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S^1 Actions and Elliptic Genera*

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Abstract. A proof is given of Witten's conjectures for the rigidity of the index of the Dirac-Ramond operator on the loop space of a spin manifold which admits an S^1 symmetry.

1. Introduction

When M is a connected, compact, oriented, even dimensional, spin Riemannian manifold, one can define the Dirac operator, ∂, to act on the space of smooth sections of the bundle of complex spinors, $S(T^*M)\to M$. The index of this operator can be defined by using Clifford multiplication on $S(T^*M)$ by $(i)^{n(n+1)/2}\cdot\omega$, with ω being the image in the Clifford algebra of the volume form on M and with $n=\dim(M)$. This defines a covariantly constant involution, γ, of $S(T^*M)$. As an involution of $C^\infty(S(T^*M))$, γ anti-commutes with the Dirac operator. Then,

$$\operatorname{Ind}(\partial,\gamma)\equiv\dim(\ker(\partial|_{\ker(\gamma-1)}))-\dim(\ker(\partial|_{\ker(\gamma+1)}))\,. \tag{1.1}$$

Now, suppose that M admits an isometric action of S^1. Here, the index of ∂ has a refinement which is the S^1 equivariant index. That is, use the S^1 action to decompose $C^\infty(S(T^*M))=\oplus_k C^\infty(S(T^*M),k)$ where the double cover of S^1 acts on $C^\infty(S(T^*M),k)$ as multiplication by λ^k; $\lambda\in S^1$. As ∂ and γ commute with the S^1 action, they both preserve $C^\infty(S(T^*M),k)$ and with this understood, the S^1-equivariant index of ∂ is, by definition, the set of integers, $\{\operatorname{Ind}(\partial,\gamma,k)\}$, which is obtained by replacing $\ker(\gamma\pm1)\cap C^\infty(S(T^*M))$ in Eq. (1.1) with $\ker(\gamma\pm1)\cap C^\infty(S(T^*M),k)$.

The S^1-equivariant index can be generalized in the usual way by twisting the dirac operator with a vector bundle over M. Thus, when $V\to M$ is a complex vector bundle, one can define the index of the Dirac operator on $S(T^*M)\otimes V$, $\operatorname{Ind}(\partial, V,\gamma)$, by replacing $\ker(\gamma\pm1)\subset C^\infty(S(T^*M))$ with $\ker(\gamma\pm1)\subset C^\infty(S(T^*M)\otimes V)$. And, if a finite cover of the S^1 action on M has a lift to V, one can consider the S^1 equivariant

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index of the Dirac operator on $S(T^*M)\otimes V$. This is a set of integers $\{\mathrm{Ind}(\partial, V, \gamma, k)\}$ which is defined by replacing $\ker(\gamma\pm 1)\subset C^\infty(S(T^*M))$ in Eq. (1.1) with $\ker(\gamma\pm 1)\cap C^\infty(S(T^*M)\otimes V, k)$.

Atiyah and Hirzebruch [A–H] proved that $\mathrm{Ind}(\partial, \gamma, k)=0$ for all k when S^1 acts non-trivially on a connected spin manifold. Witten [W1], considering the Rarita-Schwinger operator, asked whether for $k\neq 0$, one could prove that $\mathrm{Ind}(\partial, T^*M, \gamma, k)=0$. Landweber and Stong [L–S] proved that $\mathrm{Ind}(\partial, T^*M, \gamma, k)=0$ for all k when the S^1-action is non-trivial and is assumed to be of odd type and to be semi-free. [Odd type means that the action does not lift to an action on $S(T^*M)$; and being semi-free means that the stabilizer of a point is S^1 or is 1.]

In fact, Landweber and Stong considered the following formal power series in a variable q with values in $\mathrm{Vect}(M)$:

$$F_D(q;\, T^*M)\equiv \bigotimes_{0<m\,\mathrm{even}} \mathrm{Sym}(q^m\cdot T^*M) \bigotimes_{0<m\,\mathrm{odd}} \Lambda^*(q^m\cdot T^*M)\,, \tag{1.2}$$

where $\mathrm{Sym}(a\cdot E)=1+a\cdot E+a^2\cdot \mathrm{Sym}^2(E)+\ldots$, and where $\Lambda^*(a\cdot E)=1+a\cdot E+a^2\cdot\Lambda^2(E)+\ldots$. Landweber and Stong [L–S] proved that when the S^1 action on M is of odd type and semi-free, then

$$\mathrm{Ind}(\partial, F_D(q;\, T^*M), \gamma, k)=0 \quad \text{for} \quad k\neq 0\,. \tag{1.3}$$

Later, Ochanine [O1] proved Eq. (1.3) for all semi-free actions. Ochanine has also proved Eq. (1.3) for certain kinds of non-semi-free actions [O2].

Witten recognized the power series in Eq. (1.2) as coming from physic's string theory [W2] (see also [W3]). On the basis of heuristic arguments, Witten proposed that for any S^1 action, $\mathrm{Ind}(\partial, F_D(q;\, T^*M), \gamma, k)=0$ for $k\neq 0$. Furthermore, Witten suggested that a similar assertion should hold in greater generality. He considered replacing T^*M in Eq. (1.3) with a real, oriented vector bundle $V\to M$ to which the S^1 action has a lift.

Two additional requirements on V were made; their statement requires a digression: Introduce the universal S^1 bundle, S^∞, the unit sphere in a complex, separable Hilbert space. The classifying space for S^1 is $S^\infty/S^1\equiv BS^1=\mathbb{CP}^\infty$. If S^1 acts on a manifold M, then one can form the quotient $S^\infty\times_{S^1}M$ as a fiber bundle with fiber M over $\mathbb{CP}^\infty$.

If $V\to M$ is a vector bundle to which the S^1 action lifts, then one can construct the vector bundle $S^\infty\times_{S^1}V\to S^\infty\times_{S^1}M$. The characteristic classes of $S^\infty\times_{S^1}V$ in the cohomology of $S^\infty\times_{S^1}M$ are called the S^1-equivariant characteristic classes of V. Of particular interest are the 2nd Stiefel-Whitney class, w_2, and 1/2 of the 1st Pontrjagin class, $1/2\cdot p_1$.

Here, a word of explanation is in order. Let M be a manifold, and let $V\to M$ be a real, oriented vector bundle of dimension $d>2$ with fiber metric. The principal $SO(d)$ bundle of oriented, orthonormal frames in V is the pull-back of the universal $SO(d)$ principal bundle over $BSO(d)$ by a map $f: M\to BSO(d)$. The characteristic classes $w_2(V)$ and $p_1(V)$ are the pull-backs by f of the universal $w_2\in H^2(BSO(d);\, \mathbb{Z}/2\cdot\mathbb{Z})$ and the universal $p_1\in H^4(BSO(d);\mathbb{Z})$.

For $d>2$, introduce the Lie group $\mathrm{Spin}(d)$. $\mathrm{Spin}(d)$ is the simply connected, double cover of $SO(d)$; for $d>2$, one has $SO(d)=\mathrm{Spin}(d)/\mathrm{Center}(\mathrm{Spin}(d))$. The cohomology of $B\,\mathrm{Spin}(d)$ has $H^4(B\,\mathrm{Spin}(d);\mathbb{Z})\approx\mathbb{Z}$ with generator q_1. The class $p_1\in H^4(BSO(d);\mathbb{Z})$ pulls up to $H^4(B\,\mathrm{Spin}(d);\mathbb{Z})$ as $2\cdot q_1$.

If $V\to X$ is a real, oriented vector bundle with $w_2(V)=0$, then the classifying map f lifts to a map $f: X\to B\,\mathrm{Spin}(d)$. For such bundles V, the characteristic class, $1/2\cdot p_1(V)\in H^4(X;\mathbb{Z})$ is defined to be f^*q_1. The class f^*q_1 is independent of the choice of the lift of f (Dan Freed showed the author a proof).

Witten considered real, oriented vector bundles $V\to M$ for which

$$\begin{aligned} &w_2\left(S^\infty \underset{S^1}{\times} (V-T^*M)\right)=0\in H^2\left(S^\infty \underset{S^1}{\times} M;\mathbb{Z}/(2\mathbb{Z})\right),\\ &\tfrac{1}{2}\cdot p_1\left(S^\infty \underset{S^1}{\times} (V-T^*M)\right)=0\in H^4\left(S^\infty \underset{S^1}{\times} M;\mathbb{Z}\right). \end{aligned} \tag{1.4}$$

And, under these conditions, Witten investigated the following formal power series in the complex K-theory of M:

$$\begin{aligned} F_D(q;V)&\equiv \bigotimes_{0<m\,\mathrm{even}} \mathrm{Sym}(q^m\cdot T^*M) \bigotimes_{0<m\,\mathrm{odd}} \Lambda^*(q^m\cdot V),\\ F_S(q;V)&\equiv S(T^*M) \bigotimes_{0<m\,\mathrm{even}} \mathrm{Sym}(q^m\cdot T^*M) \bigotimes_{0<m\,\mathrm{even}} \Lambda^*(q^m\cdot V),\\ F_E(q;V)&\equiv (S_+(T^*M)-S_-(T^*M)) \bigotimes_{0<m\,\mathrm{even}} \mathrm{Sym}(q^m\cdot T^*M)\\ &\quad\times \bigotimes_{0<m\,\mathrm{even}} \Lambda^*(-q^m\cdot V), \end{aligned} \tag{1.5}$$

where $S_\pm(T^*M)\equiv(\gamma\pm 1)\cdot S(T^*M)$, and the difference, $(S_+(T^*M)-S_-(T^*M))$, is defined in the real, oriented K-theory of M. In [W2] and [W3], heuristic arguments are given to justify the conjecture that when Eq. (1.4) holds, $\mathrm{Ind}(\partial, F_*(q;V),\gamma,k)=0$ for all $k\neq 0$.

In [B–T], Raoul Bott and the author proved Witten's conjectures using ideas from elliptic function theory. The proof in [B–T] was based on a first proof by the author which was more closely tied to the original loop space arguments of Witten. It is the purpose of this article to provide an account of that first proof of Witten's assertions.

The precise results are stated in Theorem 1.3 below. To state the theorem, a second digression is required: Since $\mathbb{Z}/(n\mathbb{Z})$ is a subgroup of S^1, the universal bundle for $\mathbb{Z}/(n\mathbb{Z})$ can be taken to be S^∞; with the classifying space $B\mathbb{Z}/(n\mathbb{Z}) = S^\infty/(\mathbb{Z}/(n\mathbb{Z}))$. If S^1 acts on M, so does $\mathbb{Z}/(n\mathbb{Z})$, and one can construct $S^\infty\times_{\mathbb{Z}/(n\mathbb{Z})}M$. For a vector bundle $V\to M$ on which S^1 acts, one can construct $S^\infty\times_{\mathbb{Z}/(n\mathbb{Z})}V\to S^\infty\times_{\mathbb{Z}/(n\mathbb{Z})}M$.

If $M'\subset M$ is fixed under a subgroup $\Gamma\subseteq S^1$, then $S^\infty\times_\Gamma M'=B\Gamma\times M'$.

Definition 1.1. Let M be a compact, oriented spin manifold on which S^1 acts. Let $V\to M$ be a real, oriented vector bundle to which the S^1 action has a lift. Require that $w_2(V)=0$. Let $\Gamma\subseteq S^1$ be a subgroup, and let $M(\Gamma)\subset M$ denote the fixed point set of Γ. The vector bundle V will be called *Γ-compatible with T^*M* if the following is true:

(1) The restriction to $H^2(B\Gamma\times M(\Gamma);\ \mathbb{Z}/(2\mathbb{Z}))$ of $w_2(S^\infty\times_\Gamma(V-T^*M))$ vanishes. (2) The restriction to $H^4(B\Gamma\times M(\Gamma);\mathbb{Z})$ of $1/2\cdot p_1(S^\infty\times_\Gamma(V-T^*M))$ is the pull back from $M(\Gamma)$ of $1/2\cdot p_1(V-T^*M)\in H^4(M(\Gamma);\mathbb{Z})$.

The vector bundle V will be called *strongly compatible with* T^*M if it is Γ-compatible for all subgroups $\Gamma \subseteqq S^1$.

Lemma 1.2. *Let* M *be a compact, oriented spin manifold on which* S^1 *acts. Let* $V \to M$ *be a real, oriented vector bundle to which the* S^1 *action has a lift. Assume that* $w_2(V)=0$. *A sufficient condition for* V *to be strongly compatible with* T^*M *is for Eq.* (1.4) *to hold.*

Proof of Lemma 1.2. There is a natural map $\pi: S^\infty \times_{\mathbb{Z}/(n\mathbb{Z})} M \to S^\infty \times_{S^1} M$, and it is not hard to check that $S^\infty \times_{\mathbb{Z}/(n\mathbb{Z})} V = \pi^*(S^\infty \times_{S^1} V)$.

The purpose of this article is to prove the following theorem:

Theorem 1.3. *Let* M *be a compact, oriented spin manifold on which* S^1 *acts. Let* $V \to M$ *be a real, oriented vector bundle to which the* S^1 *action has a lift. Require that* $w_2(V)=0$. *Require that* V *be strongly* S^1*-compatible with* T^*M *in the sense of Definition 1.1. For* $*=D, S$ *or* E, *let* $F_*(q; V)$ *be as defined in Eq.* (1.5). *Then* $\operatorname{Ind}(\partial, F_*(q; V), \gamma, k)=0$ *for all* $k \neq 0$.

The conditions in the theorem are not necessarily optimal, see Proposition 10.1.

The proof of Theorem 1.3 is strongly motivated by Witten's heuristic arguments in [W2, W3]. Indeed, the proof amounts to finding a suitable context for Witten's ideas. Here, the following observations are in order: Witten's arguments arise in consideration of a formal "Dirac operator" on the space of free loops on M, $\mathscr{L}M$. And thus, one might conjecture that such an operator must be constructed to obtain the proof of Theorem 1.3.

However, the manipulations in [W2] take place, for the most part, on the normal bundle, $\mathscr{N}M$, to the embedding of M into $\mathscr{L}M$ as the space of constant loops. And, it turns out that a Dirac operator on the normal bundle to $M \subset \mathscr{L}M$ is easy to construct, and is all that Theorem 1.3 requires.

The normal bundle to M is isomorphic to the underyling real vector bundle of the infinite dimensional complex vector bundle

$$\mathscr{N}M \equiv \bigoplus_{0<n\in\mathbb{Z}} T^*M \otimes \mathbb{C}. \tag{1.6}$$

(The topology on $\mathscr{N}M$ is the direct limit topology, see the Appendix.)

The Dirac operator, D_t, on $\mathscr{N}M$ is constructed in Sect. 3. (See also [W3].) It can be thought of as the usual Dirac operator on M, but twisted with an infinite dimensional vector bundle over M. Alternately, one can consider it as a countable set of "standard" Dirac operators,

$$\{\partial + A_m(h): C^\infty(S(T^*M)\otimes R_m(h)) \to C^\infty(S(T^*M)\otimes R_m(h))\},$$

indexed by integers m and h. The vector bundle $R_m(h) \to M$ is a finite dimensional vector bundle which is constructed out of T^*M by taking various tensor products. Here, $A_m(h)$ is a section of $\operatorname{End}(R_m(h))$ which is naturally constructed from exterior and interior product of covectors in T^*M.

In Sect. 4, this operator is generalized by twisting with various vector bundles over the normal bundle to M in $\mathscr{L}M$. These vector bundles are constructed out of tensor products of exterior products of V. The end result is still a countable set of "standard" Dirac operators. Particular generalizations are used in the proof of Theorem 1.3; these are discussed in Sect. 5.

The Fredholm properties of D_t are established by considering it as the countable set of "standard" Dirac operators. This makes the analysis completely classical. However, when manipulating D_t, it is most efficient to consider it as an honest operator on $\mathcal{N}M$; indeed, to do otherwise would waste the great simplifications obtained from the super string – theoretic formalism. But, it should be stressed that all of the manipulations are ultimately justified by returning to the representation of D_t as a countable set of standard operators on a compact manifold.

The Fredholm properties of the Dirac operator on the normal bundle to M are derived in a general setting in a separate Appendix. The main conclusions are the following: There is a canonical circle action on $\mathscr{L}M$ (as opposed to the geometric circle action which is induced by the S^1 action on M) which acts on $\mathscr{L}M$ via translation of the domain S^1. The decomposition in Eq. (1.6) of $\mathcal{N}M$ gives the character decomposition for this action. The double cover of the canonical S^1 action lifts to an action on the domain of D_t which commutes with D_t. The domain of the D_t decomposes into character subspaces (with integral and half integral weights) under the canonical S^1 action, and the restriction of D_t to each character subspace is Fredholm.

Given the vector bundle V, there are two relevant choices for D_t which differ in how they are twisted over $\mathcal{N}M$. These are unprimed or primed in this article; in the physics literature, they give the Ramond and the Neveu-Schwarz versions of the supercharge for the right movers in the underlying string theory. For each Dirac operator, there are two involutions of the domain with which to define the index. The four different constructions are described in Sect. 5.

The two versions of D_t each have two indices: Let $q\in S^1$. For integer or half integer $m<0$, both versions of D_t and both involutions have zero index on the character q^m subspace of their domain. On the character q^m $(m\geqq 0)$ subspace of the domain, the index of D_t equals the q^{2m} component of $\mathrm{Ind}(\partial, F_S(q,V),\gamma)$ or $\mathrm{Ind}(\partial, F_E(q,V),\gamma)$ for the two unprimed indices. On the character q^m $(m\geqq 0)$ subspace of the domain, the index of D_t equals the q^{2m} component of $\mathrm{Ind}(\partial, F_D(q,V),\gamma)$ or $\mathrm{Ind}(\partial, F_D(-q,V),\gamma)$ for the two primed indices.

