantum mechanics here there exist at least two vacuum states.

We suggest also a formulation of the p-adic quantum mechanics in terms of the probability measures on the space of distributions. This formulation is formally similar to the Euclidean one of usual quantum mechanics but qualitative properties of these theories are different. In particular the propagator for a massive particle has no exponential decay at infinity but a power type one. It seems that the p-adic approach can be useful for the calculation of the critical exponents in the models of field theory and statistical physics.

In Sect. 2 some necessary results on p-adic mathematics are presented. In Sect. 3 the p-adic classical mechanics is described in which time coordinates and momentum are p-adic. We give a formula for solutions of the dynamical equations for the harmonic oscillator.

In Sect. 4 we consider a formulation of p-adic quantum mechanics with complex-valued wave functions which is based on the Weyl representation. A quantization procedure of p-adic classical mechanics, in particular a functional integral approach over the p-adic numbers, is discussed. The evolution operators for the free particle and for the harmonic oscillator are constructed. The existence of at least two generalized vacuum states for prime p of the form $4l+1$ is proved. An approach to study spectral properties of p-adic quantum mechanical systems is proposed. In Sect. 5 another version of the p-adic quantum mechanics analogous to the Euclidean formulation of the usual (real) one is described. In particular, it is shown that the propagator for a massive particle has power decay at infinity, namely $|t|_p^{-3}$, but no exponential one as in the real case.

2. Some Results on p-Adic Mathematics

Let Q be the field of rational numbers. It is known, see for example [18], that any norm on Q is equivalent to the usual absolute value or to a p-adic norm. The p-adic norm is defined in the following way. Let p be a prime number, $p=2,3,5,\ldots$. Any non-zero rational number x can be represented in the form $x=p^{\nu}m/n$, where m and n are integers which are not divisible by p and ν is an integer. Then the p-adic norm is $|x|_p=p^{-\nu}$. This norm satisfies inequality $|x+y|_p \leqq \max(|x|_p, |y|_p)$, i.e. is a non-Archimedean one. The completion of Q with respect to the p-adic norm defines the

p-adic field Q_p. Any p-adic number can be uniquely represented in the canonical series

$$x=p^{\nu}\sum_{n=0}^{\infty} a_n p^n, \tag{2.1}$$

where a_n are integers, $0 \leqq a_n \leqq p-1$, $a_0 \neq 0$. The series (2.1) converges in the p-adic norm because

$$|a_n p^n|_p = p^{-n}.$$

The p-adic exponential is defined by the series

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}, \qquad x\in Q_p \tag{2.2}$$

which converges for $|x|_p<1$ if $p>2$ and for $|x|_2<1/2$.

The complex valued p-adic exponential is given by

$$\chi(x)=\exp(2\pi i\{x\}), \qquad x\in Q_p, \tag{2.3}$$

where $\{x\}$ is the rational part of x which is defined by the expansion (2.1)

$$\{x\}=p^{\nu}\sum_{0\leqq n\leqq \max\{-1,\,-\nu-1\}} a_n p^n. \tag{2.4}$$

The function (2.3) is an additive character on Q_p

$$\chi(x+y)=\chi(x)\chi(y).$$

On Q_p there is a translationally invariant Haar measure dx with the properties

$$d(ax)=|a|_p dx, \qquad a\neq 0; \qquad \int_{|x|_p\leqq 1} dx=1. \tag{2.5}$$

Since the value $|x|_p$ is discrete and takes the countable set of numbers $|x|_p=p^{-\nu}$, $\nu\in\mathbb{Z}$, then the space Q_p can be represented as a disjoint union of spheres $|x|_p=p^{-\nu}$, $x\in Q_p$. So, for any complex valued function $f\in L_1(Q_p)$ we have

$$\int_{Q_p} f(x)dx=\sum_{\nu=-\infty}^{\infty}\int_{|x|_p=p^{\nu}} f(x)dx. \tag{2.6}$$

We shall use the following integrals, see for example [19, 20]:

$$\int_{|x|_p=p^{\nu}} dx=p^{\nu}\left(1-\frac{1}{p}\right), \qquad \nu\in\mathbb{Z}, \tag{2.7}$$

$$\int_{|x|_p=p^{\nu}} \chi(\xi x)dx=\begin{cases} p^{\nu}(1-p^{-1}), & |\xi|_p\leqq p^{-\nu};\ \nu\in\mathbb{Z}, \\ -p^{\nu-1}, & |\xi|_p=p^{-\nu+1}, \\ 0, & |\xi|_p>p^{-\nu+1}, \end{cases} \tag{2.8}$$

$$\int_{Q_p} f(|x|_p)\chi(\xi x)dx=\left(1-\frac{1}{p}\right)\sum_{\nu=\frac{\ln|\xi|}{\ln p}}^{\infty} f(p^{-\nu})p^{-\nu}-f\left(\frac{p}{|\xi|_p}\right)\frac{1}{|\xi|_p}, \tag{2.9}$$

$$\int_{Q_p}\chi(\varepsilon x^2)dx=1, \qquad \int_{Q_p}\chi(\varepsilon p x^2)dx=\begin{cases}\left(\dfrac{\varepsilon_0}{p}\right)\sqrt{p}, & p\equiv 1\,(\mathrm{mod}\,4) \\ i\left(\dfrac{\varepsilon_0}{p}\right)\sqrt{p}, & p\equiv 3\,(\mathrm{mod}\,4)\end{cases}, \tag{2.10}$$

where $\varepsilon=\varepsilon_0+\varepsilon_1 p+\ldots$, $\left(\frac{\varepsilon_0}{p}\right)$ the Legendre symbol. For calculation of the integrals (2.10) we use the equalities [18, 23]

$$\sum_{k=0}^{p-1}\exp\left[2\pi i\frac{mk}{p}\right]=0,\quad \sum_{k=0}^{p-1}\exp\left[2\pi i\frac{mk^2}{p}\right]=\begin{cases}\left(\frac{m}{p}\right)\sqrt{p}, & p\equiv 1\,(\mathrm{mod}\,4)\\ i\left(\frac{m}{p}\right)\sqrt{p}, & p\equiv 3\,(\mathrm{mod}\,4)\end{cases},$$

if m is not divisible by p.