The preceding assertions hold in some generality. In the unprimed case, the assertions hold on any oriented, compact Riemannian manifold and with any real, oriented vector bundle V as long as $w_2(V)=w_2(T^*M)$. In the primed case, the assertions only require that $w_2(T^*M)=0$. No conditions on the Pontrjagin classes of V are required for the constructions, nor for the assertions in the last paragraph to hold. (See Proposition 5.4.)

When S^1 acts on M, the construction, being functorial, yields an S^1 equivariant theory. For a standard Dirac operator with S^1 equivariance, the Atiyah-Bott [A–B] generalizations of the Lefschetz fixed point formula allow for the index to be calculated from the geometric data at the components of the fixed point set of the action. Since the operator D_t decomposes into a countable set of standard Dirac operators, the Atiyah-Bott formula can be applied to D_t.

It is convenient to use an interpretation of the Atiyah-Bott formula which is due to Witten [W1]. Witten views the contribution from each component of the fixed point set as coming from the S^1 equivariant index of a suitable Dirac operator which is defined on the normal bundle to the fixed point set. This construction of Witten is the finite dimensional analog of the Dirac operator D_t on $\mathcal{N}M$. Because the fixed point formula is crucial to the proof of Theorem 1.3, and because Witten's version gives a model for the later constructions, this version is presented in the next section as a warm up for the later constructions.

One by one, localize the countable set of operators which make up D_t. Using Witten's interpretation of the Atiyah-Bott formula, the result is a Dirac operator, Q, defined on the normal bundle, $\mathcal{N}N$, in $\mathcal{L}M$ to the normal bundle, $N \subset M$ to each component, $\Sigma \subset M$, of the fixed point set of the S^1 action. Alternately, one may think of $\mathcal{N}N$ as an infinite dimensional vector bundle over Σ, in which case Q becomes a countable set of "standard" Dirac operators on Σ. The Atiyah-Bott fixed point formula for D_t in terms of the operator Q is described in Sect. 6. To summarize: Let $\{\Sigma[i]\}$ label the connected components of the fixed point set of the S^1 action. Each $\Sigma[i]$ has its corresponding operator $Q[i]$, and

$$\sum_{\Sigma[i]} \operatorname{index}(Q[i]) = \operatorname{index}(D_t) \tag{1.7}$$

holds as an equality of $S^1 \times S^1$ equivariant indices. The first S^1 is the canonical S^1 action on the loop space, and the second S^1 is the geometric S^1 action from M.

Since the geometric S^1 acts on M by isometries, the normal bundle N to a component of the fixed point set is naturally a complex vector bundle (this is described in the next section.) It decomposes as $\oplus_{0<\upsilon} N(\upsilon)$, with each $N(\upsilon) \to \Sigma$ a complex vector bundle on which the geometric S^1 acts as multiplication by ξ^υ, $\xi \in S^1$.

As an isomorphism of real bundles over Σ, one has

$$\mathcal{N}N \approx \Big(\bigoplus_{0<n\in\mathbb{Z}} (T^*\Sigma \otimes \mathbb{C})\Big) \oplus \bigoplus_{0<\upsilon\ n\in\mathbb{Z}} N(\upsilon). \tag{1.8}$$

Equation (1.8) gives a decomposition of $\mathcal{N}N$ into character subspaces for the two commuting S^1-actions: The canonical S^1 action, sends $q \in S^1$ to q^n on the n^{th} copy of $N(\upsilon)$; and the geometric S^1 sends $\xi \in S^1$ to ξ^υ on the n^{th} copy of $N(\upsilon)$. Let P and K denote the respective generators; they define automorphisms of $\mathcal{N}M$.

Equation (1.8) indicates that $\mathcal{N}N$ possesses a non-trivial $\mathbb{Z}$ subgroup of bundle automorphisms. The generator, $\imath$, acts by sending the n^{th} copy of $N(\upsilon)$ to the $(n+\upsilon)^{\text{th}}$ copy of $N(\upsilon)$. The following commutation rules are evident:

$$\imath P \imath^{-1} = P + K, \qquad \imath K \imath^{-1} = K. \tag{1.9}$$

The import of this group of automorphisms is suggested by the arguments of Witten in [W2, W3]. Interpreting Witten, one should ask whether $\imath$ lifts to define an automorphism of the domain of the operator Q on $\mathcal{N}N$.

The behavior of $\imath$ vis-á-vis the operator Q is considered in Sect. 7. There is an obstruction to lifting $\imath$ to the domain of Q, it is a component of $1/2 \cdot p_1(S^\infty \times_{S^1} (V - T^*M))$. The vanishing of this characteristic class insures the lift. Given a lift, one computes

$$\imath Q \imath^{-1} = Q + \mathcal{K}, \tag{1.10}$$

where $\mathcal{K}$ is Clifford multiplication on the domain of Q by K.

As previously remarked, only upon restriction to an eigenspace of P does D_t become Fredholm. This consideration makes the lift of commutation relations in Eq. (1.9) a crucial issue. But, there is an obstruction to the lifting; the second one lifts automatically, but the first one lifts if and only if a different component of $1/2 \cdot p_1(S^\infty \times_{S^1}(V - T^*M))$ vanishes.

Since an index of Q is the ultimate goal, it is important to consider how $\imath$ behaves with respect to an automorphism, ℓ, of the domain of Q which defines the index. For the automorphisms in question,

$$\imath \ell \imath^{-1} = (-1)^\mu \ell \tag{1.11}$$

defines $\mu \in \{0, 1\}$. The value of μ is computable from the geometric data at the fixed point set.

Equations (1.9–11) allow, in principle, for the comparison of the index of Q on the $P = m$ eigenspace of its domain with $\pm$ the index of Q on the $P = m + k$ eigenspace. Indeed, were there a Fredholm deformation of $Q + \mathscr{K}$ to Q, such would follow automatically.

Each $\Sigma[i]$ has its corresponding operator $Q[i]$, and the fixed point formula equates the index of D_t on the $P = m$, $K = k$ subspace of its domain with the sum, over i, of the index of $Q[i]$ on the $P = m$, $K = k$ subspace of $Q[i]$'s domain.

Each component $\Sigma[i]$ has an "anomaly" $(-1)^{\mu[i]}$, with $\mu[i]$ defined for $\Sigma[i]$ by Eq. (1.11). If M is a spin manifold and if $w_2(V) = 0$, then the anomaly is independent of the label i. With the anomaly independent of i, and with a Fredholm deformation of $Q + \mathscr{K}$ to Q, Eq. (1.7) implies an equality up to sign between the index of D_t on the $P = m$, $K = k$ subspace of its domain with the index of D_t on the $P = m + k$, $K = k$ subspace.

Such an equality is the crux of Witten's argument in [W2, W3]. With it, Theorem 1.3 follows automatically: As previously mentioned, the index of D_t vanishes on the $P < 0$ subspace of its domain.

Buried under the rug here is the assertion that there exists a Fredholm deformation of $Q + \mathscr{K}$ to Q. Technically this assertion is false. Since $\mathscr{K}$ is a lower order term with respect to Q, one might be tempted to consider it as a compact perturbation to Q. However, it is only on compact manifolds that a lower order term is automatically irrelevant. On a non-compact manifold (for example $\mathscr{N}N$), symbol degeneracy can occur in spatial directions.

As a function of $\alpha \in [0, 1]$, the operator $Q + \alpha \cdot \mathscr{K}$ fails to be Fredholm at $\alpha \in \Omega \equiv \{r \in [0, 1] : r \cdot \upsilon \in \mathbb{Z}$ for those $0 < \upsilon \in \mathbb{Z}$ which have $N(\upsilon) \neq \emptyset\}$. Given $\alpha_0 \in \Omega$, let n_0 denote the smallest, positive integer for which $\alpha_0 \cdot n_0 \in \mathbb{Z}$. The Fredholm failure of $Q + \alpha_0 \cdot \mathscr{K}$ is due to the "delocalization" of the operator along the submanifold, $M(n_0)$, of M which is fixed by the $\mathbb{Z}/(n_0\mathbb{Z})$ subgroup of S^1.

Provided that the conditions in Theorem 1.3 hold, there exists an operator $D_{n_0 t}$ on $\mathscr{N}M(n_0)$ which localizes under the S^1 action on $M(n_0)$ to $Q + \alpha_0 \cdot \mathscr{K}$. This means that the jump in the index of $Q + \alpha \cdot \mathscr{K}$ as α crosses α_0 is compensated by jumps at the other components of the fixed point set of the S^1 action. In particular, the compensation is due to jumps at those components which are contained in the same component of $M(n_0)$ as Σ. Thus, under Theorem 1.3's assumptions,

$$\sum_{\Sigma[i]} \operatorname{index}(Q[i] + \alpha \cdot \mathscr{K}[i]) \tag{1.12}$$

is independent of α.

This last assertion is proved for semi-free S^1 actions in Sect. 8, and for general S^1 actions in Sect. 9. The proof of Theorem 1.3 is assembled in Sect. 10.

Sections 2–10 of this paper are devoted to a self-contained proof of Theorem 1.3.

2. Localization

Part 1. The Dirac Operator

Let M be a compact, oriented, even dimensional Riemannian manifold and suppose that the group S^1 acts on M as a group of isometries. Then, the S^1 action is generated by a vector, $K_M \in C^\infty(TM)$; a vector field which obeys the Killing equation. With respect to a local coordinate system on a neighborhood of a point in M, write $K_M \equiv K^\alpha \partial_\alpha$, write the metric as $g \equiv g_{\beta\sigma} dx^\beta \otimes dx^\sigma$; and then the Killing equation is

$$\nabla_\alpha(g_{\beta\sigma}K^\sigma) + \nabla_\beta(g_{\alpha\sigma}K^\sigma) = 0\,, \tag{2.1}$$

where ∇_α is the Riemannian metric's covariant derivative in the direction of ∂_α.

When M is an even dimensional spin manifold, the Dirac operator is defined on smooth sections of the bundle of spinors, $S \equiv S(T^*M) \to M$. This is a complex vector bundle over M of complex dimension 2^p with $p \equiv \dim(M)/2$. (See [A–B–S].) In local coordinates, the Dirac operator is

$$\partial_0 \equiv dx^\alpha \cdot \nabla_\alpha\,. \tag{2.2}$$

Here, Clifford multiplication by the basis covectors $\{dx^\alpha\}$ in T^*M obeys $dx^\alpha \cdot dx^\beta + dx^\beta \cdot dx^\alpha = -2 \cdot g^{\alpha\beta}$. Since Clifford multiplication by a covector is an anti-hermitian endomorphism of S, the formal L^2-adjoint of ∂_0, ∂_0^*, is equal to ∂_0.

Suppose γ is a fiber preserving, covariantly constant involution of S which anti-commutes with Clifford multiplication by the odd elements of the Clifford algebra. The involution has eigenvalues ± 1. Define the index of ∂_0 to be

$$\mathrm{Ind}(\partial_0, \gamma) \equiv \dim \ker(\partial_0|_{\ker(\gamma-1)}) - \dim \ker(\partial_0|_{\ker(\gamma+1)})\,. \tag{2.3}$$

For $t \in \mathbb{R}$, Witten [W1] introduces $\underline{K}$ as the 1-form which is metrically dual to the Killing vector K, and he then considers the modified operator

$$\partial_t \equiv \partial_0 + i \cdot t \cdot \underline{K}\,, \tag{2.4}$$

on $C^\infty(S)$ where $\underline{K}$ acts by Clifford multiplication. Since γ anticommutes with ∂_t, the index of ∂_t is well defined, and is independent of t.

Define

$$i \cdot K \equiv \nabla_K - \tfrac{1}{4} \cdot d\underline{K} \tag{2.5}$$

as a first order differential operator on $C^\infty(S)$. (Here, ∇_K is covariant differentiation along K_M and the 2-form $d\underline{K}$ acts by Clifford multiplication.) When the S^1 action lifts to an S^1 action on S, this first order operator is the generator. In any case, K is defined, symmetric and

$$[K, \partial_t] = 0\,. \tag{2.6}$$

Equation (2.6) implies that the eigenspaces of ∂_t can be decomposed into subspaces on which K acts by multiplication. Let $C^\infty(S,k)$ denote the subspace of the space of smooth sections of S on which K acts with eigenvalue k. Then $L^2(S) = \oplus_k L^2(S,k)$, where $L^2(S,k)$ is the L^2-completion of $C^\infty(S,k)$.

Since ∂_t commutes with K, it maps $C^\infty(S,k)$ into itself. Let ∂_{tk} denote the restriction of ∂_t to $C^\infty(S,k)$. Note that K commutes with the involution γ. This means that the integer

$$\mathrm{Ind}(\partial_0, \gamma, \mathrm{k}) \equiv \mathrm{Ind}(\partial_{tk}, \gamma) \tag{2.7}$$

is well defined. Standard Fredholm theory implies that the left-hand side of Eq. (2.7) is independent of t.

The Atiyah-Bott fixed point theorem [A–B] (see also [A–Se]) asserts that $\mathrm{Ind}(\partial_0, \gamma, k)$ can be computed from geometric data at the fixed point set of the S^1-action. Witten observed that the fixed point theorem can be obtained naturally by considering the large $|t|$ limit of the right-hand side of Eq. (2.7).

To obtain Witten's proof of the fixed point formula, one should consider the Weitzenbock formula for ∂_t^2: Restricted to $C^\infty(S,k)$,

$$\partial_t^2 = \nabla^*\nabla + t^2|K_M|^2 - 2kt + \mathscr{R} + t \cdot i \cdot d\underline{K}\,, \tag{2.8}$$

where $\nabla^*\nabla$ is the trace Laplacian, $\mathscr{R}$ is a curvature endomorphism and the two form $d\underline{K}$ acts again by Clifford multiplication. (This is a calculation for the reader; see the appendix of [F–U] for help.) As $|t| \to \infty$, one expects that all eigenvalues of the self-adjoint, non-negative operator ∂_t^2 will tend to ∞, except for a finite number of small eigenvalues, whose corresponding eigenvectors will remain localized near the fixed point set of the S^1 action; near where $K_M = 0$. When M is compact, and finite dimensional, this occurs:

Proposition 2.1. *Let M be a compact, oriented, spin Riemannian manifold on which S^1 acts isometrically. Let $S \to M$ denote the bundle of spinors on M. Suppose that a finite cover of the S^1 action on M lifts to an action on S. Fix an eigenvalue k of the differential operator K in Eq.* (2.5)*; and fix a real number t. For $R \geqq 1$, let $N(R,t) \equiv \{x \in M : |K_M|(x) > R/|t|^{1/2}\}$. Suppose that $\psi \in C^\infty(S,k)$ and $\partial_t\psi = \mu \cdot \psi$, with $|\mu| < t^{1/2}$. At $x \in N(R,t)$,*

$$|\psi|(x) \leqq z(k) \cdot \exp(-c(k) \cdot |t|^{1/2} \cdot R \cdot \mathrm{dist}(x, \Sigma)^2)\,,$$

where z and c are independent of t, R, and ψ.

Proof of Proposition 2.1. Let β be a cut off function on M which is zero if $\mathrm{dist}(\cdot, \Sigma) > R/|t|^{1/2}$, and which is identically 1 if $\mathrm{Dist}(\cdot, \Sigma) < R/2 \cdot |t|^{1/2}$. Require that $|d\beta| < 8 \cdot |t|^{1/2}/R$. Then,

$$\partial_t((1-\beta) \cdot \psi) = \mu \cdot (1-\beta) \cdot \psi - \sigma(d\beta) \cdot \psi\,. \tag{2.9}$$

This last equation plus Eq. (2.8) imply that

$$\|\nabla((1-\beta)\cdot\psi)\|_{L^2}^2 + |t| \cdot R^2 \cdot \|(1-\beta)\cdot\psi\|_{L^2}^2 \leqq z \cdot |t| \cdot (1+|k|) \cdot \|\psi\|_{L^2}^2\,. \tag{2.10}$$

Equation (2.10) implies that

$$\|(1-\beta)\cdot\psi\|_{L^2}^2 \leqq z \cdot (1+|k|)/R^2 \cdot \|\psi\|_{L^2}^2\,. \tag{2.11}$$

Now, let $f \equiv |\psi|$. Then f obeys

$$d^*d(f^2/2) + |df|^2 + t^2 \cdot |K|^2 \cdot f^2 - z \cdot |t| \cdot (1+|k|) \cdot f^2 \leqq z \cdot |d\beta| \cdot |\nabla\psi| + |\nabla d\beta| \cdot |\psi|\,, \quad (2.12)$$

from which the proposition follows with the maximum principle and a suitable comparison function.

Part 2. The Normal Bundle's Dirac Operator

Witten's proof of the fixed point formula arises by using the localization assertion of Proposition 2.1 to compare the operator ∂_t with a suitable operator which is defined on the normal bundle to the fixed point set of the S^1 action.

To define this new operator, some preliminary observations are in order: Recall that the fixed point set of the S^1 action is a smooth submanifold $\Sigma \subset M$. Let $\pi: N_\Sigma \to \Sigma$ denote the normal bundle to Σ. There exists $\varepsilon > 0$ and a diffeomorphism of the ε-ball in N_Σ with a neighborhood, $O \subset M$, of Σ. Let v denote a point in N_Σ. Then said diffeomorphism sends the point v to $\exp_{\pi(v)}(v)$, where $\exp: TM \to M$ is the exponential map. The diffeomorphism is equivariant with respect to the S^1action on M and on $N_\Sigma \subset TM|_\Sigma$.

The vector K_M vanishes on Σ, so on O, K_M has the following expansion:

$$K_M(v) \equiv \nabla_v K_M + \mathcal{O}(|v|^2)\,. \quad (2.13)$$

Note, because K_M is a Killing vector, $\nabla_v K_M$ defines a vector in N_Σ. In fact, with respect to the Riemannian metric, the assignment of $v \in N_\Sigma$ to $\nabla_v K_M$ defines a non-degenerate, skew-adjoint endomorphism, ∇K_M, of N_Σ which is covariantly constant along Σ.

With this understood, it is natural to use ∇K_M to define a complex structure on $N_\Sigma \otimes \mathbb{C}$. That is, $N_\Sigma \otimes \mathbb{C} \approx N \oplus \underline{N}$, where $N \to \Sigma$ is the subbundle of $N_\Sigma \otimes \mathbb{C}$ which is spanned at each point by vectors v for which

$$\nabla_v K_M = -i \cdot \upsilon \cdot v \quad \text{with} \quad \upsilon > 0\,. \quad (2.14)$$

A priori, the set $\{\upsilon > 0 : -i \cdot \upsilon$ is an eigenvalue of ∇K_M on $N_\Sigma\}$ is a set of dim N_Σ integers after counting multiplicity; these integers are called the "exponents" of the S^1-action at Σ. Note that when υ is an exponent at Σ, then $N_\upsilon \equiv \{v \in N : \nabla_v K_M = -i \cdot \upsilon \cdot v\}$ is a well defined subbundle of N, and N decomposes as

$$N = \bigoplus_{\upsilon > 0} N_\upsilon\,. \quad (2.15)$$

Since ∇K_M is covariantly constant, the isomorphism $N_\Sigma \otimes \mathbb{C} \approx N \oplus \underline{N}$ and that in Eq. (2.15) are both preserved by parallel transport.

Note that this complex structure orients the fiber of N at each point x in Σ. Together with an orientation of M, this defines an orientation of Σ (if Σ is an isolated point, an orientation is just a sign, ± 1). This orientation will be implicity assumed in what follows.

When M is spin, it is convenient to describe the spin bundle of M on the tubular neighborhood O of Σ in the following way: Via the exponential map, pull the spin bundle $S(T^*M)$ back to N_Σ; this identifies it with the spin bundle $S(T^*N_\Sigma)$. Parallel transport along the normal geodesics to Σ constructs an isomorphism between $S(T^*M) \to N_\Sigma$ and the spin bundle $\pi^*(S(T^*M)|_\Sigma)$, where $\pi: N_\Sigma \to \Sigma$ is the projection. This isomorphism will be implicitly assumed.

Let $\operatorname{Fr} M$ denote the bundle of positively oriented, orthonormal frames in TM. If $n = \dim M$, then $\operatorname{Fr} M$ is a principal $SO(n)$ bundle over M. Restricted to Σ, $\operatorname{Fr} M|_\Sigma \approx \operatorname{Fr}\Sigma \otimes P_N$; where $\operatorname{Fr}\Sigma \to \Sigma$ is the principal $SO(n-2d)$ bundle of orthonormal frames in $T\Sigma$; and where $P_N \to \Sigma$ is the principal $U(d)$ bundle of unitary frames in the (complex) vector bundle $N \to \Sigma$.

As M is spin, $\operatorname{Fr} M$ lifts to a principal $\operatorname{Spin}(n)$ bundle, $\operatorname{Fr}' M \to M$. The fact that $\operatorname{Fr} M|_\Sigma$ is spin means that the second Stieffel-Whitney class of $T\Sigma \oplus N_\Sigma$ is zero. Note that w_2 of a direct sum is the sum of the w_2's from each summand when the summands are oriented. Also, $w_2(N_\Sigma) = c_1(N) \pmod 2$. Here, c_1 is the first Chern class. Finally, $c_1(N) = c_1(\Lambda^d N)$, where $\Lambda^d N \to \Sigma$ is the determinant line bundle of N.

Thus, Σ inherits a spin structure from M if and only if $\Lambda^d N$ admits a square root; that is, if and only if $c_1(\Lambda^d N) = 0 (\operatorname{mod} 2)$. In any case, a $\operatorname{spin}_{\mathbb{C}}$-structure on Σ is defined by the line bundle $\Lambda^d N^*$; one can construct the $\operatorname{spin}_{\mathbb{C}}$ bundle, S_Σ, from $T^*\Sigma \oplus (\Lambda^d N^*)^{-1}$.

Here, a digression concerning $\operatorname{spin}_{\mathbb{C}}$-structures is in order. Let X be a smooth manifold and let $V \to X$ be a real, oriented $2r$-dimensional vector bundle with fiber metric. The bundle of positively oriented, orthonormal frames in V, $\operatorname{Fr} V \to X$ is a principal $SO(2 \cdot r)$ bundle over M. The second Stiefel-Whitney class, $w_2(V)$, is the obstruction to the existence of a principal $\operatorname{Spin}(2 \cdot r)$ bundle, $\operatorname{Fr}' V \to X$ with the property that $\operatorname{Fr} V = \operatorname{Fr} V' / \{\pm 1\}$.

When $w_2(V) = 0$, the spin representation, ϱ, of $\operatorname{Spin}(2 \cdot r)$ on the complex vector space $\Delta \equiv \Lambda^*(\mathbb{C}^r)$ (see [A–B–S]) defines an associated vector bundle over X, the bundle of spinors $S(V) \equiv \operatorname{Fr}'(V) \times_\varrho \Delta \to X$.

Suppose that $w_2(V)$ is the mod(2) reduction of an integral class. Then, there exists a complex line bundle, $L \to X$ whose first Chern class obeys $c_1(L)_{\operatorname{mod}(2)} = w_2(V)$. When such a line bundle exists, a $\operatorname{spin}_{\mathbb{C}}$-bundle from $V \oplus L$ can be constructed.

This construction starts with the observation that the bundle of positively oriented, orthonormal frames in $V \oplus L$ which respect the splitting is a principal $SO(2 \cdot r) \times U(1)$ bundle, $\operatorname{Fr}_s(V \oplus L) \to M$. Introduce the Lie group $\operatorname{Spin}^c(2r) \equiv \operatorname{Spin}(2 \cdot r) \times_{\{\pm 1\}} U(1)$ as in [A–B–S] and introduce the fibration of groups

$$\{\pm 1\} \to \operatorname{Spin}(2 \cdot r) \underset{\{\pm 1\}}{\times} U(1) \to SO(2 \cdot r) \times U(1). \tag{2.16}$$

The condition $w_2(V) = c_1(L)_{\operatorname{mod}(2)}$ is necessary and sufficient for the existence of a principal $\operatorname{Spin}(2 \cdot r) \times_{\{\pm 1\}} U(1)$ bundle $\operatorname{Fr}'_s(V \oplus L) \to X$ with the property that $\operatorname{Fr}'_s(V \oplus L)/\{\pm 1\} = \operatorname{Fr}_s(V \oplus L)$.

The $\operatorname{spin}_{\mathbb{C}}$ bundle constructed from $V \oplus L$ is the complex vector bundle $S(V; L) \equiv \operatorname{Fr}'_s(V \oplus L) \times_{\varrho \otimes i} \Delta \to X$. Here, $i: U(1) \to U(2^r)$ is the center.

In the situation at hand, use $T^*\Sigma \oplus (\Lambda^d N^*)^{-1} \to \Sigma$ to construct the $\operatorname{spin}_{\mathbb{C}}$ bundle, $S_\Sigma \equiv S(T^*\Sigma; (\Lambda^d N^*)^{-1})$. Then, the spin bundle $S(T^*M)|_\Sigma \to \Sigma$, is isomorphic to the tensor product bundle $S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*$.

With this construction understood, let us remark that a similar construction can be made with $\underline{N} \to \Sigma$, the conjugate bundle to N. Since $c_1(\Lambda^d \underline{N}) = -c_1(\Lambda^d N)$, a $\operatorname{spin}_{\mathbb{C}}$-structure on Σ is also defined by the line bundle $\Lambda^d \underline{N}$. The $\operatorname{spin}_{\mathbb{C}}$ bundle from $S(T^*\Sigma; (\Lambda^d \underline{N}^*)^{-1})$ is denoted by $\underline{S}_\Sigma$ and using $\underline{S}_\Sigma$, one then constructs the spin bundle $\underline{S}_\Sigma \otimes_{\mathbb{C}} \Lambda^* \underline{N}^*$. However, there exists a natural, $\mathbb{C}$-linear isomorphism

$\underline{S}_\Sigma\otimes_{\mathbb{C}}\Lambda^*\underline{N}^* \approx S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*$ (complex conjugation gives a $\mathbb{C}$-anti-linear isomorphism) as will now be explained.

Indeed, observe that $\underline{S}_\Sigma \approx S_\Sigma\otimes\Lambda^d N^*$ and $\Lambda^*\underline{N}^*\otimes(\Lambda^d\underline{N}^*)^{-1}\approx\Lambda^*N^*$. To construct this last isomorphism, use the fact that $\Lambda^d\underline{N}^*\wedge\Lambda^d N^*$ $(\approx\Lambda^{2d}N^*_R\otimes\mathbb{C})$ has a canonical section to conclude that $\Lambda^d\underline{N}^*\approx(\Lambda^d N^*)^*\approx(\Lambda^d N^*)^{-1}$. Then, note that the Hodge star provides an isomorphism $\Lambda^p\underline{N}^*\otimes\Lambda^d N^*\approx(\Lambda^{d-p}\underline{N}^*)^*$. Finally, use the hermitian metric to make an isomorphism between $(\Lambda^{d-p}\underline{N}^*)^*$ and $\Lambda^{d-p}N^*$.

Since $\underline{S}_\Sigma\otimes_{\mathbb{C}}\Lambda^*\underline{N}^*$ and $S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*$ are conjugate bundles, the $\mathbb{C}$-linear isomorphism $\underline{S}_\Sigma\otimes_{\mathbb{C}}\Lambda^*\underline{N}^*\approx S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*$ induces a $\mathbb{C}$-antilinear involution, τ, of $S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*$. The involution τ defines a real structure on $S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*$.

Now, through Clifford multiplication, T^*N_Σ becomes a subbundle of $\mathrm{End}(S(T^*N_\Sigma))\to N_\Sigma$ and this is conveniently described using the preceding identification of the spin bundle as $\pi^*(S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*)\to N_\Sigma$. To begin, observe that the complexified tangent space of N_Σ splits using the Riemannian connection as the direct sum

$$\pi^*T\Sigma_{\mathbb{C}}\oplus\pi^*N\oplus\pi^*\underline{N}\,, \tag{2.17}$$

where $\pi^*T\Sigma_{\mathbb{C}}$ is identified with the horizontal subspace $H\subset T(N_\Sigma)_{\mathbb{C}}$ as defined by the Riemannian connection on the normal bundle N_Σ.

The complexified cotangent bundle of N_Σ correspondingly splits as

$$\pi^*T^*\Sigma_{\mathbb{C}}\oplus T^*N\oplus T^*\underline{N}\,, \tag{2.18}$$

where $T^*N\approx\pi^*N^*\to N$ is dual to π^*N in Eq. (2.17) and annihilates the horizontal space H as well as $\pi^*\underline{N}$. The space $T^*\underline{N}$ is likewise dual to $\pi^*\underline{N}$.

With this understood, identify T^*N with π^*N^* and identify $S(T^*N_\Sigma)$ with $\pi^*S_\Sigma\otimes\pi^*\Lambda^*N^*$. Now exterior multiplication embeds π^*N^* as a subbundle of $\mathrm{End}(\pi^*S_\Sigma\otimes\pi^*\Lambda^*N^*)$ and this is Clifford multiplication when viewed in terms of T^*N and $S(T^*N_\Sigma)$.

The Riemannian metric on M induces a hermitian metric on $T^*N_{\Sigma\mathbb{C}}$ for which the splittings in Eqs. (2.17, 18) are orthogonal, and this same hermitian metric identifies $\pi^*\underline{N}^*$ with π^*N. Interior multiplication embedds π^*N in $\mathrm{End}(\pi^*S_\Sigma\otimes\pi^*\Lambda^*N^*)$ and this is the Clifford multiplication embedding of $T^*\underline{N}$ into $\mathrm{End}(S(T^*N_\Sigma))$ after $\pi^*\underline{N}^*$ and $T^*\underline{N}$ are identified. Alternately, use exterior multiplication to embedd $\pi^*\underline{N}$ in $\mathrm{End}(\pi^*\underline{S}_\Sigma\otimes_{\mathbb{C}}\pi^*\Lambda^*\underline{N}^*)$ and then observe that under the $\mathbb{C}$-linear isomorphism $\underline{S}_\Sigma\otimes_{\mathbb{C}}\Lambda^*\underline{N}^*\approx S_\Sigma\otimes_{\mathbb{C}}\Lambda^*N^*$, this embedding and that of π^*N in $\mathrm{End}(\pi^*S_\Sigma\otimes\pi^*\Lambda^*N^*)$ by interior product are the same.

Complex conjugation defines a $\mathbb{C}$-anti-linear isomorphism between π^*N^* and $\pi^*\underline{N}^*$ and hence, with the Riemannian metric, a $\mathbb{C}$-anti-linear isomorphism between π^*N^* and π^*N. With both π^*N^* and π^*N in $\mathrm{End}(\pi^*S_\Sigma\otimes\pi^*\Lambda^*N^*)$, this $\mathbb{C}$-anti-linear isomorphism is induced by the anti-linear automorphism of $\mathrm{End}(\pi^*S_\Sigma\otimes\pi^*\Lambda^*N^*)$ which sends a matrix to its hermitian adjoint.

Let e be a 1-form in $T^*N_{\Sigma\mathbb{C}}$ which is pulled back by π from $T^*\Sigma_{\mathbb{C}}$. To define Clifford multiplication on $\pi^*(S_\Sigma\otimes\Lambda^*N^*)$ by e, introduce the degree operator (Fermion number operator), $(-1)^{F_0}$ on Λ^*N^*. For $\omega\in\Lambda^*N^*$ a p-form, set $(-1)^{F_0}\omega \equiv(-1)^p\cdot\omega$. Then, extend this operator to Λ^*N^* by linearity. Now, define Clifford multiplication by e on $s\otimes\omega$ to yield $(e\cdot s\otimes(-1)^{F_0}\omega)$. Then, Clifford multiplication by e anti-commutes with Clifford multiplication by any elements of π^*N^* and $\pi^*\underline{N}^*$ as required.

To define the normal bundle Dirac operator, $D_t: C^\infty(S(T^*N_\Sigma)) \to C^\infty(S(T^*N_\Sigma))$, it is necessary to take derivatives of spinors. One can distinguish between directional derivatives which are tangent to the fibers of $\pi: N_\Sigma \to \Sigma$, and then, the horizontal derivatives. For $x \in \Sigma$ note that the restriction to a fiber $N_\Sigma|_x$ of $\pi^*(S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*)$ is naturally isomorphic (up to an action of the unitary group $U(d)$) to $S_\Sigma|_x \otimes \Lambda^* \mathbb{C}^{d*}$ and this has an obvious flat connection which allows differentiation along the fiber $N_\Sigma|_x$.

As x varies, this differentiation along the fiber takes a section over N_Σ of $\pi^*(S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*)$ and gives one of

$$\pi^*(S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*) \otimes (T^*N \oplus T^*\underline{N}) \approx \pi^*(S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*) \otimes (\pi^* N^* \oplus \pi^* \underline{N}^*) .$$

Differentiation followed by the Clifford multiplication map from

$$\pi^*(S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*) \otimes (\pi^* N^* \oplus \pi^* \underline{N}^*) \quad \text{to} \quad \pi^*(S_\Sigma \otimes_{\mathbb{C}} \Lambda^* N^*)$$

defines a first order operator, ∂, which restricts to each fiber $N_\Sigma|_x$ as the Dirac operator on $N_\Sigma|_x$.

A derivative ∇ for the horizontal directions is defined as follows: Consider a decomposable section $u = s \otimes \omega : N \to \pi^*(S_\Sigma \otimes \Lambda^* N^*)$. Let $v \in H$ be a horizontal vector. Then set

$$\nabla_v(s \otimes \omega) \equiv (\pi^* \nabla^S)_v s \otimes \omega + s \otimes (\pi^* \nabla^P)_v \omega , \tag{2.19}$$

where $\pi^* \nabla^S$ is the covariant derivative on $\pi^*(S_\Sigma)$ which is defined by the pull back to N of the spin connection on $S_\Sigma \to \Sigma$; and $\pi^* \nabla^P$ is the covariant derivative on $\pi^* \Lambda^* N^*$ which is defined by pull back to N of the Levi-Civita connection on the normal bundle to Σ. Thus ∇ takes a section of $\pi^*(S_\Sigma \otimes \Lambda^* N^*)$ and gives one of $\pi^*(S_\Sigma \otimes \Lambda^* N^*) \otimes \pi^* T^* M$.

Use ∇ followed by the Clifford multiplication map from

$$\pi^*(S_\Sigma \otimes \Lambda^* N^*) \otimes \pi^* T^* M \quad \text{to} \quad \pi^*(S_\Sigma \otimes \Lambda^* N^*)$$

to define the first order operator D_Σ. Then, the normal bundle Dirac operator is

$$D_0 \equiv D_\Sigma + \partial . \tag{2.20}$$

Remember that the Dirac operators in Eq. (2.4) are indexed by a parameter t. There is a corresponding 1-parameter family of normal bundle Dirac operators and its definition requires a closer look at the S^1-action on N_Σ.