$$\int_{Q_p}\chi(ax^2+bx)dx=\frac{\lambda_p(a)}{|a|_p^{1/2}}\chi\left(-\frac{b^2}{4a}\right);\quad p\neq 2,\tag{2.11}$$

where[2]

$$\lambda_p(a)=\begin{cases}1 & \text{if}\quad v \text{ is even},\\ \left(\frac{a_0}{p}\right) & \text{if}\quad v \text{ is odd, } p\equiv 1\,(\mathrm{mod}\,4),\\ i\left(\frac{a_0}{p}\right) & \text{if}\quad v \text{ is odd, } p\equiv 3\,(\mathrm{mod}\,4),\end{cases}\qquad a=p^v(a_0+a_1p+\ldots).$$

The proof of formula (2.11) is carried out as in the real case by means of reduction to the integrals (2.10) because any $a\in Q_p$ is: or c^2, or pc^2, εc^2 or $p\varepsilon c^2$, where $c\in Q_p$ and $|\varepsilon|_p=1$, $\left(\frac{\varepsilon_0}{p}\right)=-1$. The detailed calculations are contained in paper [20].

In the Hilbert space $L_2(Q_p)$ which is the space of complex valued square integrable functions on Q_p we introduce the standard inner product and the norm

$$(\psi,\varphi)=\int_{Q_p}\psi(x)\overline{\varphi(x)}dx,\qquad \|\psi\|^2=(\psi,\psi).$$

The Fourier transform for a function $\psi\in L_2(Q_p)$ is defined by the formula

$$\tilde{\psi}(\xi)=\int_{Q_p}\psi(x)\chi(\xi x)dx.$$

The Fourier inversion formula and the Parsevale-Steklov formula hold

$$\psi(x)=\int_{Q_p}\tilde{\psi}(\xi)\chi(-\xi x)d\xi,\qquad \|\psi\|=\|\tilde{\psi}\|.$$

Some useful results from the theory of generalized functions of type $Q_p\to\mathbb{C}$ can be found in [19, 20] and of type $Q_p\to Q_p$ in [18, 21, 22].

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[2] The following properties of $\lambda_p(a)$ are obvious:

$$|\lambda_p(a)|=1,\quad \lambda_p(a)\lambda_p(-a)=1,\quad \lambda_p(c^2a)=\lambda_p(a);\quad c\neq 0,\ p\neq 2$$

Let $\mathscr{D}(Q_p)$ be a space of complex valued functions $\varphi(x)$ with compact support on Q_p such that $\varphi(x)=\varphi(x+a)$, $x\in Q_p$, $|a|_p\leqq p^N$ for some integer $N=N(\varphi)$. The space $\mathscr{D}(Q_p)$ provides a natural topology. Let $\mathscr{D}'(Q_p)$ be the dual space to the space $\mathscr{D}(Q_p)$ – the space of generalized functions.

Equation $x^2=a$, where $a=p^{2N}(a_0+a_1p+\dots)$, $a_0\neq 0$, $0\leqq a_j\leqq p-1$, $p\neq 2$, has a solution in Q_p if and only if a_0 is a square residue mod p, i.e. $a_0\equiv x_0^2 \pmod p$. In this case there exist two solutions of the form $x=\pm p^N(x_0+x_1p+\dots)$. In particular when $a=-1$ the equation $x^2=-1=p-1+(p-1)p+(p-1)p^2+\dots$ has a solution if and only if the Legendre symbol

$$\left(\frac{-1}{p}\right)=(-1)^{\frac{p-1}{2}}$$

is equal to 1 (see [23]), i.e. when the prime number p has the form $p=4\ell+1$, for example for $p=5,13,17,\dots$.

3. Classical p-Adic Mechanics

Let us start with consideration of the classical p-adic Hamiltonian equations

$$\dot{p}_i=-\frac{\partial H}{\partial q^i},\qquad \dot{q}^i=\frac{\partial H}{\partial p_i},\qquad i=1,\dots,n, \tag{3.1}$$

where all variables: coordinates $q=q(t)$, momentum $p=p(t)$, the Hamiltonian $H=H(p,q)$ and time t take values in Q_p.[3] We understand here the notion of derivation in the following sense [1]:

$$\left|\frac{x(t+\Delta t)-x(t)}{\Delta t}-\dot{x}(t)\right|_p\to 0,\qquad |\Delta t|_p\to 0.$$

It is known that there is a theorem on the local analytical solvability of the ordinary analytical differential equation over Q_p [22].

We consider first the simplest case of a free particle with the Hamiltonian

$$H=\frac{1}{2m}p^2, \tag{3.2}$$

here $m\in Q_p$, $m\neq 0$.

Hamilton's equations

$$\dot{p}=0,\qquad \dot{q}=\frac{1}{m}p;\qquad p(0)=p,\qquad q(0)=q$$

have a unique *analytical* solution[4] for $t\in Q_p$,

$$p(t)=p,\qquad q(t)=q+\frac{p}{m}t. \tag{3.3}$$

[3] We use the same symbol p for the notation of a prime number and for a momentum. We hope that it does not lead to misunderstanding

[4] We don't consider here a more exotic solution. It is known that the equation $\dot{x}=0$ in Q_p has not only the solution $x=\text{const}$ but also other solutions

Let us also present a solution for the harmonic oscillator with the Hamiltonian

$$H=\frac{1}{2m}p^2+\frac{1}{2}m\omega^2q^2, \tag{3.4}$$

where $m, \omega \in Q_p$, $m \neq 0$.

The equations of motion

$$\dot{p}=-m\omega^2 q, \qquad \dot{q}=\frac{1}{m}p; \qquad p(0)=p, \qquad q(0)=q$$

have an analytical solution which is analogous to the solution over the field of real numbers

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{m\omega} p \sin \omega t + q \cos \omega t \\ p \cos \omega t - q m\omega \sin \omega t \end{pmatrix} = T_t \begin{pmatrix} q \\ p \end{pmatrix}, \tag{3.5}$$

where

$$T_t = \left\| \begin{matrix} \cos \omega t & \frac{1}{m\omega} \sin \omega t \\ -m\omega \sin \omega t & \cos \omega t \end{matrix} \right\|. \tag{3.6}$$

It is clear that the matrix $T_t \in SO(2, Q_p)$. In Eqs. (3.5), (3.6) according to Eq. (2.2),

$$\sin \omega t = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\omega t)^{2n-1}}{(2n-1)!}, \qquad \cos \omega t = \sum_{n=0}^{\infty} (-1)^n \frac{(\omega t)^{2n}}{(2n)!}. \tag{3.7}$$

The series (3.7) are convergent for $|\omega t|_p < 1$, $p > 2$ and for $|\omega t|_2 < 1/2$. We will denote this region by $\mathcal{D}_p$.