Recall that the S^1-action decomposed the complex vector bundle N into a direct sum of complex bundles according to Eq. (2.15) such that on N_υ, the action sends $\lambda \in S^1$ to multiplication by λ^υ. The generator of this S^1 action on N_υ is the restriction to N_υ of the vector field, K_M; it is clearly vertical with respect to the projection $\pi: N_\upsilon \to \Sigma$. Identify the vertical subbundle of the total tangent space to N_υ with $\pi^* N_\upsilon \oplus \pi^* \underline{N}_\upsilon$ and then K_M restricts to N_υ as the section which sends the point ζ to $(-i \cdot \upsilon \cdot J(\zeta), i \cdot \upsilon \cdot J(\underline{\zeta}))$, where $J: N_\upsilon \to \pi^* N_\upsilon$ is the canonical embedding.

The metric dual to K_M is the 1-form K; on N_υ, this 1-form annihilates the horizontal subspace H and so can be identified as a section of $\pi^* N_\upsilon^* \oplus \pi^* \underline{N}_\upsilon^*$. This is the section which sends ζ to $(i \cdot \upsilon \cdot J(\zeta), -i \cdot \upsilon \cdot J^*(\underline{\zeta}))$, where $J^*: N_\upsilon \to \pi^* N_\upsilon^*$ is the $\mathbb{C}$-anti-linear embedding which is canonically defined once the metric is used to provide the $\mathbb{C}$-anti-linear isomorphism between N_υ and N_υ^*. Use Clifford multiplication to embed $\pi^*N_\upsilon^* \oplus \pi^*\underline{N}_\upsilon^*$ in $\mathrm{End}(\pi^*S_\Sigma \otimes_{\mathbb{C}} \pi^*\Lambda^*N)$ so that $\underline{K}$ can define a section over N_Σ of $\mathrm{End}(\pi^*S_\Sigma \otimes_{\mathbb{C}} \pi^*\Lambda^*N)$.

With these constructions understood, introduce a family of Dirac operators on N_Σ by defining, for each t, the first order operator

$$D_t \equiv D_\Sigma + \partial + i \cdot t \cdot \underline{K}. \tag{2.21}$$

This family of operators D_t is of fundamental interest to all that follows.

Before discussing the significance of D_t, some remarks are in order. First, note that D_Σ involves differentiation along horizontal directions in TN_Σ, while $\partial + i \cdot t \cdot \underline{K}$ differentiates along the vertical directions. So, it should not be a complete surprise that the operators D_Σ and $\partial + i \cdot t \cdot \underline{K}$ anti-commute. Thus, one can analyze D_t by first analyzing the operator $\partial + i \cdot t \cdot \underline{K}$. Then, one analyzes the restriction of D_Σ to the eigenspaces of the operator $(\partial + i \cdot t \cdot \underline{K})^2$ with which D_Σ commutes. (This technique is the old physicist's stand-by – separation of variables.)

Now, in practice, the study of $\partial + i \cdot t \cdot \underline{K}$ is simplified many-fold by the observation that, as it differentiates only vertically, $\partial + i \cdot t \cdot \underline{K}$ is determined completely by its restriction to each fiber of $\pi : N_\Sigma \to \Sigma$.

Each such fiber is a copy of $\mathbb{C}^d$, canonically up to the action on $\mathbb{C}^d$ of the unitary group $U(d)$. If one chooses the usual coordinates $(z^1, z^2, \ldots, z^d)$ for $\mathbb{C}^d$, then the S^1 action on the fiber is generated by

$$K_M \equiv -i \cdot \sum_{j=1}^{d} \upsilon(j) \cdot (z^j \cdot \partial/\partial z^j - \underline{z}^j \cdot \partial/\partial \underline{z}^j), \tag{2.22}$$

where each $\upsilon(j)$ a positive integer. (The fiber of N_υ is $\{(z) \in \mathbb{C}^d : z^j \equiv 0 \text{ if } \upsilon(j) \neq \upsilon\}$.

The restriction of $\partial + i \cdot t \cdot \underline{K}$ to $\mathbb{C}^d$ (as the fiber of N_Σ) is the operator

$$\sum_{j=1}^{d} (\Gamma^j \cdot (\partial/\partial \underline{z}^j - \tfrac{1}{2} \cdot t \cdot \upsilon(j) \cdot z^j) + \Gamma^{*j} \cdot (\partial/\partial z^j + \tfrac{1}{2} \cdot t \cdot \upsilon(j) \cdot \underline{z}^j)), \tag{2.23}$$

where Γ^j denotes interior multiplication by $\partial/\partial z^j$ on $\Lambda\mathbb{C}^{d*}$, while Γ^{*j} denotes exterior multiplication by dz^j. It is not hard to find an explicit diagonalization of the operator in Eq. (2.23); this exercise is a useful one which is left to the reader.

As a final remark, the operator K of Eq. (2.5) also involves only vertical differentiations so it too is determined completely by its restriction to $\mathbb{C}^d$ as the fiber of $N_\Sigma \to \Sigma$. This restriction is

$$K = -\sum_j \upsilon(j) \cdot (z^j \partial_j - \underline{z}^j \underline{\partial}_j + \tfrac{1}{2}(\Gamma^{*j} \cdot \Gamma^j - 1)). \tag{2.24}$$

It is straightforward to calculate that the commutator of K with D_t vanishes.

Part 3. The Localization Theorem

As $N_\Sigma \to \Sigma$ is oriented by its complex structure, there is a differential form, ω, on N_Σ of degree $2 \cdot d$ which is uniquely determined by the following two conditions: First, it restricts to each fiber $\mathbb{C}^d$ as the canonical volume element $\prod_{j=1}^{d} (i/2 \cdot dz^j \wedge d\underline{z}^j)$, and second, interior product by horizontal vectors annihilates ω.

Define the L^2-inner product, $\langle \cdot, \cdot \rangle_{L^2}$, on the space of compactly supported, C^∞ sections over N of $\pi^*(S_\Sigma \otimes \Lambda^*N^*)$ by integrating, over N, the pointwise inner product of two sections; use the top dimensional form $\pi^* d\,\mathrm{vol}_\Sigma \wedge \omega$ to define this integration. (Here, $d\,\mathrm{vol}_\Sigma$ is the Riemannian volume element for Σ. If Σ is an isolated point, $d\,\mathrm{vol}_\Sigma = \pm 1$ with $+1$ taken iff the complex orientation on $N_R \approx TM|_\Sigma$ agrees with the induced orientation from M.)

For fixed integer or half-integer k, let $C^\infty_{\mathrm{cpt}}(S,k)$ denote the space of smooth, compactly supported sections of $\pi^*(S_\Sigma \otimes \Lambda^* N^*)$ over N on which the operator K acts with eigenvalue k. Let $L^2(S,k)$ denote the completion of said space with the L^2 norm. Let $H^1(S,k)$ denote the completion $C^\infty_{\mathrm{cpt}}(S,k)$ using the inner product $\langle D_t(\cdot), D_t(\cdot)\rangle_{L^2} + \langle \cdot , \cdot \rangle_{L^2}$. Automatically, D_t defines a bounded operator from $H^1(S,k)$ to $L^2(S,k)$.

Furthermore, one has, as a special case of Proposition A.1 in the Appendix,

Proposition 2.2. *For $t \neq 0$, and for fixed integer or half-integer k, the operator $D_t : H^1(S,k) \to L^2(S,k)$ is Fredholm.*

Since M and so N_Σ is an even dimensional manifold, one can define a character valued index of the operator D_t by restricting γ. An example is $\gamma = (i)^\sigma \cdot d\,\mathrm{vol}_M$, where $d\,\mathrm{vol}_M$ denotes here the image in the Clifford algebra of the volume form on M, and where $\sigma = n \cdot (n+1)/2$ with $n = \dim_{\mathbb{R}}(M)$. Along Σ, the restriction of γ has the following decomposition: Introduce the endomorphism $(-1)^{F_0}$ of $\Lambda^* N^*$ which acts on $\Lambda^p N^*$ as $(-1)^p$. Next, observe that Clifford multiplication by $(i)^{\sigma+d} \cdot d\,\mathrm{vol}_\Sigma$ defines a fiber preserving, covariantly involution, γ_Σ, of S_Σ. Now, the covariantly constant involution γ of Eq. (2.3) restricts to $\pi^*(S_\Sigma \otimes \Lambda^* N^*)$, where it sends $s \otimes \omega$ to $\pi^* \gamma_\Sigma s \otimes (-1)^{F_0} \omega$. Remark that γ anticommutes with the odd elements of the Clifford algebra, and so it anticommutes with both D_Σ and with $\partial + i \cdot t \cdot \underline{K}$, while it commutes with the operator K.

Define the index of D_t on $H^1(S,k)$ to be

$$\mathrm{Ind}(D_t, \gamma, k) \equiv \dim(\ker D_t|_{\ker(\gamma - 1)}) - \dim(\ker D_t|_{\ker(\gamma+1)}) . \tag{2.25}$$

Proposition 2.3. *The number $\mathrm{Ind}(D_t, \gamma, k)$ is a locally constant function of t in $\mathbb{R} \setminus \{0\}$. For fixed, integer $\upsilon > 0$, let $N^*_\upsilon \to \Sigma$ denote the sub-bundle of N^* on which K_M acts with eigenvalue υ, let $\underline{N}^*_\upsilon$ denote the complex conjugate bundle and let $c_1 \equiv \sum_\upsilon \upsilon \cdot \dim_{\mathbb{C}} N_\upsilon$. For complex ζ, define a formal power series in ζ with coefficients in $\mathrm{Vect}(\Sigma)$ by $R(\zeta) \equiv \otimes_\upsilon (\oplus_{m \geq 0} \zeta^{m\upsilon} \cdot \mathrm{Sym}_m(\underline{N}^*_\upsilon))$; and, for fixed k, let $R_k \to \Sigma$ denote the coefficient of ζ^k in the expansion of $\zeta^{1/2 \cdot c_1} \cdot R(\zeta)$.*

(1) *Σ is an isolated point: For $t > 0$ and for $k \geqq 1/2 \cdot c_1$, the index of D_t on $H^1(S,k)$ equals R_k. For $t > 0$, and for $k < 1/2 \cdot c_1$, the index of D_t is zero.*

(2) *Σ has positive dimension: For $t > 0$ and for $k \geqq 1/2 \cdot c_1$, the index of D_t on $H^1(S,k)$ equals the index of the* $\mathrm{spin}_{\mathbb{C}}$*-Dirac operator on Σ when coupled to R_k. For $t > 0$, and for $k < 1/2 \cdot c_1$, the index of D_t is zero.*

(3) *In both cases above, the index of D_t for $t < 0$ is obtained from that of D_t for $t > 0$ after changing k to $-k$ and after changing $\underline{N}^*_\upsilon$ to N^*_υ.*

Proof of Proposition 2.3. Here, one need only look at the Weitzenbock formula for D_t (this appears as a special case of Proposition A.1 and Lemma A.2), and then take t large to reduce the index calculation to an algebraic computation.

It still remains to compare the normal bundle Dirac operator with the Dirac operator ∂_t. To do this, let $\{\Sigma_i\}$ denote the connected components of the fixed point set of the S^1 action. The comparison yields Witten's [W1] interpretation of the Atiyah-Bott-Lefschetz [A–B] formula:

Proposition 2.4. *Let M be a compact, oriented, spin manifold which admits an action of S^1. Let Eq. (2.7) define $\mathrm{Ind}(\partial_0, \gamma, k)$, the character valued index of the Dirac operator. Now, let $\{\Sigma_i\}$ denote the connected components of the fixed point set of the S^1 action. For each Σ_i, construct the normal bundle Dirac operator, $D_t(i)$, as an operator on sections over the normal bundle to Σ_i of the pulled back, bundle of spinors. Then*

$$\mathrm{Ind}(\partial_0, \gamma, k) = \sum_i \mathrm{Ind}(D_t(i), \gamma, k).$$

Proof of Proposition 2.4. For an indirect proof, compute the left-hand side of the equality using the formulas in [A–B] and [A–Se]; and compute the right-hand side of the equality using the Atiyah-Singer index theorem and compare.

Alternately, one can prove the equality directly using the localization theorem, Proposition 2.1. Indeed, due to that proposition, $\mathrm{Ind}(\partial_0, \gamma, k)$ can be computed from the left-hand side of Eq. (2.7) for any value of t. In particular, for $|t|$ sufficiently large, the kernel and cokernel of the ∂_{tk} are supported almost entirely in the tubular neighborhoods of the components of the fixed point set. As $|t|$ gets larger, these neighborhoods get smaller and smaller. Pulled back to the normal bundle of a component, Σ, of the fixed point set, the operators D_t and ∂_t agree to leading order in an expansion in the distance from Σ.

As $|t|$ gets large, the small eigenvalue eigenvectors of D_t on $L^2(S, k)$ are also supported (but for an exponentially small tail) within distance $\mathcal{O}(|t|^{1/2})$ in N of Σ. The proof of this assertion is obtained by mimicking the proof of Proposition 2.1: Use the Weitzenbock formula D_t^2 on N, a special case of the Weitzenbock formula in Eq. (A.10).

Meanwhile, the gaps in the L^2-spectrum of both ∂_{tk} and of D_t are not shrinking with increasing t; again, this is a consequence of the Weitzenbock formulas for these operators. For D_t, the formula is a special case of Eq. (A.10).

The equality of $\mathrm{Ind}(\partial_0, \gamma, k)$ with $\Sigma_i \mathrm{Ind}(D_t(i), \gamma, k)$ follows from these last facts; one can compare the small eigenvalue eigenspaces ∂_0^2 and $\{D_t[i]^2\}$ directly: Use the fact that the eigenvectors are localized near the $\Sigma[i]$, but for an exponential error. One can also view this equality as a consequence of the excision property of the index for elliptic operators (see [A–Si]).

Part 4. Coupling to Vector Bundles

Let $V \to M$ be a complex vector bundle to which the S^1 action on M lifts. When M is a spin manifold, the localization theorem, and the fixed point formula of Propositions 2.1–2.4 generalize to give a formula for the S^1-character valued index of the Dirac operator on $C^\infty(S \otimes V)$.

More generally, when M is not assumed to be a spin manifold, one can consider an oriented, real vector bundle $Y \to M$ with $w_2(Y) = w_2(T^*M)$ and the Dirac operator on the bundle of spinors, $S(U)$, built from the bundle $U \equiv T^*M \oplus Y$. Require of U that a finite cover of the S^1 action on M lifts to an action on U. One can also consider a real, oriented vector bundle $Y \to M$ and a complex line bundle $L \to M$ with the property that the vector bundle $U \equiv T^*M \oplus Y$ obeys $w_2(U) = c_1(L)_{\mathrm{mod}(2)}$. Then, the bundle U has a spin$_{\mathbb{C}}$-structure, and one can consider the

Dirac operator on sections of the spin$_{\mathbb{C}}$ bundle $S(U; L)$. Assume that a finite cover of the S^1 action on M lifts to an action on Y and to an action on L. For notational convenience, set $S^0(U) \equiv S(U)$ or $S(U; L)$.

To discuss localization formulae in these more general contexts, endow V, Y, and L with invariant fiber metrics and invariant, metric-compatible connections. With these connections, the Dirac operator is defined by Eq. (2.2) using the direct product connection on $S^0(U) \otimes V$ to define the covariant derivative. Equation (2.4) defines the family of Dirac operators ∂_t on $C^\infty(S^0(U) \otimes V)$. More generally, let A denote a covariantly constant, self-adjoint endomorphism of V. Assume that Clifford multiplication by T^*M on $S^0(U)$ has been extended to Clifford multiplication on $S^0(U) \otimes V$ in such a way that the extension of A to $S^0(U) \otimes V$ as $1 \otimes A$ anticommutes with multiplication by elements in T^*M. (Section 3 provides examples.) Consider the family of operators

$$\partial_t \equiv \partial_0 + i \cdot t \cdot \underline{K} + A \tag{2.26}$$

on $C^\infty(S^0(U) \otimes V)$.

To consider the S^1 action, the following observations are necessary: Suppose that $E \to M$ is a vector bundle to which a finite cover of the S^1 action has a lift. Give E an invariant metric and an invariant metric compatible connection. The action of the Lie algebra of S^1 on E induces an action on $C^\infty(E)$ whose generator is the first order differential operator

$$K_E \equiv \nabla_K - \sigma_E . \tag{2.27}$$

Here, ∇_K is covariant differentiation along K_M and $\sigma_E \in C^\infty(\operatorname{End} E)$ is a skew-symmetric endomorphism which obeys

$$\nabla \sigma_E = i(K_M) \cdot F \tag{2.28}$$

with $i(K_M) \cdot F \in C^\infty(T^*M \otimes \operatorname{End} E)$ denoting the interior product between K_M and the curvature, $F \in C^\infty(\Lambda^2 T^*M \otimes \operatorname{End} E)$, of the connection on E.

In the present circumstances, $E \equiv S^0(U) \otimes V$. Assume that $K_{S^0(U) \otimes V}$ and A commute as endomorphisms of $C^\infty(S^0(U) \otimes V)$.

To define the S^1 equivariant index of ∂_t, assume that V admits a covariantly constant involution (denoted by θ) which anticommutes with A and which commutes with the S^1-action on V. As the dimension of M is even, $S^0(U) \otimes V$ admits a covariantly constant involution which anti-commutes with D_t; namely $\ell \equiv \gamma \otimes \theta$. With ℓ replacing γ, Eq. (2.3) defines the index of ∂_t.