We note that if $t, t' \in \mathcal{D}_p$ then $t + t' \in \mathcal{D}_p$ and $\mathcal{D}_p$ is an additive group. For such t and t' the matrices T_t satisfy the group relation

$$T_t T_{t'} = T_{t+t'}. \tag{3.8}$$

On the phase space $V = Q_p \times Q_p$ we define a skew-symmetric bilinear form $B: V \times V \to Q_p$ of the form

$$B(z, z') = -pq' + p'q, \tag{3.9}$$

where $z = (q, p) \in V$, $z' = (q', p') \in V$. The pair (V, B) defines a symplectic space.

We have then

$$B(T_t z, T_t z') = B(z, z'), \tag{3.10}$$

i.e. the dynamics of the oscillator defines a one-parametric group of symplectic automorphisms of the space (V, B). It is also true for the dynamics of a free particle.

4. p-Adic Quantum Mechanics

4.1. The Weyl Representation

In this section we construct the p-adic quantum mechanics in which states are described by complex-valued wave functions $\psi \in L_2(Q_p)$.

The standard quantum mechanics starts with a representation of the well-known Heisenberg commutation relation

$$[\hat{q}, \hat{p}] = i$$

in the space $L_2(\mathbb{R})$. In the Schrödinger representation the operators $\hat{q}$ and $\hat{p}$ are realised by multiplication and differentiation respectively. However in the p-adic quantum mechanics we have $x \in Q_p$ and $\psi(x) \in \mathbb{C}$, and therefore the operator $\psi(x) \to x\psi(x)$ of multiplciation by x has no meaning. Fortunately in this situation there is a possibility to use the Weyl representation. Recall that in the Weyl representation in the space $L_2(\mathbb{R})$ a pair of unitary operators is considered

$$e^{i\hat{p}q} : \psi(x) \to \psi(x+q), \qquad e^{i\hat{q}p} : \psi(x) \to e^{ixp}\psi(x).$$

In this form it is possible to construct the following generalization to the p-adic case. We consider in the space $L_2(Q_p)$ the unitary operators

$$U_q : \psi(x) \to \psi(x+q), \qquad V_p : \psi(x) \to \chi(2px)\psi(x),$$

where $q, p, x \in Q_p$ and χ is the additive character on Q_p (see Sect. 2).

A family of unitary operators

$$W(z) = \chi(-qp)U_q V_p, \qquad z = (q, p) \in Q_p^2 \tag{4.1}$$

satisfies the Weyl relation

$$W(z)W(z') = \chi(B(z, z'))W(z+z'),$$

where $B(z, z')$ is the symplectic form (3.9). The operator $W(z)$ acts by the following way

$$W(z)\psi(x) = \chi(pq + 2px)\psi(x+q),$$

i.e.

$$W(z)\psi(x) = \int_{Q_p} W(z; x, y)\psi(y)dy,$$

where the kernel is

$$W(z; x, y) = \chi(pq + 2px)\delta(x - y + q). \tag{4.2}$$

In the standard quantum mechanics the utilization of the Weyl representation is technically convenient, and it is widely used in mathematics see [24–26]. As we saw from the above discussion in p-adic quantum mechanics the use of the Weyl representation is the most appropriate way for the construction canonical commutation relations.

We consider now a question on the description of dynamics in the p-adic quantum mechanics. In the standard quantum mechanics one starts with the quantum Hamiltonian and then one constructs an operator of evolution $U(t)$. From our discussion it is clear that in the p-adic quantum mechanics one needs to construct directly a unitary group $U(t)$. It is understood we shall use a classical p-adic Hamiltonian for heuristic arguments. As is known the usual quantization procedure is the following. For each function f from some class, defined on the phase space, one associates a corresponding operator $\hat{f}$ in $L_2(\mathbb{R})$. This quantization map $f \to \hat{f}$ has to satisfy some natural conditions, see [24, 25]. In general, the quantization procedure is ambiguous and different quantizations exist.

If the function $f(p,q)$ is the Fourier transform of a function $\varphi(\alpha,\beta)$,

$$f(p,q)=\int_{\mathbb{R}^2} e^{i(\alpha p+\beta q)}\varphi(\alpha,\beta)d\alpha d\beta=\tilde{\varphi}(p,q)\,, \tag{4.3}$$

then the Weyl quantization is the construction of the operator

$$\hat{f}=\int_{\mathbb{R}^2} e^{i(\alpha\hat{p}+\beta\hat{q})}\varphi(\alpha,\beta)d\alpha d\beta\,,$$

where $\hat{p}$ and $\hat{q}$ are the momentum and position operators. Such quantization theory is closely connected to the theory of pseudo-differential operators. This quantization procedure can be generalized to the p-adic case. Let $f(p,q)$ be a complex-valued function on the p-adic phase space $V=Q_p^2$ and from $\mathscr{D}(Q_p^2)$. It can be represented as the Fourier transform (see Sect. 2)

$$f(p,q)=\int_{Q_p^2} \chi(\alpha p+\beta q)\varphi(\alpha,\beta)d\alpha d\beta=\tilde{\varphi}(p,q)\,,\qquad \varphi\in\mathscr{D}(Q_p^2)\,.$$

In analogy with (4.3) to any such function one corresponds an operator in $L_2(Q_p)$

$$\hat{f}=\int_{Q_p^2} W(\alpha,\beta)\varphi(\alpha,\beta)d\alpha d\beta\,,$$

where $W(\alpha,\beta)=W(z)$ is the unitary operator (4.1), $z=(\alpha,\beta)$. The function $f(p,q)$ is called the symbol of the operator $\hat{f}$. Note an essential difference of such quantization of the p-adic theory from the standard real theory. In the p-adic theory we cannot quantize polynomial functions $f(p,q)$ since such functions take values in Q_p but not in $\mathbb{C}$.

In standard quantum mechanics usually one starts with the construction of the Hamiltonian operator and then one proves its selfadjointness. Then one constructs the operator of evolution. In the p-adic quantum mechanics we can proceed in the following way. Let $H(p,q)$ be a classical p-adic Hamiltonian.

We put

$$U_0(t)=\int_{Q_p^2} W(\alpha,\beta)\chi(tH(\alpha,\beta))d\alpha d\beta\,.$$

It is natural to define the operator of evolution which corresponds to the classical Hamiltonian $H(p,q)$ as

$$U(t)=\lim_{N\to\infty} U_0\left(\frac{t}{N}\right)^N,$$

if this limit does exist in $L_2(Q_p)$.

In standard quantum mechanics such a construction gives in fact the operator of evolution. It would be very interesting to prove the existence of $U(t)$ for some class of p-adic Hamiltonians $H(p,q)$. Perhaps this limit will depend on a choice of subsequence N.

As is known in standard quantum mechanics the symbol $U(t)$ can be given in terms of the Feynman functional integral. It is natural to *suspect* that in the p-adic quantum mechanics the corresponding kernel will be expressed as the functional integral

$$K_t(x,y)=\int\chi\left(\frac{1}{h}\int_0^t L(q,\dot{q})dt\right)\prod_t dq(t)\,, \tag{4.4}$$

where integration is performed over classical p-adic trajectories with the boundary conditions $q(0)=y, q(t)=x$. Here $L(q,\dot{q})$ is a classical p-adic Lagrangian, $L(q,\dot{q})\in Q_p$ and $h\in Q_p$. The integral $\int_0^t L dt=S(t)$ in the formula (4.4) is understood as a function which is inverse to the operation of differentiation, i.e. $\frac{d}{dt}S(t)=L$, $S(0)=0$, $S(t)\in Q_p$. It would be interesting to give an intrinsic definition of such integral. Here the theory of integration from [27] may be useful.

Here we are not going to analyze a general case but instead we consider rigorously the simplest cases of the free particle and harmonic oscillator. In these cases it will be shown that as in the standard quantum mechanics the kernel $K_t(x,y)\sim\chi(S_{cl}(t))$, where S_{cl} is the action calculated on the classical p-adic trajectory (compare [28]).

4.2. Free Particle

We construct the dynamics of the free quantum particle which corresponds to the classical Hamiltonian (3.2) by means of the Fourier transformation. Let ψ be from $L_2(Q_p)$ and $\tilde{\psi}(k)$ is its Fourier-transformation. As is known (see Sect. 3) the Fourier transformation $F:\psi\to\tilde{\psi}$ is an unitary operator in $L_2(Q_p)$. The evolution operator in momentum representation is given by the formula

$$\tilde{U}(t)\tilde{\psi}(k)=\chi\left(\frac{k^2}{4m}t\right)\tilde{\psi}(k). \tag{4.5}$$

Let us calculate the kernel of the evolution operator in x-representation. Using the theorem on the Fourier transformation of the convolution (justification see [20]) from (4.5) we have

$$\begin{aligned}\tilde{U}(t)\psi(x)&=F^{-1}[\tilde{U}(t)F\psi(x)]=F^{-1}\left[\chi\left(\frac{k^2}{4m}t\right)F\psi\right]\\&=F\left[\chi\left(\frac{k^2}{4m}t\right)\right]*\psi=\int_{Q_p}K_t(x-y)\psi(y)dy,\end{aligned} \tag{4.6}$$

where

$$K_t(\xi)=F\left[\chi\left(\frac{k^2}{4m}t\right)\right](\xi). \tag{4.7}$$

Note that the formulas (4.5) and (4.6) give a group of unitary operators $U(t)$ for any $t\in Q_p$ and the relation is fulfilled

$$U(t+t')=U(t)U(t'). \tag{4.8}$$

From (4.7) we have

$$K_t(\xi)=\int_{Q_p}\chi\left(\frac{k^2}{4m}t+k\xi\right)dk. \tag{4.9}$$

Taking into account the equality (2.11) we have

$$K_t(\xi)=\lambda_p\left(\frac{t}{m}\right)\left|\frac{m}{t}\right|_p^{1/2}\chi\left(-\frac{m}{t}\xi^2\right), \quad t\neq 0. \tag{4.10}$$

It is easy to see that

$$U(t)W(z)U(t)^{-1}=W(T_t z), \tag{4.11}$$

where $T_t z=(q(t),p(t))=\left(q+\frac{p}{m}t,p\right)$ is the classical trajectory. The formula (4.11) has the form as in the standard quantum mechanics.

From Eqs. (4.8) and (4.10) such a property of $\lambda_p(a)$ follows:

$$\lambda_p(a)\lambda_p(b)\lambda_p\left(-\frac{a+b}{ab}\right)=\lambda_p(a+b), \qquad p\neq 2. \tag{4.11'}$$

However the evolution of the Gaussian packet which is given by the formula

$$I_t(x)=\int_{Q_p} K_t(x-y)\exp[-\tfrac{1}{2}|y-a|_p^2]dy \tag{4.12}$$

is radically different. Calculations show that, see [20], $p\neq 2$,

$$I_t(x)=(1-p^{-1})S\left(-\frac{1}{2}\left|\frac{t}{m}\right|_p,\frac{1}{p}\right), \qquad |x-a|_p^2\leqq\left|\frac{t}{m}\right|_p; \tag{4.13a}$$

$$\begin{aligned}I_t(x)=\exp&\left[-\frac{1}{2}|x-a|_p^2\right]+\left|\frac{t}{m}\right|_p^{1/2}|x-a|_p^{-1}\chi\left(-\frac{m}{t}(x-a)^2\right)\\ &\times\left[\left(1-\frac{1}{p}\right)S\left(-\frac{1}{2}\left|\frac{t}{m}\right|_p^2|x-a|_p^{-2},\frac{1}{p}\right)\right.\\ &\left.-\exp\left[-\frac{p^2}{2}\left|\frac{t}{m}\right|_p^2|x-a|_p^{-2}\right]\right], \qquad |x-a|_p^2>\left|\frac{t}{m}\right|_p.\end{aligned} \tag{4.13b}$$

Here we suppose that $-\frac{m}{t}=c^2$, $c\in Q_p$. The function S is

$$S(\alpha,q)=\sum_{n=0}^{\infty}\frac{\alpha^n}{n!(1-q^{1+2n})}=\sum_{k=0}^{\infty}q^k\exp[\alpha q^{2k}], \qquad |q|<1.$$