By virtue of Eq. (2.28), the operators $K_{S^0(U) \otimes V}$ and ∂_t commute; and so the eigenspaces of ∂_t can be decomposed into representations of S^1. Letting $C^\infty(S^0(U) \otimes V, k)$ denote the subspace of $C^\infty(S^0(U) \otimes V)$ on which $K_{S^0(U) \otimes V}$ acts with eigenvalue k, the S^1-character valued index of ∂_t (which is independent of t) is then defined by Eq. (2.7) after replacing γ by ℓ.

This S^1-character valued index can be computed from the local data at the fixed point set; there is localization to the fixed point set of the eigenfunctions of ∂_t as t gets large. As before, this fact is made evident with the Weitzenbock formula for ∂_t^2. The old Weitzenbock formula, Eq. (2.8) is changed somewhat; by the addition of curvature terms coming from Y, L, and V; and by the addition of a term A^*A; and by a change in the term which is linear in t involving $\sigma_{S^0(U) \otimes V}$. But the term which is quadratic in t remains the same, so the localization result of Proposition 2.1 still holds.

The fixed point formula for the S^1-character valued index can be derived by comparing ∂_t with the analog of the operator D_t in Eq. (2.21). To state the formula, the following comments are required: Let $E \to M$ be a vector bundle to which a finite cover of the S^1 action lifts. Again, suppose that E has an invariant metric and metric compatible connection. Upon restriction to a component, Σ, of the fixed point set, the endomorphism σ_E of E in Eq. (2.27) is covariantly constant. This follows from Eq. (2.28).

The eigenvalues of σ_E on the complexification of any given fiber are a set of rational numbers $\{\upsilon\}$; and for fixed eigenvalue υ, the set $E_\upsilon \equiv \ker(\sigma_E - \upsilon) \subset E \otimes \mathbb{C}|_\Sigma$ defines a smooth vector bundle over Σ. If E is real, then the eigenvalues come in $\pm$ pairs and complex conjugation identifies E_υ with $\underline{E}_{-\upsilon}$. In particular, E_0 always has a real structure, and the underlying real bundle will be denoted by E_{0R}. Thus, if E is real, there is an isomorphism of real bundles $E|_\Sigma \approx E_{0R} \oplus_{0<\upsilon} E_\upsilon$.

If E is complex, then σ_E is already diagonalizable on E with eigenvalues $\{\upsilon\}$ and one has $E|_\Sigma \approx \oplus_\upsilon E_\upsilon$.

Let $Y \to M$ be a real, oriented vector bundle to which a finite cover of the S^1 action on M lifts. Suppose that $U \equiv T^*M \oplus Y$ is oriented and spin. Restrict to Σ, and define

$$L_\Sigma \equiv \Big(\bigotimes_{0<\upsilon} \det(N_\upsilon^*) \bigotimes_{0<\upsilon} \det(Y_\upsilon^*) \Big)^{-1}.$$

Or, suppose that $L \to M$ is a complex line bundle to which a finite cover of the S^1 action on M lifts. Suppose that $U \equiv T^*M \oplus Y$ is spin$_\mathbb{C}$ using the line bundle L to define the spin$_\mathbb{C}$-structure. Restrict to Σ and define

$$L_\Sigma \equiv L \otimes \Big(\bigotimes_{0<\upsilon} \det(N_\upsilon^*) \bigotimes_{0<\upsilon} \det(Y_\upsilon^*) \Big)^{-1}.$$

Upon restriction to Σ, one has

$$S^0(U)|_\Sigma \approx S(T^*\Sigma \oplus Y_{0R}; L_\Sigma) \bigotimes_{0<\upsilon} \Lambda^*(N_\upsilon^*) \bigotimes_{0<\upsilon} \Lambda^*(Y_\upsilon^*). \tag{2.29}$$

The new Dirac operator is defined initially on the set of smooth, compactly supported sections of $\pi^*(S^0(U) \otimes V|_\Sigma)$ over the normal bundle $N \to \Sigma$. This new operator is

$$D_t \equiv D_\Sigma + \partial + i \cdot t \cdot \underline{K} + A. \tag{2.30}$$

In analogy with the case where V is trivial, define $C^\infty_{\text{cpt}}(S^0(U) \otimes V|_N, k)$ to be the space of smooth, compactly supported sections of the complexification of $S^0(U) \otimes V|_N$ on which $K_{S^0(U) \otimes V}$ acts with eigenvalue k. Then, define the spaces $L^2(S^0(U) \otimes V|_N, k)$ and $H^1(S^0(U) \otimes V|_N, k)$ as the completions of $C^\infty_{\text{cpt}}(S^0(U) \otimes V|_N, k)$ with the norms which come from the metrics $\langle \cdot, \cdot \rangle_{L^2}$ and $\langle D_t(\cdot), D_t(\cdot) \rangle_{L^2} + \langle \cdot, \cdot \rangle_{L^2}$, respectively. By construction, D_t extends to define a bounded operator from $H^1(S(U) \otimes V|_N, k)$ to the $L^2(S(U) \otimes V|_N, k)$.

Since the endomorphism A commutes with K_V it preserves the decomposition $V = V(0)_R \oplus_{0<\upsilon} V_\upsilon$. Since A is covariantly constant, $V_{\upsilon A} \equiv \ker(A)|V(\upsilon)$ defines a smooth vector bundle over Σ.

The analogs of Propositions 2.2–2.4 in the twisted case follow. They are proved by generalizing in a straightforward way the proofs of Propositions 2.2–2.4; the details are left to the reader.

Proposition 2.5. *Let M be a compact, oriented manifold which admits an isometric S^1 action. Let $Y \to M$ be a real, oriented vector bundle and let $L \to M$ be a complex line bundle. Assume that the S^1 action on M has a finite cover which lifts to Y and to L. Let $U \equiv T^*M \oplus Y$ and assume that $w_2(U)=0$ or else that $w_2(U)=c_1(L)_{\mathrm{mod}(2)}$. Let $S^0(U) \to M$ denote the spin bundle $S(U)$ or the* spin$_{\mathbb{C}}$ *bundle $S(U;L)$. Let Σ be a connected component of the fixed point set of the S^1 action on M, and denote by $N \to \Sigma$ the complex normal bundle. For $t \neq 0$, and eigenvalue k of K on $C^\infty_{\mathrm{cpt}}(S^0(U) \otimes V|_N, k)$, the operator $D_t : H^1(S^0(U) \otimes V|_N, k) \to L^2(S^0(U) \otimes V|_N, k)$ in Eq.* (2.30) *is Fredholm. Let ℓ be an involution of $S^0(U) \otimes V$ with $\ell^2 = 1$. Require that ℓ commute with K, and anti-commute with A and with multiplication by odd elements in the Clifford algebra. Define the index of D_t on $C^\infty(S^0(U) \otimes V|_N, k)$ to be*

$$\mathrm{Ind}(D_t, \ell, k) \equiv \dim(\ker D_t|_{\ker(\ell-1)}) - \dim(\ker D_t|_{\ker(\ell+1)}) .$$

This index is a locally constant function on $\mathbb{R} \backslash \{0\}$.

The analog of Proposition 2.4 in the present context is the twisted version of Witten's interpretation of the Atiyah-Bott formula:

Proposition 2.6. *Make the same assumptions as in Proposition 2.5. Define the character valued index of the Dirac operator ∂_t on $C^\infty(S^0(U) \otimes V)$ as in Eq.* (2.7) *using the involution ℓ. Let $\{\Sigma_i\}$ denote the connected components of the fixed point set of the S^1 action. For each Σ_i, construct the normal bundle Dirac operator, $D_t(i)$, as defined in Eq.* (2.30). *Then*

$$\mathrm{Ind}(\partial_t, \ell, k) = \sum_i \mathrm{Ind}(D_t(i), \ell, k) .$$

3. The Dirac Operator on the Normal Bundle to M

Let M be a compact, oriented Riemannian manifold of dimension n and let $\mathscr{L}M$ denote the space of loops on M. The constant loops give an embedding of M inside $\mathscr{L}M$, and the normal bundle fiber over $x \in M$ is $\mathscr{L}_0 TM|_x \subset \mathrm{Maps}(S^1; TM|_x)$. Dense inside this space is the total space of a real vector bundle

$$\mathscr{N}M \approx \bigoplus_{n>0} TM_{\mathbb{C}} . \tag{3.1}$$

Since $\mathscr{N}M$ has a natural complex structure, consider it as a complex vector bundle over M. See the Appendix for a description of the topology of $\mathscr{N}M$.

To make this isomorphism explicit, restrict attention to an open set U over which TM admits an orthonormal basis, $e \equiv \{e_a\}_{a=1}$. For y in U, a vector in $\mathscr{L}TM|_y$ is some $x(t) \cdot e(y)$ with $x(t) \equiv (x^a(t))^n_{a=1} : S^1 \to \mathbb{R}^n$. Coordinates for $\mathscr{N}M|_U$ are obtained using the Fourier components of $x(\cdot)$. That is, a point in $\mathscr{N}M|_U$ is specified by the data $Y \equiv \{y, x_m\}_{m>0}$, where y is a point in U, where $x_m \equiv (x^a_m)^n_{a=1}$ is a vector in $\mathbb{C}^n$. The point $Y \equiv \{y, x_m\}$ has only finitely many $\{x_m\}$ not zero and it parametrizes the point

$$\sum_{m>0} (x_m e^{-imt} + \underline{x}_m e^{imt}) \cdot e(y) \in \mathscr{L}TM|_U . \tag{3.2}$$

Here, $\underline{x}_m$ is the complex conjugate of x_m. If the orthonormal frame is changed, $e(y) \equiv \lambda(y) \cdot e'(y)$, with $\lambda : U \to SO(n)$, then the normal coordinates change as

$$x'_m \equiv \lambda^T(y) \cdot x_m . \tag{3.3}$$

The horizontal subbundle of $T\mathcal{N}M$ is defined via the isomorphism in Eq. (3.1). Explicitly, this is the kernel of the following set of $\mathbb{C}^n$-valued differential forms on $T\mathcal{N}M$: For $m>0$, set

$$\theta_m \equiv dx_m + \omega \cdot x_m, \qquad \underline{\theta}_m \equiv d\underline{x}_m + \omega \cdot \underline{x}_m, \tag{3.4}$$

where $\omega \equiv (\omega_b^a(y))$ is the Levi-Civita connection matrix of 1-forms. (Note, $\omega_b^a = -\omega_a^b$.) This horizontal bundle is spanned by the vectors $v \equiv \{v_a\}$ where

$$v_a \equiv e_a - \omega_{ba}^c \cdot \sum_{m>0} (x_m^b \cdot \partial/\partial x_m^c + \underline{x}_m^b \cdot \partial/\partial \underline{x}_m^c). \tag{3.5}$$

Thus the tangent space to $\mathcal{N}M|_U$ is spanned by the vectors $\{v, \partial/\partial x_m, \partial/\partial \underline{x}_m\}_{m>0}$. These vectors define an orthonormal basis for $T\mathcal{N}M|_U$, just as $\{e, \theta_m, \underline{\theta}_m\}$ define an orthonormal basis for $T^*\mathcal{N}M|_U$.

A convenient space of functions on $\mathcal{N}M$ is parametrized by the infinite dimensional vector bundle over M,

$$\mathscr{B} \equiv \bigotimes_{n \neq 0} \mathrm{Sym}(T^*M \otimes \mathbb{C}), \tag{3.6}$$

where $\mathrm{Sym}(T^*M \otimes \mathbb{C}) \equiv \oplus_k \mathrm{Sym}_k(T^*M \otimes \mathbb{C})$. (The topology on $\mathscr{B}$ is described in the Appendix.) Indeed, a section of $\oplus_{n \neq 0} T^*M \otimes \mathbb{C}$ defines a function on $\mathcal{N}M$ which is linear ($n>0$) or anti-linear ($n<0$) in the fiber coordinate. More generally, a section of $\mathscr{B}$ defines a function on $\mathcal{N}M$ which is a polynomial in the fiber coordinates and the complex conjugate coordinates.

Note that $\mathscr{B} \to M$ inherits an obvious connection from its representation in Eq. (3.6).

To facilitate calculations, it is convenient to introduce a set of differential operators along the fiber of $\mathcal{N}M$. Alternately, when $\mathcal{N}M$ is viewed as a vector bundle over M, a differential operator along the fiber becomes a subbundle of $\mathrm{End}(\mathscr{B})$. Define, for $m>0$, the physicist's "right moving creation and annihilation operators" on $\mathcal{N}M \to U$ to be

$$a_m^* \equiv i(\partial/\partial \underline{x}_m - t \cdot m \cdot x_m) \quad \text{and} \quad a_m \equiv i(\partial/\partial x_m + t \cdot m \cdot \underline{x}_m), \tag{3.7a}$$

and similarly define the left moving

$$\underline{a}_m^* \equiv i(\partial/\partial x_m - t \cdot m \cdot \underline{x}_m) \quad \text{and} \quad \underline{a}_m \equiv i(\partial/\partial \underline{x}_m + t \cdot m \cdot x_m). \tag{3.7b}$$

The commutation rules for these differential operators are

$$\begin{aligned} a_m^a a_n^{*b} - a_n^{*b} a_m^a &= 2 \cdot t \cdot m \cdot \delta^{ab} \cdot \delta_{mn}, \\ a_m^a a_n^b - a_n^b a_m^a &= a_m^{*a} a_n^{*b} - a_n^{*b} a_m^{*a} = 0, \end{aligned} \tag{3.8}$$

and similarly for the left moving operators. The right moving operators all commute with the left moving operators.

There are additional commutation relations with the horizontal vector fields:

$$\begin{aligned} a_m^a v_b - v_b a_m^a &= -\omega_{ab}^c \cdot a_m^c, \qquad a_m^{*a} v_b - v_b a_m^{*a} = -\omega_{ab}^c \cdot a_m^{*c}, \\ \underline{a}_m^a v_b - v_b \underline{a}_m^a &= -\omega_{ab}^c \cdot \underline{a}_m^c, \qquad \underline{a}_m^{*a} v_b - v_b \underline{a}_m^{*a} = -\omega_{ab}^c \cdot \underline{a}_m^{*c}. \end{aligned} \tag{3.9}$$

To define the correct domain for the Dirac operator on $\mathcal{N}M$, it is necessary to introduce an additional function on $\mathcal{N}M$; this being the "Bosonic" generating functional. It sends the point $Y \equiv (y, x_m)$ to

$$\Phi_0(Y) \equiv \exp\left(-t \cdot \sum_{m>0} m \cdot |x_m|^2\right). \tag{3.10}$$

This is globally defined on M because of the change of variable rule between open sets. Note that the set of operators $\{a_m, \underline{a}_m\}$ annihilate Φ_0.

At $x \in M$, the Bosonic Fock space will be $\underline{\mathscr{B}}|_x \otimes \Phi_0$; it is generated by finite linear combinations of functions of the form

$$\Pi(a_{m(1)}^{a(1)*} \dots a_{m(J)}^{a(J)*}) \cdot \Pi(\underline{a}_{m'(1)}^{a'(1)*} \dots \underline{a}_{m'(J')}^{a'(J')*}) \cdot \Phi_0 . \tag{3.11}$$

When $\mathscr{N}M$ is viewed as a vector bundle over M, then the creation and annihilation operators define a subbundle over M of End$(\underline{\mathscr{B}} \otimes \Phi_0)$. Equation (3.8) describes the commutators between the members of this set of endomorphisms, and Eq. (3.9) describes the commutators with the covariant derivative on $\underline{\mathscr{B}} \otimes \Phi_0$.

Define a metric on $\underline{\mathscr{B}}|_x \otimes \Phi_0$ by requiring that

$$\langle \Phi_0, \Phi_0 \rangle = 1 , \tag{3.12}$$

and that a_m^{a*} be the adjoint of a_m^a and likewise for $\underline{a}_m^{a*}$ and $\underline{a}_m^a$. Equivalently, one can say that the restriction of Φ_0 to each fiber $\mathscr{N}M|_x$ defines a Gaussian measure; the functions in $\underline{\mathscr{B}}|_x$ being measurable, and this inner product is the L^2 inner product with respect to the Gaussian measure.

The family of Fock spaces, $\{\underline{\mathscr{B}}|_x\}$ which are parameterized by M fit together to form a vector bundle over M; $\underline{\mathscr{B}} \to M$.

The bundle of spinors over $\mathscr{N}M$ is defined after introducing the "Fermionic" Fock space at $x \in M$ (see e.g. [W3]). Introduce

$$\begin{aligned} \mathscr{T}_+ &\equiv \text{Finite linear combinations of } \{\theta_m : m > 0\} , \\ \mathscr{T}_- &\equiv \text{Finite linear combinations of } \{\underline{\theta}_m : m > 0\} . \end{aligned} \tag{3.13}$$

The Fermionic Fock space is

$$\underline{\mathscr{F}}|_x \equiv \Lambda^* \mathscr{T}_+ . \tag{3.14}$$

The topology here is described in the Appendix. The vector spaces $\{\underline{\mathscr{F}}|_x : x \in M\}$ fit together to define a vector bundle $\underline{\mathscr{F}} \to M$; with

$$\underline{\mathscr{F}} \approx \bigotimes_{n>0} \Lambda^*(T^*M \otimes \mathbb{C}). \tag{3.15}$$

The space $\underline{\mathscr{F}}|_x$ is a Clifford module for the complex Clifford algebra, Cliff$(\mathscr{T}_+ \oplus \mathscr{T}_-)|_x$. Indeed, for $m>0$, let $\Gamma_m^* \equiv \sqrt{2} \cdot \theta_m$ act on $\underline{\mathscr{F}}|_x$ by wedge product on $\Lambda^* \mathscr{T}_+$. For $m>0$, let $\sqrt{2} \cdot \underline{\theta}_m$ act as interior product by $\sqrt{2} \cdot \partial/\partial x_m$; and denote this endomorphism of $\underline{\mathscr{F}}|_x$ by Γ_m. (In physics lingo, these define the creation operators for the right moving fermions.) The set $\{\Gamma, \Gamma^*\}$ generate a subbundle of End$(\underline{\mathscr{F}})$.