Asymptotics of the function $I_t(x)$ for $|x|_p\to\infty$ has the form

$$I_t(x)\sim\frac{1}{2}\left|\frac{t}{m}\right|_p^{5/2}\frac{p^4+p^3}{p^2+p+1}\frac{1}{|x|_p^3}\chi\left(-\frac{m}{t}(x-a)^2\right) \tag{4.14}$$

and for $|t|_p\to\infty$,

$$I_t(x)=c\left(\left|\frac{t}{m}\right|_p\right)\frac{|m|_p^{1/2}}{|t|_p^{1/2}},$$

$$\left(1-\frac{1}{p}\right)\left(\frac{1}{\sqrt{e}}+\frac{\pi}{\sqrt{2}\ln p}-2\frac{1}{\ln p}\int_{1/p}^{p}e^{-\alpha^2/2}d\alpha\right)<c(t)<\left(1-\frac{1}{p}\right)\left(\frac{1}{\sqrt{e}}+\frac{\pi}{\sqrt{2}\ln p}\right). \tag{4.15}$$

Recall the expression for the Gaussian packet in the usual quantum mechanics

$$\psi(x,t)=\frac{1}{\left(1+i\frac{\hbar t}{m}\right)^{1/2}}\exp\ -\frac{x^2}{2\left(1+i\frac{\hbar t}{m}\right)}\ . \tag{4.16}$$

The function (4.16) has exponential decay for $|x|\to\infty$ in contrast to (4.14). The behaviour of (4.16) for $|t|\to\infty$ is similar to (4.15).

4.3. Harmonic Oscillator

We construct to describe the dynamics of the harmonic oscillator a group of unitary operators $U(t)$ in $L_2(Q_p)$ which satisfies the condition [cf. (4.11)]

$$U(t)W(z)U(t)^{-1}=W(T_t z), \tag{4.17}$$

where the classical evolution $T_t z$, $z=(q,p)$ is given by the formula (3.5). The expression for the kernel of the operator of evolution in the case of real numbers is well known, see for example [29]. We propose the following kernel of the operator of evolution for case of p-adic numbers:

$$U(t)\psi(x)=\int_{Q_p} K_t(x,y)\psi(y)dy,$$

where $K_0(x,y)=\delta(x-y)=\lim_{t\to 0} K_t(x,y)$;

$$K_t(x,y)=\lambda_p\left(\frac{t}{m}\right)\left|\frac{m}{t}\right|_p^{1/2}\chi\left(m\omega\left(-\frac{x^2+y^2}{\operatorname{tg}\omega t}+2\frac{xy}{\sin\omega t}\right)\right) \tag{4.18}$$

for $t\in\mathscr{D}_p\backslash\{0\}$, $p\neq 2$.

The expression (4.18) has no sense when $\sin\omega t=0$. Note here that $\sin\omega t$ for $|\omega t|_p<1$ vanishes only for $t=0$. It follows from the equalities $|\sin x|_p=|x|_p$, $|\cos x|_p=1$; $|x|_p<1$ which can be easily checked.

Let us prove the group property (4.8)

$$\int_{Q_p} K_t(x,y)K_{t'}(y,x')dy=K_{t+t'}(x,x'), \qquad t,t'\in\mathscr{D}_p-\{0\}. \tag{4.19}$$

Indeed we have

$$\begin{aligned}
\int_{Q_p} K_t(x,y)K_{t'}(y,x')dy &= \lambda_p\left(\frac{t}{m}\right)\lambda_p\left(\frac{t'}{m}\right)\left|\frac{m}{t}\right|_p^{1/2}\left|\frac{m}{t'}\right|_p^{1/2} \\
&\times\int_{Q_p}\chi\left(m\omega\left(-\frac{x^2+y^2}{\operatorname{tg}\omega t}+\frac{2xy}{\sin\omega t}-\frac{y^2+x'^2}{\operatorname{tg}\omega t'}+\frac{2yx'}{\sin\omega t'}\right)\right)dy \\
&=\lambda_p\left(\frac{t}{m}\right)\lambda_p\left(\frac{t'}{m}\right)\left|\frac{m^2}{tt'}\right|_p^{1/2}\chi\left(m\omega\left(-\frac{x^2}{\operatorname{tg}\omega t}-\frac{x'^2}{\operatorname{tg}\omega t'}\right)\right) \\
&\times\int_{Q_p}\chi\left(m\omega\left(y^2\left(-\frac{1}{\operatorname{tg}\omega t}-\frac{1}{\operatorname{tg}\omega t'}\right)+2y\left(\frac{x}{\sin\omega t}+\frac{x'}{\sin\omega t'}\right)\right)\right)dy \\
&=\lambda_p\left(\frac{t}{m}\right)\lambda_p\left(\frac{t'}{m}\right)\lambda_p\left(-\frac{m\omega}{\operatorname{tg}\omega t}-\frac{m\omega}{\operatorname{tg}\omega t'}\right)\left|\frac{m^2}{tt'}\right|_p^{1/2}\left|m\omega\left(\frac{1}{\operatorname{tg}\omega t}+\frac{1}{\operatorname{tg}\omega t'}\right)\right|_p^{-1/2} \\
&\times\chi\left(m\omega\left(-\frac{x^2}{\operatorname{tg}\omega t}-\frac{x'^2}{\operatorname{tg}\omega t'}+\left(\frac{x}{\sin\omega t}+\frac{x'}{\sin\omega t'}\right)^2\left(\frac{1}{\operatorname{tg}\omega t}+\frac{1}{\operatorname{tg}\omega t'}\right)^{-1}\right)\right) \\
&=\lambda_p\left(\frac{\operatorname{tg}\omega t}{m\omega}+\frac{\operatorname{tg}\omega t'}{m\omega}\right)\left|\frac{m}{t+t'}\right|_p^{1/2}\chi\left(m\omega\left(-\frac{x^2+x'^2}{\operatorname{tg}\omega(t+t')}+\frac{2xx'}{\sin\omega(t+t')}\right)\right) \\
&=K_{t+t'}(x,x').
\end{aligned}$$

Here we have used the formulas (2.11), (4.11′) and usual trigonometric formulas which take place over the field of p-adic numbers and also

$$\lambda_p\left(\frac{1}{m\omega}(\operatorname{tg}\omega t+\operatorname{tg}\omega t')\right)=\lambda_p\left(\frac{1}{m\omega}\operatorname{tg}(\omega t+\omega t')(1-\operatorname{tg}\omega t\operatorname{tg}\omega t')\right)$$
$$=\lambda_p\left(\frac{1}{m\omega}\operatorname{tg}\omega(t+t')\right)=\lambda_p\left(\frac{t+t'}{m}\right).$$

For proving Eq. (4.17) we calculate the integral

$$\int_{Q_p^2} K_t(x,u)W(z;u,v)K_{-t}(v,y)du\,dv=W(T_tz;x,y)\,, \tag{4.20}$$

where $W(z;u,v)$ is the kernel (4.2). Here we have used the formulas

$$\delta(ax)=\frac{1}{|a|_p}\delta(x)\,,\qquad a\neq 0\,;\qquad \int_{Q_p}\chi(u\xi)du=\tilde{1}=\delta(\xi)\,.$$

4.4. Generalized Vacuum and Spectrum

An important question is the existence of a vacuum state, i.e. a function from $L_2(Q_p)$ which satisfies the condition $U(t)\psi_0=\psi_0$. As is known for the quantum harmonic oscillator in the case of real numbers $\psi_0(x)=\exp[-\frac{1}{2}|x|^2]$. In p-adic quantum mechanics after performing calculations for $\psi_0(x)=\exp[-\frac{1}{2}|x|_p^2]$ analogously to the integral (4.12) one gets $\left(\text{for } -\frac{m}{t}=c^2,\ c\in Q_p\right)$,

$$U(t)\psi_0(x)=\int_{Q_p}K_t(x,y)\exp\left[-\frac{1}{2}|y|_p^2\right]dy=S\left(-\frac{1}{2}\left|\frac{t}{m}\right|_p,\frac{1}{p}\right)\chi\left(-\frac{1}{2}m\omega x^2\sin 2\omega t\right) \tag{4.21}$$

for $|x|_p\leqq|t/m|_p^{1/2}$. For other x the answer will be given by the formula analogous to (4.13b). So we have just seen that the function $\exp[-\frac{1}{2}|x|_p^2]$ is not a vacuum state. Nevertheless we have a generalized vacuum state

$$\psi_0(x)=\chi(\tau m\omega x^2)\,. \tag{4.22}$$

Here $\tau\in Q_p$, $\tau^2=-1$. The function (4.22) satisfies the equation $U(t)\psi_0=\psi_0$ for arbitrary t, $|\omega t|_p<1$ because

$$U(t)\psi_0(x)=\int_{Q_p}K_t(x,y)\chi(\tau m\omega y^2)dy$$
$$=\lambda_p\left(\frac{t}{m}\right)\left|\frac{m}{t}\right|_p^{1/2}\int_{Q_p}\chi\left(m\omega\left(-\frac{x^2+y^2}{\operatorname{tg}\omega t}+2\frac{xy}{\sin\omega t}+\tau y^2\right)\right)dy$$
$$=\lambda_p\left(-\frac{t}{m}\right)\lambda_p\left(\frac{t}{m}\right)\chi(\tau m\omega x^2)=\psi_0(x)\,.$$

This function does not belong to the space $L_2(Q_p)$, since $|\psi_0(x)|=1$, but it can be considered as an element from a corresponding nested Hilbert space. In formula (4.22) τ is a solution of the equation $\tau^2=-1$ in Q_p. This equation has exactly two solutions for the prime p of the form $p=4l+1$ and only for such a prime p, see Sect. 2. Therefore for such a prime p we have at least two vacuum states (degeneration of vacuum).[5]

We discuss now the question on the spectrum of the p-adic harmonic oscillator. As is known the quantum harmonic oscillator in the case of real numbers has the discrete spectrum and the corresponding eigenfunctions are expressed in terms of the Hermitian polynomials. In p-adic quantum mechanics according to this approach we have to express the spectral properties in terms of the group $U(t)$. To this end we describe the spectral theory of the harmonic oscillator in the case of real numbers as follows: We have as is known (for $\omega=m=1$),

$$\int_{\mathbb{R}} K_t(x,y)H_n(y)e^{-\frac{1}{2}y^2}dy=e^{\mathrm{i}nt}H_n(x)e^{-\frac{1}{2}x^2}, \tag{4.23}$$

where $H_n(x)$ are the Hermite polynomials and $K_t(x,y)$ is the kernel of the evolution operator. We multiply the equality (4.23) by $\xi^n/n!$ and sum up over n. Then we find

$$\int_{\mathbb{R}} K_t(x,y)\exp[2\xi y-\xi^2-\tfrac{1}{2}y^2]dy=\exp[2\xi e^{it}x-(\xi e^{it})^2-\tfrac{1}{2}x^2] \tag{4.24}$$

because the generating function for the Hermitian polynomials has the form

$$\exp[2\xi y-\xi^2]=\sum_{n=0}^{\infty}\frac{\xi^n}{n!}H_n(y).$$

Now we consider the case of p-adic numbers. In this case the Hermitian polynomials $H_n(x)$ of the p-adic variable $x\in Q_p$ take values in Q_p and formula (4.23) has no meaning in $L_2(Q_p)$. However the formula (4.24) can be extended to the p-adic case because the character χ is analogous to the real exponent and one can prove the equality

$$\int_{Q_p} K_t(x,y)\chi(\tau m\omega(4\xi y-2\xi^2-y^2))dy=\chi(\tau m\omega(4\xi e^{-\tau\omega t}x-2(\xi e^{-\tau\omega t})^2-x^2)). \tag{4.25}$$

Indeed we have

$$\begin{aligned}
&\int_{Q_p} K_t(x,y)\chi(\tau m\omega(4\xi y-2\xi^2-y^2))dy=\lambda_p\left(\frac{t}{m}\right)\left|\frac{m}{t}\right|_p^{1/2}\chi\left(m\omega\left(-\frac{x^2}{\operatorname{tg}\omega t}-\tau\xi^2\right)\right)\\
&\quad\times\int_{Q_p}\chi\left(m\omega(-y^2(\tau+\operatorname{c\,tg}\omega t))+2m\omega y\left(\frac{x}{\sin\omega t}+2\tau\xi\right)\right)dy\\
&=\lambda_p\left(\frac{t}{m}\right)\lambda_p(-m\omega(2\tau+\operatorname{c\,tg}\omega t))\left|\frac{m}{t}\right|_p^{1/2}|m\omega(\operatorname{c\,tg}\omega t+2\tau)|_p^{-1/2}\\
&\quad\times\chi\left(m\omega\left(-\frac{x^2}{\operatorname{tg}\omega t}-2\tau\xi^2+\left(\frac{x}{\sin\omega t}+2\tau\xi\right)^2\sin\omega t e^{-\tau\omega t}\right)\right)\\
&=\chi(\tau m\omega(4\xi e^{-\tau\omega t}x-2(\xi e^{-\tau\omega t})^2-x^2))
\end{aligned}$$