The following anti-commutation relations hold:

$$\begin{aligned} \Gamma_n^{b*} \Gamma_m^a + \Gamma_m^a \Gamma_n^{b*} &\equiv 2 \cdot \delta^{ab} \cdot \delta_{mn} , \\ \Gamma_n^{b*} \Gamma_m^{a*} + \Gamma_m^{a*} \Gamma_n^{b*} &\equiv 0 , \\ \Gamma_n^b \Gamma_m^a + \Gamma_m^a \Gamma_n^b &\equiv 0 , \end{aligned} \tag{3.16}$$

Define the "Fermion number operator", $(-1)^{F_0}$, on $\mathscr{F}$ by declaring it to be equal to $(-1)^p$ on $\Lambda^p\mathscr{T}_+$. Note that $(-1)^{F_0}$ anti-commutes with each of $\{\Gamma, \Gamma^*\}$.

A metric on $\mathscr{F}$ is defined by requiring that

$$\langle 1, 1\rangle = 1\,, \tag{3.17}$$

and that Γ^* be the adjoint of Γ. From Eq. (3.16), $\mathscr{F}$ inherits an obvious connection, which is metric compatible. For computational purposes, it is useful to note that the resulting covariant derivative enjoys the following commutation relations with the set $\{\Gamma, \Gamma^*\}$: Over an open set U where $TM|_U$ has been trivialized,

$$\nabla_a \Gamma_m^b - \Gamma_m^b \nabla_a = -\omega_{ca}^b \Gamma_m^c\,, \qquad \nabla_a \Gamma_m^{b*} - \Gamma_m^{b*} \nabla_a = -\omega_{ca}^b \Gamma_m^{c*}\,. \tag{3.18}$$

Let $\mathrm{Cliff}(T^*M)\to M$ denote the bundle of Clifford algebras over M which is isomorphic as a vector space to Λ^*T^*M. Let $S\to M$ be a complex vector bundle and $\mathrm{Cliff}(T^*M)$ module; that is, there is a bundle map of $\mathrm{Cliff}(T^*M)$ into $\mathrm{End}(S)$ which gives a representation that is faithful on T^*M. For example, if M is a spin manifold, then $S\to M$ could be taken to be the bundle of spinors, $S(T^*M)$ on M.

Assume that S has a metric and a metric compatible connection. Require of the connection that the bundle map from $\mathrm{Cliff}(T^*M)$ into $\mathrm{End}(S)$ be covariantly constant.

Let $\mathscr{S}\equiv(S\otimes\mathscr{F})$. The operator $(-1)^{F_0}$ can be used to extend the Clifford multiplication by allowing covectors $\zeta\in T^*M|_x$ to act on $\mathscr{S}|_x$ according to the rule $\zeta(s\otimes\omega)\equiv(\zeta\cdot s\otimes(-1)^{F_0}\cdot\omega)$. With the action defined in this way, the basis co-vectors $\{e\}$ anti-commute with each of $\{\Gamma, \Gamma^*\}$ and define a Clifford sub-algebra amongst themselves. Use the same notation e to denote the Clifford element which is defined by the covector e.

Define a metric on $\mathscr{S}|_x$ by using the metrics on S and on $\mathscr{F}$.

Define the vector bundles $\mathscr{E}\equiv\mathscr{B}\otimes\mathscr{S}$. Note that $\mathscr{E}\otimes\Phi_0$ inherits a fiber metric; denote it by $\langle\cdot,\cdot\rangle$. The space $C^\infty(\mathscr{E}\otimes\Phi_0)$ of smooth sections over M of $\mathscr{E}\otimes\Phi_0$ now has the L^2 metric

$$\langle\cdot,\cdot\rangle_{L^2}\equiv\int_M\langle\cdot,\cdot\rangle\cdot d\,\mathrm{vol}\,. \tag{3.19}$$

Let $L^2(\mathscr{E}\otimes\Phi_0)$ denote the completion of $C^\infty(\mathscr{E}\otimes\Phi_0)$ with the norm which comes from the metric above.

To define the Dirac-Ramond operator, note that $\mathscr{E}\otimes\Phi_0$ inherits a metric compatible connection from the connections on S, $\mathscr{B}\otimes\Phi_0$ and $\mathscr{F}$. A covariant derivative, ∇, is defined on $C^\infty(\mathscr{E}\otimes\Phi_0)$ from this connection. With respect to a local trivialization of TM over an open set U in M:

$$\begin{aligned}\nabla_a \equiv \nabla_a^S &+ \tfrac{1}{2}\cdot\omega_{ca}^b\cdot\sum_{m>0}\Gamma_m^{b*}\Gamma_m^c\\ &-\omega_{ba}^c\cdot\sum_{m>0}(x_m^b\cdot\partial/\partial x_m^c + \underline{x}_m^b\cdot\partial/\partial\underline{x}_m^c)\,,\end{aligned} \tag{3.20}$$

where ∇^S is the connection on S, and where $\{\omega_{ba}^c\}$ are the components of the connection form with respect to the given frame for $TM|_U$.

The Dirac-Ramond operator on $\mathscr{N}M$ is

$$D_t\equiv e^a\cdot\nabla_a+\sum_{m>0}(\Gamma_m^*\cdot a_m+\Gamma_m\cdot a_m^*)\,. \tag{3.21}$$

Here, e^a denotes Clifford multiplication by the 1-form which is dual to the orthonormal vector e_a. Thus, $e^a \cdot \nabla_a$ is the Dirac operator on M, but coupled to the vector bundle $\mathscr{B} \otimes \mathscr{F}$.

It is an immediate consequence of Eqs. (3.9, 18, 20) that D_t defines an endomorphism of $C^\infty(\mathscr{E} \otimes \Phi_0)$.

Define the Hilbert space $H^1(\mathscr{E} \otimes \Phi_0)$ to be the completion of $C^\infty(\mathscr{E} \otimes \Phi_0)$ in the norm which comes from the metric

$$\langle \cdot , \cdot \rangle_{H^1} \equiv \langle D_t \cdot , D_t \cdot \rangle_{L^2} + \langle \cdot , \cdot \rangle_{L^2} . \tag{3.22}$$

Then, D_t defines a bounded map from $H^1(\mathscr{E} \otimes \Phi_0)$ to $L^2(\mathscr{E} \otimes \Phi_0)$.

Consider now the action of the circle S^1 on $C^\infty(\mathscr{E} \otimes \Phi_0)$. Note that S^1 fixes Φ_0. The action on $\mathscr{B}$ is generated by $i \cdot P_B$, with

$$P_B \equiv - \sum_{m>0} m \cdot (x_m \cdot \partial/\partial x_m - \underline{x}_m \cdot \partial/\partial \underline{x}_m) . \tag{3.23}$$

The S^1 action on $\mathscr{F}$ is generated by $i \cdot P_F$, with

$$P_F \equiv -\tfrac{1}{2} \cdot \sum_{m>0} m \cdot \Gamma_m^* \cdot \Gamma_m . \tag{3.24}$$

This S^1 action fixes the spin bundle S. Thus, S^1 acts on $\mathscr{E} \otimes \Phi_0$, and on $C^\infty(\mathscr{E} \otimes \Phi_0)$ with generator $i \cdot P$ with

$$\begin{aligned} P \equiv P_B + P_F &= - \sum_{m>0} m \cdot (x_m \cdot \partial/\partial x_m - \underline{x}_m \cdot \partial/\partial \underline{x}_m + \tfrac{1}{2} \cdot \Gamma_m^* \cdot \Gamma_m) \\ &= -\tfrac{1}{2} \cdot \sum_{m>0} ((a_m^* \cdot a_m - \underline{a}_m^* \cdot \underline{a}_m) + m \cdot \Gamma_m^* \cdot \Gamma_m) . \end{aligned} \tag{3.25}$$

It will prove convenient to decompose P as $P_L - P_R$, where P_R, P_L are its right and left moving parts:

$$P_R \equiv \tfrac{1}{2} \cdot \sum_{m>0} (a_m^* \cdot a_m + m \cdot \Gamma_m \cdot \Gamma_m^*), \qquad P_L \equiv \tfrac{1}{2} \cdot \sum_{m>0} \underline{a}_m^* \cdot \underline{a}_m . \tag{3.26}$$

Both P_R, P_L define covariantly constant endomorphisms of $\mathscr{E}$; and hence of $C^\infty(\mathscr{E} \otimes \Phi_0)$. The two endomorphisms commute, and as endomorphisms of $C^\infty(\mathscr{E} \otimes \Phi_0)$, both commute with D_t.

Restricted to a fiber $\mathscr{E}|_x$, both P_R and P_L are symmetric, and negative semi-definite. Let $\mathscr{E}_{mh}|_x$ denote the subspace of $\mathscr{E}|_x$ on which P_R acts with eigenvalue $h \geqq 0$ and P_L acts with eigenvalue $m+h \geqq 0$; m being the eigenvalue of P. This is a finite dimensional vector space (see Lemma A.2), and the family of vector spaces $\{\mathscr{E}_{mh}|_x : x \in M\}$ defines a finite dimensional vector bundle, $\mathscr{E}_{mh} \to M$. The bundle $\mathscr{E}_{mh}$ is isomorphic to $S \otimes R_m(h)$, where $R_m(h) \to M$ is the coefficient of $z^{m+h} w^{-h}$ in the following formal power series with coefficients in $\mathrm{Vect}(M)$:

$$\begin{aligned} &\bigotimes_{n \geqq 0} \left(\bigoplus_{i \geqq 0} z^{in} \cdot \mathrm{sym}^i(T^*M) \right) \bigotimes_{n' \geqq 0} \left(\bigoplus_{j \geqq 0} w^{-jn'} \cdot \mathrm{sym}^j(T^*M) \right) \\ &\quad \times \bigotimes_{p \geqq 0} \left(\bigoplus_{k \geqq 0} w^{-kp} \cdot \Lambda^k(T^*M) \right) . \end{aligned} \tag{3.27}$$

The decomposition $\mathscr{E} = \oplus_{h \geqq 0, m \geqq -h} \mathscr{E}_{mh}$ is orthogonal, and compatible with the connection on $\mathscr{E}$. For an eigenvalue m for P on $\mathscr{E}$, let $\mathscr{E}_m \equiv \oplus_{h \geqq -m} \mathscr{E}_{mh}$ denote the eigenspace; an infinite dimensional vector bundle over M.

Since D_t commutes with both P_R and P_L, it decomposes as the direct sum $D_t = \oplus_{h \geqq 0, m \geqq -h} D_{tmh}$, with each $D_{tmh}: C^\infty(\underline{\mathscr{E}}_{mh} \otimes \Phi_0) \to C^\infty(\underline{\mathscr{E}}_{mh} \otimes \Phi_0)$ an operator of the form

$$\partial + A_m(h), \tag{3.28}$$

where ∂ is the Dirac operator coupled to $R_m(h)$, and where $A_m(h)$ is a covariantly constant endomorphism of $R_m(h)$.

Define $L^2(\underline{\mathscr{E}}_m \otimes \Phi_0)$ and $H^1(\underline{\mathscr{E}}_m \otimes \Phi_0)$ accordingly.

The properties of $D_t: H^1(\underline{\mathscr{E}}_m \otimes \Phi_0) \to L^2(\underline{\mathscr{E}}_m \otimes \Phi_0)$ are described in the Appendix; see Propositions A.1 and A.3. The following proposition summarizes:

Proposition 3.1. *Let M be a compact, oriented Riemannian manifold, and let $S \to M$ be a finite dimensional complex vector bundle on which T^*M acts faithfully by Clifford multiplication. Construct the vector bundle $\underline{\mathscr{E}} \otimes \Phi_0 \to M$, and $D_t: C^\infty(\underline{\mathscr{E}} \otimes \Phi_0) \to C^\infty(\underline{\mathscr{E}} \otimes \Phi_0)$ as described above. Let m be an eigenvalue of P on $\underline{\mathscr{E}}$. Then*
(1) *D_t extends to define a Fredholm operator, D_t, from $H^1(\underline{\mathscr{E}}_m \otimes \Phi_0)$ to $L^2(\underline{\mathscr{E}}_m \otimes \Phi_0)$.*
(2) $\operatorname{coker}(D_t) \subset H^1(\underline{\mathscr{E}}_m)$; *and* $\operatorname{coker}(D_t) = \ker(D_t)$.
(3) $\operatorname{coker}(D_t) = \ker(D_t) \subset H^1(\underline{\mathscr{E}}_{m0})$, *the eigenspaces of P_R with $q=0$. In particular, both vector spaces are empty for $m<0$.*

Suppose that the bundle S decomposes as $S_+ \oplus S_-$ which are the ± 1 eigenspaces of a covariantly constant bundle involution, γ, which anti-commutes with the odd elements in the Clifford algebra's action on S. (See Sect. 4 for an example.) Let $\ell \equiv \gamma \otimes (-1)^{F_0}$. This defines an involution of $\underline{\mathscr{E}}_m \otimes \Phi_0$, and hence, one of $L^2(\underline{\mathscr{E}}_m \otimes \Phi_0)$. Also, since ℓ anti-commutes with D_t, it defines an involution of $H^1(\underline{\mathscr{E}}_m \otimes \Phi_0)$.

Define the index of D_{mt} by (see Eq. (A.28))

$$\operatorname{Ind}(D, \underline{\mathscr{E}}_m, \ell) \equiv \dim \ker(D_{mt}|_{\ker(\ell-1)}) - \dim \ker(D_{mt}|_{\ker(\ell+1)}). \tag{3.29}$$

The following proposition is a direct corollary to Proposition 3.1 and Proposition A.3:

Proposition 3.2. *For $m<0$, $\operatorname{Ind}(D, \underline{\mathscr{E}}_m, \ell) = 0$; and for $m \geqq 0$, $\operatorname{Ind}(D, \underline{\mathscr{E}}_m, \ell)$ is equal to the index of the Dirac operator from $C^\infty(S_+ \otimes S_m)$ to $C^\infty(S_- \otimes S_m)$, where $S_m \to M$ is the coefficient of q^m in the following formal power series with coefficient in* Vect$M$:

$$S(q) \equiv \bigotimes_{n>0} \left(\bigoplus_{k \geqq 0} q^{nk} S^k(TM) \right).$$

Here $S^k(TM)$ denotes the k^{th} symmetric power of the tangent bundle to M.

4. Some Twisted Dirac-Ramond Operators

The Dirac-Ramond and Neveu-Schwarz operators are obtained by twisting the operator D_t from the preceding section with specific vector bundles over the loop space. It is convenient to introduce the construction in some generality by using the following generic setting: Let M be a smooth, oriented manifold. Let $E, V \to M$ be real, oriented, finite dimensional vector bundles. Suppose that

$$E \approx E[0]_R \bigoplus_{0<v<r} E[v] \oplus E[r]_R$$

is a real isomorphism which decomposes E into sub-bundles. Require the $E[0]_R = TM$; and that when $0<\upsilon<r$, $E[\upsilon]$ is intrinsically a complex vector bundle. Let $E[0]$ and $E[r]$ denote the complexifications of $E[0]_R$ and $E[r]_R$. Suppose that

$$V \approx V[0]_R \bigoplus_{0<\upsilon<r} V[\upsilon] \oplus V[r]_R$$

is a direct sum decomposition of V into real sub-bundles, with $V[\upsilon]$ naturally complex for $\upsilon \neq 0, r$. An important special case is to let $E \equiv E[0] = TM$ and let V be the zero dimensional vector bundle, $M \times \{0\}$. By convention, direct summing with $M \times \{0\}$ is the identity on Vect(M). Also, tensor product with $M \times \{0\}$ gives $M \times \{0\}$. Setting both $E \equiv TM$ and $V \equiv M \times \{0\}$ will recover the construction in the preceding section. Examples of the general construction are provided in the next two sections.

Choose a metric and a metric compatible connection on E and V which respect the subbundle decomposition and which will induce real metrics on the $\upsilon = 0, r$ subbundles and hermitian metrics on the $\upsilon \neq 0, r$ subbundles.

For each $\upsilon \in \{1, \ldots, r\}$, choose $\alpha(\upsilon) \in [0, 1)$. Define the infinite dimensional vector bundle

$$\mathcal{N}E \equiv \bigoplus_{0<n\in\mathbb{Z}} TM_{n\mathbb{C}} \bigoplus_{0<n\in\mathbb{Z}+\alpha(r)} E[r]_n \oplus \bigoplus_{0<\upsilon<r} \bigoplus_{n\in\mathbb{Z}+\alpha(\upsilon)} E[\upsilon]_n \to M\,. \tag{4.1}$$

Here, the subscript "$_n$" is an indexing label of the bundle in question. This is an infinite dimensional vector bundle over M; see the Appendix for a discussion of the topology on $\mathcal{N}E$.