[5] Recently it was shown that there exists a vacuum state belonging to $L_2(Q_p)$ [31, 32]. Note also that Zelenov [33] extended all these results for the case $p=2$

as $\lambda_p(-m\omega(2\tau+\mathrm{c\,tg}\,\omega t))=\lambda_p\left(-m\omega\frac{1}{\sin\omega t}\right)=\lambda_p\left(-\frac{m}{t}\right)=\lambda_p\left(-\frac{t}{m}\right)$, $|\omega t|_p<1$.
Here $\xi,\tau\in Q_p$ and $\tau^2=-1$.

In fact it seems to us that such an approach corresponds to the expansion of a representation of the group $U(t)$ on irreducible ones. Note also that the presented theory of the p-adic quantum harmonic oscillator is closely related to the theory of representations of the group $SL_2(Q_p)$ [19, 26].

5. Probability Theory (Euclidean) Formulation of the p-Adic Quantum Mechanics

We consider now a formulation of the p-adic quantum mechanics which corresponds to the classical p-adic mechanics with real-valued coordinates and a p-adic time, i.e. a classical coordinate $x(t)$ is a function $x:Q_p\to\mathbb{R}$. Let $d\mu$ be a Gaussian probability measure on the space of real distributions $\mathscr{D}_r'(Q_p)$ with mean zero and covariance

$$\langle x(f)x(g)\rangle=\int_{\mathscr{D}_r'(Q_p)} x(f)x(g)d\mu(x)=\int_{Q_p^2} f(t)G(t-\tau)g(\tau)dtd\tau\,,$$

where

$$G(t)=\int_{Q_p}\chi(tk)\frac{dk}{|k|_p^2+m^2}\,,\qquad m\in\mathbb{R}\,,\qquad m\neq 0\,;\qquad f,g\in\mathscr{D}_r(Q_p)\,. \tag{5.1}$$

This measure is defined on Borel cylinder sets in $\mathscr{D}_r'(Q_p)$

$$U_{B,f_1,\ldots,f_n}=\{\varphi\in\mathscr{D}_r'(Q_p):(\varphi(f_1),\ldots,\varphi(f_n))\in B\subset\mathbb{R}^n\}$$

in the following way:

$$\mu(U_{B,f_1,\ldots,f_n})=\int_B\exp[-\tfrac{1}{2}\langle x,G^{-1}x\rangle]d\kappa\Big/\int_{\mathbb{R}^n}\exp[-\tfrac{1}{2}\langle x,G^{-1}x\rangle]dx\,.$$

Here $f_1,\ldots,f_n\in\mathscr{D}_r(Q_p)$, B is a Borel set in $\mathbb{R}^n$,

$$\langle x,G^{-1}x\rangle=\sum_{i,j=1}^{n}x_iG_{ij}^{-1}x_j\,,$$

$$G_{ij}=\int_{Q_p^2}f_i(t)G(t-\tau)f_j(\tau)dtd\tau\,,\qquad i,j=1,\ldots,n\,.$$

The characteristic functional of $d\mu$ is

$$\int_{\mathscr{D}_r'} e^{ix(f)}d\mu(x)=\exp[-\tfrac{1}{2}G(f,f)]\,,$$

where

$$G(f,f)=\int_{Q_p} f(t)G(t-\tau)f(\tau)dtd\tau\,.$$

The measure $d\mu$ corresponds to the quantum harmonic oscillator. The anharmonic oscillator is described by the measure

$$dv(x)=\exp\left[-\lambda\int_{Q_p}:x^4(t):dt\right]d\mu(x)\Big/\int_{\mathscr{D}_r'}\exp\left[-\lambda\int_{Q_p}:x^4(t):dt\right]d\mu(x)\,. \tag{5.2}$$

Here $\lambda \geqq 0$. The classical equation of motion in this approach is derived from the action

$$S = \int_{Q_p} (|k|_p^2 + m^2)\,|\tilde{x}(k)|^2 dk + \lambda \int_{Q_p} x^4(t)dt\,, \tag{5.3}$$

where $\tilde{x}(k)$ is the Fourier transform of $x(t)$, see Sect. 2. Note that this action takes values in the field of real numbers in contrast to the case considered above in which the Lagrangian and the action were p-adic valued.

The expressions (5.1)–(5.3) formally look like the corresponding ones for the case of real numbers, see [30], and the corresponding Feynman diagram techniques in perturbation theory can be constructed in an analogous way. However the properties of the p-adic theory are essentially different. Let us consider in particular the behaviour of the propagator (5.1) when $|t|_p \to \infty$. We have (see [20])

$$\int_{Q_p} \frac{\chi(tk)dk}{|k|_p^2 + m^2} = \left(1 - \frac{1}{p}\right) \frac{|t|_p}{p^2 + m^2|t|_p^2} \sum_{n=0}^{\infty} p^{-n} \frac{p^2 - p^{-2n}}{p^{-2n} + m^2|t|_p^2}\,, \qquad t \neq 0$$

and for $|t|_p \to \infty$ one gets

$$\int_{Q_p} \frac{\chi(tk)dk}{|k|_p^2 + m^2} \sim \frac{p^3(p+1)}{p^2+p+1} \frac{1}{m^4|t|_p^3}\,,$$

i.e. power decay at infinity. Recall that in the real case one has an exponential decay

$$\int_{\mathbb{R}} \frac{e^{itk}dk}{k^2+m^2} = \frac{\pi}{m} e^{-m|t|}$$

for $|t| \to \infty$.