The physicist's Bosonic Fock space at each $x \in M$ is the space of finite, complex-valued polynomials on the underlying real vector space of $\mathcal{N}E$. This defines a vector bundle over M which is isomorphic to

$$\begin{aligned}\mathscr{B}_E \equiv & \bigotimes_{0 \neq n\in\mathbb{Z}} \mathrm{Sym}(TM^*_{n\mathbb{C}}) \bigotimes_{0<\upsilon<r} \Big\{ \bigotimes_{n\in\mathbb{Z}+\alpha(\upsilon)} \mathrm{Sym}(E[\upsilon]^*_n) \\ & \times \bigotimes_{n\in\mathbb{Z}+\alpha(\upsilon)} \mathrm{Sym}(\underline{E[\upsilon]^*_n}) \Big\} \bigotimes_{n\in\mathbb{Z}+\alpha(r)} \mathrm{Sym}(E[r]^*_n)\,. \end{aligned} \tag{4.2}$$

Here, the complex conjugate bundle is indicated by underlining. Complex conjugation is a $\mathbb{C}$-antilinear isomorphism between $TM^*_{n\mathbb{C}}$ and $TM^*_{-n\mathbb{C}}$, between $E[r]^*_n$ and $E[r]^*_{-n}$, and between $E[\upsilon]^*_n$ and $\underline{E[\upsilon]^*_n}$.

There is a natural, covariantly constant endomorphism of the vector bundle $\mathcal{N}E \to M$ which is generated by

$$\begin{aligned} P_E \equiv & -\sum_{0<n\in\mathbb{Z}} n \cdot (x_n \cdot \partial/\partial x_n - \underline{x}_n \cdot \partial/\partial \underline{x}_n) \\ & - \sum_{0<\upsilon<r} \sum_{n\in\mathbb{Z}+\alpha(\upsilon)} n \cdot (z[\upsilon]_n \cdot \partial/\partial z[\upsilon]_n - \underline{z[\upsilon]_n} \cdot \partial/\partial \underline{z[\upsilon]_n}) \\ & - \sum_{0<n\in\mathbb{Z}+\alpha(r)} n \cdot (z[r]_n \cdot \partial/\partial z[r]_n - \underline{z[r]_n} \cdot \partial/\partial \underline{z[r]_n})\,. \end{aligned} \tag{4.3}$$

Here, $x_n \equiv \{x_n^a\}$ are local coordinates on $TM_{n\mathbb{C}}$ which are defined by the choice of a local orthonormal frame, $\{e\}$, for TM. Likewise, $z[\upsilon]_n \equiv \{z[\upsilon]_n^\alpha\}$ are local coordinates on $E[\upsilon]_n$ which are defined by the choice of a local orthonormal frame for $E[\upsilon]$. Note that P_E lifts to an endomorphism of $\mathscr{B}_E$.

The physicist's Fermionic Fock space bundle over M has left movers and right movers together. To construct this bundle, begin by defining a vector bundle $\mathcal{N}V \to M$. For this purpose, choose $\beta(\upsilon) \in (0,1)$ for each $\upsilon \in \{0, \dots, r\}$. Then,

$$\mathcal{N}V \equiv \bigoplus_{0>n\in\mathbb{Z}+\beta(0)} V[0]_n \bigoplus_{0>n\in\mathbb{Z}+\beta(r)} V[r]_n \bigoplus_{0<\upsilon<r}\ \bigoplus_{n\in\mathbb{Z}+\beta(\upsilon)} V[\upsilon]_n. \tag{4.4}$$

Define the physicist's right moving, Fermionic Fock space bundle

$$\begin{aligned}\underline{\mathcal{F}}_E \equiv \Lambda\Bigg(&\bigoplus_{0<n\in\mathbb{Z}} TM^*_{n\mathbb{C}} \bigoplus_{0<n\in\mathbb{Z}+\alpha(r)} E[r]^*_n \\ &\otimes \bigoplus_{0<\upsilon<r}\Bigg(\bigoplus_{n\in\mathbb{Z}+\alpha(\upsilon)} E[\upsilon]^*_n \bigoplus_{0<n\in\mathbb{Z}-\alpha(\upsilon)} \underline{E[\upsilon]^*_{-n}}\Bigg)\Bigg).\end{aligned} \tag{4.5}$$

Define the physicist's left moving, Fermionic Fock space bundle

$$\begin{aligned}\underline{\mathcal{G}}_V \equiv \Lambda\Bigg(&\bigoplus_{0>n\in\mathbb{Z}+\beta(0)} V[0]^*_n \bigoplus_{0>n\in\mathbb{Z}+\beta(r)} V[r]^*_n \\ &\otimes \bigoplus_{0<\upsilon<r}\Bigg(\bigoplus_{0>n\in\mathbb{Z}+\beta(\upsilon)} V[\upsilon]^*_n \bigoplus_{0>n\in\mathbb{Z}-\beta(\upsilon)} \underline{V[\upsilon]^*_{-n}}\Bigg)\Bigg).\end{aligned} \tag{4.6}$$

The vector bundle $\underline{\mathcal{F}}_E \otimes \underline{\mathcal{G}}_V$ is a complex Clifford module for the bundle of Clifford algebras which is generated by

$$\begin{aligned}&\bigoplus_{0<n\in\mathbb{Z}} TM^*_{n\mathbb{C}} \bigoplus_{0>n\in\mathbb{Z}+\beta(0)} V[0]^*_n \bigoplus_{0<n\in\mathbb{Z}+\alpha(r)} E[r]^*_n \bigoplus_{0>n\in\mathbb{Z}+\beta(r)} V[r]^*_n \\ &\qquad \bigoplus_{0<\upsilon<r}\Bigg(\bigoplus_{0<n\in\mathbb{Z}+\alpha(\upsilon)} E[\upsilon]^*_n \bigoplus_{0<n\in\mathbb{Z}-\alpha(\upsilon)} \underline{E[\upsilon]^*_{-n}}\Bigg) \\ &\qquad \bigoplus_{0<\upsilon<r}\Bigg(\bigoplus_{0>n\in\mathbb{Z}+\beta(\upsilon)} V[\upsilon]^*_n \bigoplus_{0>n\in\mathbb{Z}-\beta(\upsilon)} \underline{V[\upsilon]^*_{-n}}\Bigg).\end{aligned} \tag{4.7}$$

Introduce an orthonormal frame $e \equiv [e^a]$ for TM^* over a ball in M, and for $0 < \upsilon \leqq r$, introduce an orthonormal frame $\zeta(\upsilon) \equiv \{\zeta(\upsilon)^j\}$ for $E[\upsilon]^*$ over the same ball. For $n > 0$, and for $0 < \upsilon < r$, it is convenient to introduce the notation Γ^*_n, $\Gamma(r)^*_n$, $\Gamma(r)^*_n$, $\Gamma(r)^*_n$ for exterior multiplication on the Fock space $\underline{\mathcal{F}}_E \otimes \underline{\mathcal{G}}_V$ by $\sqrt{2} \cdot e \in TM^*_{n\mathbb{C}}$, $\sqrt{2} \cdot \zeta(\upsilon) \in E[\upsilon]^*_n$, $\sqrt{2} \cdot \underline{\zeta(\upsilon)} \in \underline{E[\upsilon]^*_{-n}}$, $\sqrt{2} \cdot \zeta(r) \in E[r]^*_n$, respectively.

Let Γ_n, $\Gamma(\upsilon)_n$, $\underline{\Gamma(\upsilon)}_n$, $\Gamma[r]_n$ denote interior multiplication on the Fock space $\underline{\mathcal{F}}_E \otimes \underline{\mathcal{G}}_V$ with $\sqrt{2} \cdot e \in TM^*_{-n\mathbb{C}}$, $\sqrt{2} \cdot \zeta(\upsilon) \in E[\upsilon]^*_n$, $\sqrt{2} \cdot \underline{\zeta(\upsilon)} \in \underline{E[\upsilon]^*_{-n}}$, $\sqrt{2} \cdot \zeta(r) \in E[r]^*_{-n}$, respectively.

The Γ^*'s are the Fock space creation operators, and the Γ's are the Fock space annihilation operators.

The metric on $V \to M$ induces metrics on $V[0]^*_R$ and $V[r]^*_R$; and it induces hermitian metrics on each $V[\upsilon]^*$ for $0 < \upsilon < r$. Choose orthonormal frames $o(0) \equiv \{o(0)^A\}$, $o(r) \equiv \{o(r)^A\}$, and $o(\upsilon) \equiv \{o(\upsilon)^A\}$ for $V[0]^*_R$ and $V[r]^*_R$ and $V[\upsilon]^*$, respectively. For $n < 0$, exterior multiplication on the Fock space $\underline{\mathcal{F}}_E \otimes \underline{\mathcal{G}}_V$ by $\sqrt{2} \cdot o(0) \in V[0]^*_n$, $\sqrt{2} \cdot o(\upsilon) \in V[\upsilon]^*_n$, $\sqrt{2} \cdot \underline{o(\upsilon)} \in \underline{V[\upsilon]^*_{-n}}$, $\sqrt{2} \cdot o(r) \in V[r]^*_n$ defines the creation operators $\Theta(0)^*_n$, $\Theta(\upsilon)^*_n$, $\underline{\Theta(\upsilon)}^*_n$, $\Theta(r)^*_n$.

For $n < 0$, interior multiplication on the Fock space $\underline{\mathcal{F}}_E \otimes \underline{\mathcal{G}}_V$ by $\sqrt{2} \cdot o(0) \in V[0]^*_{-n}$, $\sqrt{2} \cdot o(\upsilon) \in V[\upsilon]^*_n$, $\sqrt{2} \cdot \underline{o(\upsilon)} \in \underline{V[\upsilon]^*_{-n}}$, $\sqrt{2} \cdot o(r) \in V[r]^*_{-n}$ defines the annihilation operators $\Theta(0)_n$, $\Theta(\upsilon)_n$, $\underline{\Theta(\upsilon)}_n$, $\Theta(r)_n$.

The endomorphism P_E has its analog on the Fock space $\underline{\mathscr{F}}_E \otimes \underline{\mathscr{G}}_V$. This is the endomorphism given by $P_F + P_V$. Here,

$$P_F \equiv -\tfrac{1}{2} \cdot \Bigg(\sum_{0<n\in\mathbb{Z}} n \cdot \Gamma_n^* \cdot \Gamma_n + \sum_{0<n\in\mathbb{Z}+\alpha(r)} n \cdot \Gamma(r)_n^* \cdot \Gamma(r)_n$$
$$+ \sum_{0<\upsilon<r} \sum_{0<n\in\mathbb{Z}+\alpha(r)} n \cdot \Gamma(\upsilon)_n^* \cdot \Gamma(\upsilon)_n$$
$$+ \sum_{0<\upsilon<r} \sum_{0<n\in\mathbb{Z}-\alpha(r)} n \cdot \underline{\Gamma(\upsilon)}_n^* \cdot \underline{\Gamma(\upsilon)}_n \Bigg). \tag{4.8}$$

And

$$P_V \equiv -\tfrac{1}{2} \cdot \Bigg(\sum_{0>n\in\mathbb{Z}+\beta(0)} n \cdot \Theta(0)_n^* \cdot \Theta(0)_n + \sum_{0>n\in\mathbb{Z}+\alpha(r)} n \cdot \Theta(r)_n^* \cdot \Theta(r)_n$$
$$+ \sum_{0<\upsilon<r} \sum_{0>n\in\mathbb{Z}+\beta(\upsilon)} n \cdot \Theta(\upsilon)_n^* \cdot \Theta(\upsilon)_n$$
$$+ \sum_{0<\upsilon<n_0/2} \sum_{0>n\in\mathbb{Z}-\beta(\upsilon)} n \cdot \underline{\Theta(\upsilon)}_n^* \cdot \underline{\Theta(\upsilon)}_n \Bigg). \tag{4.9}$$

No spin manifold assumption about M has been made yet; but now, a spinor bundle over M must be constructed. Let $Y \to M$ be an oriented, real vector bundle. Let $L \to M$ be a complex line bundle. Assume that $w_2(Y) = w_2(T^*M)$, or that $w_2(T^*M \oplus Y) = c_1(L)_{\mathrm{mod}(2)}$. Give Y and L metrics and metric compatible connections. Let

$$U \equiv TM^* \oplus Y. \tag{4.10}$$

If $w_2(Y) \equiv w_2(T^*M)$, then U is spin; and if $w_2(T^*M \oplus Y) = c_1(L)_{\mathrm{mod}(2)}$, then U has a $\mathrm{spin}_{\mathbb{C}}$ structure which is defined by the line bundle L.

Let $S^0(U) \to M$ denote the spin bundle $S(U)$ or the $\mathrm{spin}_{\mathbb{C}}$-bundle $S(U; L)$ as the case may be. Note that $S^0(U)$ is a Clifford module for the bundle of Clifford algebras over M which is generated by TM^*. Also, $S^0(U)$ is a Clifford module for the bundle of Clifford algebras generated by Y. Clifford multiplication by TM^* anti-commutes with that by Y. Clifford multiplication by $e \in TM^* \oplus Y$ will be denoted by "e".

The total Fermionic Fock space bundle over M is defined to be

$$S^0(U) \otimes \underline{\mathscr{F}}_E \otimes \underline{\mathscr{G}}_V. \tag{4.11}$$

Clifford multiplication by the Γ's, Γ^*'s, and Θ's and Θ^*'s extends to the vector bundles in Eq. (4.11) directly. To define the Clifford module structure over TM^* and Y^*, introduce the automorphism, $(-1)^{F+G}$: On forms of homogeneous degree in $\underline{\mathscr{F}}_E \otimes \underline{\mathscr{G}}_V$, this is $(-1)^{\mathrm{degree}}$. It extends to an automorphism of $\underline{\mathscr{F}}_E \otimes \underline{\mathscr{G}}_V$. For $e \in TM^* \oplus Y$, Clifford multiplication on a decomposable element $s \otimes \psi \in S^0(U) \otimes \underline{\mathscr{F}}_E \otimes \underline{\mathscr{G}}_V$ gives $e \cdot s \otimes (-1)^{F+G} \cdot \psi$.

An additional necessity for the Dirac-Ramond construction is the density function on $\mathscr{N}E$. This is defined after choosing for each $0 < \upsilon < r$ and $n \in \mathbb{Z} + \alpha(\upsilon)$, a real number $\mu[\upsilon, n] \in \mathbb{R} \setminus \{0\}$. Assume that $\mathrm{sign}(\mu[\upsilon, n]) = \mathrm{sign}(n)$ for all but finitely many n.

Let

$$\Phi_E \equiv \exp\Bigg(-t \cdot \Bigg(\sum_{0<n\in\mathbb{Z}} n \cdot |x_n|^2 + \sum_{0<\upsilon<r} \sum_{n\in\mathbb{Z}+\alpha(\upsilon)}$$
$$\times |\mu[\upsilon, n]| \cdot |z(\upsilon)_n|^2 + \sum_{0<n\in\mathbb{Z}+\alpha(\upsilon)} n \cdot |z(r)_n|^2 \Bigg)\Bigg). \tag{4.12}$$

For later applications, let $W \to M$ denote a complex vector bundle with metric and metric compatible connection.

The domain for the Dirac-Raymond operator will be the space of smooth sections over M of $\mathscr{E}_{E,V} \otimes \Phi_E$, where

$$\mathscr{E}_{E,V} \equiv \mathscr{B}_E \otimes S^0(U) \otimes \mathscr{F}_E \otimes \mathscr{G}_V \otimes W. \tag{4.13}$$

To define inner products on the domain, observe that the bundle $S^0(U)$ has a fiber metric which is induced by the metric on T^*M, E, Y, and on V. Also, $\mathscr{F}_E \otimes \mathscr{G}_V$ has a natural fiber metric which is induced by the metrics on T^*M, E and on V; the constant 1 is declared to have unit length, and then, the Γ's are declared adjoint to the respective Γ^*'s, and likewise for the Θ's and Θ^*'s.

The connections on TM^* and E and on V induce connections on $S(U)$, and $\mathscr{F}_E \otimes \mathscr{G}_V$ which are metric compatible.

The Bosonic Fock space $\mathscr{B}_E \to M$ has a natural fiber metric which is given by the Gaussian measure which Φ_E defines on each fiber. Alternately, one could introduce the creation and annihilation operators as in the Appendix. A connection on $\mathscr{B}_E$ which is metric compatible is inherited from the connections on TM^* and on E.

Let $\langle\, ,\rangle$ denote the induced metric on $\mathscr{E}_{E,V}$, and let ∇ denote the induced covariant derivative on $C^\infty(\mathscr{E}_{E,V} \otimes \Phi_E)$.