This formulation can be generalized to the field theory as follows. Instead of the propagator (5.1) we consider the propagator

$$\langle \varphi(x)\varphi(y) \rangle = G(x-y) = \int_{Q_p^n} \chi((x-y,k)) \frac{dk}{|(k,k)|_p + m^2}\,,$$

where $(x,k) = \sum_{i=1}^{n} x_i k_i \in Q_p$. Such a theory has $O(n)$ symmetry. For $n=2$ there is also a generalization which is related to a quadratic extension of the field Q_p, i.e. instead of (k,k) one considers

$$(k,k)_\tau = k_1^2 - \tau k_2^2\,, \qquad \tau \in Q_p\,.$$

Another important case is related to the propagator $1/\|k\|^2 + m^2$.

6. Conclusion

In this paper we have suggested a mathematical approach to p-adic quantum mechanics. Of course there are many open questions; some of them were pointed out in the text, some of them are subject to further investigations. In particular it would be very interesting to construct the operator of evolution $U(t)$ for other quantum-mechanical systems and to investigate the scattering theory.

We now say a few words on a physical interpretation of the formalism proposed. The p-adic quantum mechanics with complex-valued wave functions is closer to traditional quantum mechanics than it would seem but its physical interpretation is essentially different. In standard quantum mechanics the expression $|\psi(x)|^2$ is considered as a probability density of location of a particle at a point $x \in \mathbb{R}$. It seems that in p-adic quantum mechanics one has to interpret $|\psi(x)|^2$ as a probability density of location of a "particle" at a point $x \in Q_p$. Note here that a point $x \in Q_p$ generally speaking does not belong to the usual real line $\mathbb{R}$, in spite of the fact that the dense set of rational numbers belongs to both fields. Therefore here the question is in fact on a probability of location of the "particle" outside the standard Euclidean real space. Of course it means that we fall outside the frames of standard understanding of reality in the usual quantum theory. In fact it happens even in classical p-adic mechanics in which we deal with p-adic time and coordinates.

Note here that these strange and unusual ideas which appear in p-adic quantum theory probably are inevitable if we are going to understand physics at the Planck length, for a discussion see [4]. A further development of the mathematical formalism of p-adic quantum theory will give us, we hope, a deeper understanding of the new physics at the Planck length.

References

1. Vladimirov, V.S., Volovich, I.V.: Superanalysis, Differential calculus. Theor. Math. Phys. **59**, 3–27 (1984)
2. Volovich, I.V.: p-adic space-time and string theory. Theor. Math. Phys. **71**, 337–340 (1987)
3. Volovich, I.V.: p-adic string. Class. Quant. Grav. **4**, L83–L87 (1987)
4. Volovich, I.V.: Number theory as the ultimate physical theory. Preprint CERN-TH. 4981/87
5. Grossman, B.: p-adic strings, the Weyl conjectures and anomalies. Phys. Lett. B **197**, 101–106 (1987)
6. Freund, P.G.O., Olson, M.: Non-archimedean strings. Phys. Lett. B **199**, 186–190 (1987)
7. Freund, P.G.O., Witten, E.: Adelic string amplitudes. Phys. Lett. B **199**, 191–195 (1987)
8. Volovich, I.V.: Harmonic analysis and p-adic strings. Lett. Math. Phys. **16**, 61–67 (1988)
9. Aref'eva, I.Ya., Dragovič, B., Volovich, I.V.: On the p-adic summability of the anharmonic oscillator. Phys. Lett. B **200**, 512–514 (1988)
10. Gervais, J.-L.: p-adic analyticity and Virasoro algebras for conformal theories in more than two dimensions. Phys. Lett. B **201**, 306–310 (1988)
11. Frampton, P.H., Okada, Y.: p-adic string N-point function. Phys. Lett. **60**, 484–486 (1988)
12. Marinari, E., Parisi, G.: On the p-adic five point function. Preprint ROM2F-87/38
13. Freund, P.G.O., Olson, M.: p-adic dynamical systems. Nucl. Phys. B **297**, 86–97 (1988)
14. Aref'eva, I.Ya., Dragovič, B., Volovich, I.V.: On the adelic string amplitudes. Phys. Lett. B **209**, 445–450 (1988)
15. Frampton, P.H., Okada, Y.: Effective scalar field theory of p-adic string. Phys. Rev. D (to appear)
16. Grossman, B.: Adelic conformal field theory. Preprint Rockefeller University DOE/ER/40325-16
17. Meurice, Y.: The classical harmonic oscillator on Galois and p-adic fields, preprint ANL-HEP-PR-87-114
18. Borevich, Z.I., Shafarevich, I.R.: Number theory. New York: Academic Press 1966
19. Gelfand, I.M., Graev, M.I., Pjatetski-Shapiro, I.I.: Theory of representations and automorphic functions. Moscow: Nauka 1966

20. Vladimirov, V.S.: Distributions over the field of p-adic numbers. Usp. Math. Nauk **43**, 17–53 (1988)
21. Koblitz, N.: p-adic numbers, p-adic analysis, and zeta-functions. Berlin, Heidelberg, New York: Springer 1984
22. Dwork, B.: Lectures on p-adic differential equations. Berlin, Heidelberg, New York: Springer 1982
23. Vinogradov, I.M.: Elements of Number theory. Moscow: Nauka 1965
24. Weyl, H.: The theory of groups and quantum mechanics. New York: Dover 1931
25. Karasev, M.V., Maslov, V.P.: Asymptotic and geometric quantization. Usp. Math. Nauk. **39**, 115–173 (1984)
26. Weil, A.: Sur certains groupes d'operateurs unitaries. Acta Math. **111**, 143–211 (1964)
27. Manin, Yu.I.: Non-Archimedean integration and p-adic L-functions of Jacquet and Lenglands. Usp. Math. Nauk. **31**, 5–54 (1976)
28. Feynman, R.P., Hibbs, A.R.: Quantum mechanics and path integrals. New York: McGraw-Hill 1965
29. Maslov, V.P.: Operator methods. Moscow: Nauka 1973
30. Glimm, J., Jaffe, A.: Quantum physics. A functional integral point of view. Berlin, Heidelberg, New York: Springer 1987
31. Vladimirov, V.S., Volovich, I.V.: Vacuum state in p-adic quantum mechanics. Phys. Lett. B **217**, 411–415 (1989)
32. Vladimirov, V.S., Volovich, I.V.: p-adic Schrödinger type equation, Preprint NBI-HE-87-77 (1988)
33. Zelenov, E.I.: p-adic quantum mechanics for $p=2$. Theor. Math. Phys. (to appear)

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