In local coordinates, let $e^a \cdot \nabla_a : C^\infty(\mathscr{E}_{E,V} \otimes \Phi_E) \to C^\infty(\mathscr{E}_{E,V} \otimes \Phi_E)$ denote the usual Dirac operator. Then, the Dirac-Ramond operator is

$$D_t \equiv e^a \cdot \nabla_a + T_E, \tag{4.14a}$$

where T_E is the following covariantly constant endomorphism of $\mathscr{E}_{E,V} \otimes \Phi_E$:

$$\begin{aligned} T_E \equiv i \cdot \Bigg[& \sum_{0<n\in\mathbb{Z}} (\Gamma_n \cdot (\partial/\partial \underline{x}_n - t \cdot n \cdot x_n) + \Gamma_n^* \cdot (\partial/\partial x_n + t \cdot n \cdot \underline{x}_n)) \\ & + \sum_{0<\upsilon<r} \sum_{0 \leq n \in \mathbb{Z} + \alpha(\upsilon)} (\Gamma(\upsilon)_n \cdot (\partial/\partial \underline{z(\upsilon)}_n - t \cdot \mu[\upsilon, n] \cdot z(\upsilon)_n) \\ & + \Gamma(\upsilon)_n^* \cdot (\partial/\partial z(\upsilon)_n + t \cdot \mu[\upsilon, n] \cdot \underline{z(\upsilon)}_n)) \\ & + \sum_{0<\upsilon<r} \sum_{0<n\in\mathbb{Z}-\alpha(\upsilon)} (\underline{\Gamma(\upsilon)}_n \cdot (\partial/\partial z(\upsilon)_{-n} - t \cdot \mu[\upsilon, -n] \cdot z(\upsilon)_{-n}) \\ & + \Gamma(\upsilon)_n^* \cdot (\partial/\partial z(\upsilon)_{-n} + t \cdot \mu[\upsilon, -n] \cdot z(\upsilon)_{-n}) \\ & + \sum_{0<n\in\mathbb{Z}+\alpha(r)} (\Gamma(r)_n \cdot (\partial/\partial \underline{z(r)}_n - t \cdot n \cdot z(r)_n) \\ & + \Gamma(r)_n^* \cdot (\partial/\partial z(r)_n + t \cdot n \cdot \underline{z(r)}_n))\Bigg]. \end{aligned} \tag{4.14b}$$

By construction, D_t maps $C^\infty(\mathscr{E}_{E,V} \otimes \Phi_E)$ into itself.

With these fiber metrics and connections, the endomorphisms P_E, P_F, and P_V of $\mathscr{E}_{E,V}$ are covariantly constant and symmetric. Furthermore, $P \equiv P_E + P_F + P_V$ commutes with T_E as an endomorphism of $\mathscr{E}_{E,V} \otimes \Phi_E$. As an operator on $C^\infty(\mathscr{E}_{E,V} \otimes \Phi_E)$, it commutes with D_t. The endomorphism P can be diagonalized explicitly on $\mathscr{E}_{E,V}$ and P decomposes $\mathscr{E}_{E,V}$ as the direct sum $\oplus_m \mathscr{E}_{E,Vm}$, where P acts on $\mathscr{E}_{E,Vm}$ as multiplication by m. Then, D_t restricts to an operator on $C^\infty(\mathscr{E}_{E,Vm} \otimes \Phi_E)$.

Introduce the L^2-norm and the H^1-norm on sections of $\mathscr{E}_{E,Vm} \otimes \Phi_E$;

$$\langle\, ,\rangle_{L^2} \equiv \int_M \langle\, ,\rangle \quad \text{and} \quad \langle\, ,\rangle_{H^1} \equiv \langle D_t(\,), D_t(\,)\rangle_{L^2} + \langle\, ,\rangle_{L^2}. \tag{4.15}$$

Introduce the Hilbert spaces $L^2(\mathscr{E}_{E,Vm}\otimes\Phi_E)$ and $H^1(\mathscr{E}_{E,Vm}\otimes\Phi_E)$ as the completions of $C^\infty(\mathscr{E}_{E,Vm}\otimes\Phi_E)$ in the L^2 and H^1-norms above, respectively.

The following proposition is a reassertion of Proposition A.1:

Proposition 4.1. *In the preceding construction, assume that* $\alpha(\upsilon)>0$ *for all* $\upsilon>0$. *Choose* $\mu[\upsilon,n]\equiv n$. *Construct* $\mathscr{E}_{E,V}$, Φ_E *and, for* $t>0$, *the operator* D_t. *Then* D_t *defines a Fredholm operator from* $H^1(\mathscr{E}_{E,Vm}\otimes\Phi_E)$ *and* $L^2(\mathscr{E}_{E,Vm}\otimes\Phi_E)$. *The kernel and cokernel of* D_t *vanish if* $m<0$, *and in general, define the identical vector space of smooth sections over* M *of the finite dimensional vector bundle* $\mathrm{Ker}(T_E)\cap\mathscr{E}_{E,Vm}\otimes\Phi_E$.

The index of D_t on $C^\infty(\mathscr{E}_{E,Vm}\otimes\Phi_{n_0})$ is defined using a covariantly constant involution ℓ, of $S^0(U)\otimes\mathscr{F}_E\otimes\mathscr{G}_V$. The involution ℓ is required to anti-commute with D_t. Define D_{tm} to be the restriction of D_t to $C^\infty(\mathscr{E}_{E,Vm}\otimes\Phi_E)$. Set

$$\mathrm{Ind}(D;\mathscr{E}_{E,Vm},\ell)\equiv\dim(\ker(D_{tm}|_{\ker(\ell-1)}))-\dim(\ker(D_{tm}|_{\ker(\ell+1)}))\,. \tag{4.16}$$

Proposition 4.2. *Make the same assumptions as in Proposition 4.1. Let* ℓ *be a covariantly constant involution of* $S^0(U)\otimes\mathscr{F}_E\otimes\mathscr{G}_V$ *which anti-commutes with* D_t. *Define* $\mathrm{Ind}(D;\mathscr{E}_{E,Vm},\ell)$ *by Eq.* (4.16). *Then this index is independent of* t; *and it vanishes for* $m<0$.

Proof of Proposition 4.2. The proposition follows from Propositions 4.1 and A.3.

When M and Y are even dimensional, there are two such involutions, ℓ_e and ℓ_s. The first, ℓ_e, gives an index of D which is a generalization of the Euler characteristic. The second, ℓ_s, gives an index of D which is a loop space generalization of the index of the signature operator or the Dirac operator.

To define ℓ_e, start by defining $\varepsilon_0\equiv\dim(TM\oplus Y)$. Since TM^*, Y^* are all oriented, $\det(TM^*\oplus Y^*)$ has a covariantly constant, unit norm section, ω_e, which defines the orientation of $TM^*\oplus Y^*$ along M. The image of ω_e in the Clifford algebra defines an automorphism, ω_e, which anti-commutes with the operator $e^a\nabla_a$ in Eq. (4.14a). This automorphism has square $\omega_e^2=(-1)^{\varepsilon_0(\varepsilon_0+1)/2}$, so $\gamma_e\equiv(i)^{\varepsilon_0(\varepsilon_0+1)/2}\cdot\omega_e$ defines an involution of $S^0(U)$. Define the automorphism ℓ_e on $S^0(U)\otimes\mathscr{F}_E\otimes\mathscr{G}_V$ by first considering it on decomposable elements of the type $s\otimes\omega$, with $s\in S^0(U)$ and with ω of homogeneous degree in $\mathscr{F}_E\otimes\mathscr{G}_V$. Require that

$$\ell_e(s\otimes\omega)\equiv\gamma_e\cdot s\otimes(-1)^{\mathrm{degree}(\omega)}\cdot\omega\,. \tag{4.17}$$

Then, extend the definition of ℓ_e by linearity. Extend ℓ_e to $S^0(U)\otimes\mathscr{F}_E\otimes\mathscr{G}_V\otimes W$ by ignoring W.

To define ℓ_s, use the orientation of TM and the metric to define the volume form. When M is even dimensional, Clifford multiplication by the volume form on M defines an automorphism, $d\,\mathrm{vol}_M$, of $S^0(U)$ with $d\,\mathrm{vol}_M^2=(-1)^{\dim(M)(\dim(M)+1)/2}$; thus, $\gamma_M\equiv(i)^{\dim(M)(\dim(M)+1)/2}\cdot d\,\mathrm{vol}_M$ defines an involution of $S^0(U)$. Define the automorphism ℓ_s on $S^0(U)\otimes\mathscr{F}_E\otimes\mathscr{G}_V$ by first considering it on decomposable elements of the type $s\otimes\omega\otimes\varphi$, with $s\in S^0(U)$ and with ω of homogeneous degree in $\mathscr{F}_E$; and with φ arbitrary in $\mathscr{G}_V$. Require that

$$\ell_s(s\otimes\omega\otimes\varphi)\equiv\gamma_M\cdot s\otimes(-1)^{\mathrm{degree}(\omega)}\cdot\omega\otimes\varphi\,. \tag{4.18}$$

Then, extend the definition of ℓ_s by linearity. When tensoring with the auxiliary bundle W, ignore W.

5. A Relevant Example

A geometric example of the constructions in the preceding section arises in the following way: Suppose that S^1 acts as a group of isometries of an oriented Riemannian manifold, M. For each $p \in M$, introduce the subgroup $G(p) \subset S^1$ which stabilizes p. The S^1 action is called semi-free when the stabilizer of a point in M is either $\{1\}$ or else it is S^1. Generally, the set of distinct subgroups of S^1 which appear as stabilizers of the points in M is some list from the set $\{\mathbb{Z}/n \cdot \mathbb{Z} : n \in \{1, 2, \ldots, \infty\}\}$, with $\mathbb{Z}/\infty \cdot \mathbb{Z} \equiv S^1$.

Each $n \in \{1, 2, \ldots, \infty\}$ defines a (possibly empty) subset $M(n) \subset M$ as the set of points p for which $\mathbb{Z}/n \cdot \mathbb{Z} \subset G(p)$; i.e., the set of points which are fixed by $\mathbb{Z}/n \cdot \mathbb{Z}$. For example, $M(1) = M$ and $M(\infty)$ is the fixed point set of the S^1-action. Note that if n divides n', then $\mathbb{Z}/n \cdot \mathbb{Z}$ is a subgroup of $\mathbb{Z}/n' \cdot \mathbb{Z}$ and $M(n') \subset M(n)$.

For $n_0 > 0$, the normal bundle, $NM(n_0) \to M(n_0)$ inherits a covariantly constant $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of its automorphism group. This induces a character decomposition of the complexified bundle,

$$NM(n)_{\mathbb{C}} = \bigoplus_{0 < v < n_0} NM(n_0; v). \tag{5.1}$$

Complex conjugation provides a $\mathbb{C}$-anti-linear isomorphism between $NM(n_0; v)$ and $NM(n_0; n_0 - v)$. If n is even, this produces a real structure on $NM(n_0; n_0/2)$; this bundle is the complexification of a real bundle $NM(n_0; n_0/2)_R \to M(n_0)$. Thus, $NM(n_0)$ is isomorphic as a real bundle to

$$NM(n_0) \approx NM(n_0; n_0/2)_R \bigoplus_{0 < v < n_0/2} NM(n_0; v). \tag{5.2}$$

When $n = 1$, then $NM(1) \equiv M \times \{0\}$ is the special case of the zero dimensional bundle.

The manifold $M(n_0)$ will replace the manifold M in the constructions of the preceding section. For this to proceed, it is necessary to insure the orientability of $TM(n_0)$.

Lemma 5.1. *Let M be a compact, oriented, manifold on which S^1 acts. Let $V \to M$ be a real, oriented vector bundle to which the S^1 action lifts. Assume that $w_2(V) = 0$. Let $M(n_0) \subset M$ be the fixed point set of the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of S^1. Let $V(n_0; 0)_R \to M(n_0)$ denote the subbundle of $V|_{M(n_0)}$ on which the induced $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of automorphisms acts trivially. Then $V(n_0; 0)_R$ is orientable.*

This lemma is a corollary to the main theorem in [E]; another proof is given in Sect. 10 of [B–T], and a third proof is given at the end of this section.

To motivate the construction of the preceding section, define the fiber bundle over S^1 with fiber $NM(n_0)$,

$$S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} NM(n_0). \tag{5.3}$$

Witten's discussion in [W2] suggests that one should replace the space of loops on $NM(n_0)$ with the space of sections over S^1 of the fiber bundle in Eq. (5.3). This space is naturally a fiber bundle over $\mathscr{L}M(n_0)$ whose fiber at the loop $\varphi \in \mathscr{L}M(n_0)$ is the space of smooth sections over S^1 of $S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} \varphi^* NM(n_0)$.

The constant sections of $S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} NM(n_0)$ can be identified with the point loops, $M(n_0) \subset \mathscr{L}M(n_0) \subset C^\infty(S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} NM(n_0))$. The normal bundle to $M(n_0) \subset C^\infty(S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} NM(n_0))$ has a dense subbundle which is the underlying real bundle of

$$\mathscr{N} NM(n_0) \equiv \bigoplus_{0<n\in\mathbb{Z}} TM(n_0)_{n\mathbb{C}} \oplus \Big(\bigoplus_{0<\upsilon<n_0/2} \bigoplus_{n\in\mathbb{Z}+\upsilon/n_0} NM(n_0;\upsilon)_n \Big) \times \bigoplus_{0<n\in\mathbb{Z}+1/2} NM(n_0;n_0/2)_n . \tag{5.4}$$

Here, the last term is understood to be absent when n_0 is odd.

The relationship between $\mathscr{N}NM(n_0)$ and $C^\infty(S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} NM(n_0))$ is obtained by considering Fourier components as in Sect. 3. Indeed, the metric on TM induces a metric on $NM(n_0; n_0/2)$ and for $0<\upsilon<n_0/2$, a hermitian metric on $NM(n_0 \colon \upsilon)$. Choose an orthonormal frame $e \equiv \{e_a\}$ for $TM(n_0)$ at $x \in M$; and for $0<\upsilon \leqq n_0/2$, choose an orthonormal frame $\zeta(\upsilon) \equiv \{\zeta(\upsilon)_j\}$ for $NM(n_0;\upsilon)$. Then a point $Y=(x_n, z(\upsilon)_n)$ in $\mathscr{N}NM(n_0)$ as defined in Eq. (5.4) specifies

$$\begin{aligned}\varphi(\theta) \equiv \Big(& \sum_{0<n\in\mathbb{Z}} (x_n \cdot e^{-in\theta} + \underline{x}_n \cdot e^{in\theta}) \cdot e, \sum_{0<\upsilon<n_0/2} \sum_{n\in\mathbb{Z}+\upsilon/n_0} \\ & \times (z(\upsilon)_n \cdot e^{-in\theta} \cdot \zeta(\upsilon) + \underline{z(\upsilon)_n} \cdot e^{in\theta} \cdot \underline{\zeta}(\upsilon)) + \sum_{0<n\in\mathbb{Z}+1/2} \\ & \times z(n_0/2)_n \cdot e^{-in\theta} + \underline{z(n_0/2)_n} \cdot e^{in\theta}) \cdot \zeta(n_0/2) \Big)\end{aligned} \tag{5.5}$$

as a map from $[0,2\pi]$ into $TM|_x$. Composing with the exponential map gives a section over S^1 of $S^1 \times_{\mathbb{Z}/n_0 \cdot \mathbb{Z}} NM(n_0)$.

It is natural to make the constructions in the previous section using

$$E \equiv TM = TM(n_0) \oplus NM(n_0/n_0/2) \bigoplus_{0<\upsilon<n_0/2} NM(n_0;\upsilon);$$

and using $\alpha(\upsilon) \equiv \upsilon/n_0$.

Let $V \to M$ be a real, oriented vector bundle to which the S^1 action on M lifts. Endow V with an invariant metric and a metric compatible connection. The vector bundle V along $M(n)$ inherits a covariantly constant, $\mathbb{Z}/n \cdot \mathbb{Z}$-subgroup of its automorphism group which decomposes V into its characters under the $\mathbb{Z}/n \cdot \mathbb{Z}$ action on M according to $V \otimes \mathbb{C} \equiv \oplus_{0 \leqq \upsilon < n_0} V(n_0;\upsilon)$. For $\upsilon \neq 0$, complex conjugation gives a $\mathbb{C}$-anti-linear isomorphism between $\overline{V(n_0;\upsilon)}$ and $V(n_0; n_0-\upsilon)$. Complex conjugation gives a real structure to $V(n_0;0)$ and, when n_0 is even, to $V(n_0;n_0/2)$. As a real bundle,

$$V|_{M(n_0)} \approx V(n_0;0)_R \bigoplus_{0<\upsilon<n_0/2} V(n_0;\upsilon) \oplus V(n_0;n_0/2)_R; \tag{5.6}$$

where the last term is understood to be trivial when n_0 is odd.

The construction of $\mathscr{N}V$ in Eq. (4.4) from the vector bundle V in Eq. (5.6) can be done in two ways, unprimed and primed. In the unprimed case, denote $\mathscr{N}V \equiv \mathscr{V}$, and in the primed case, denote $\mathscr{N}V \equiv \mathscr{V}'$. The two cases correspond to distinct choices of the data $\{\beta(\upsilon)\}$: In the unprimed case, $\beta(\upsilon) \equiv \upsilon/n_0$; and in the primed case, $\beta(\upsilon) \equiv \upsilon/n_0 + 1/2$. Thus,

$$\mathscr{V} \approx V(n_0;0)_{0R} \bigoplus_{0>n\in\mathbb{Z}} V(n_0;0)_n \oplus \Big(\bigoplus_{0<\upsilon<n_0/2} \bigoplus_{n\in\mathbb{Z}+\upsilon/n_0} V(n_0;\upsilon)_n \Big) \bigoplus_{0>n\in\mathbb{Z}+1/2} V(n_0;n_0/2)_n, \tag{5.7}$$

$$\mathscr{V}' \approx V(n_0;n_0/2)_{0R} \bigoplus_{0>n\in\mathbb{Z}} V(n_0;n_0/2)_n \oplus \Big(\bigoplus_{0<\upsilon<n_0/2} \bigoplus_{n\in\mathbb{Z}+\upsilon/n_0+1/2} V(n_0;\upsilon)_n \Big) \bigoplus_{0>n\in\mathbb{Z}+1/2} V(n_0;0)_n. \tag{5.8}$$

In the physics literature, the unprimed case is called Ramond, and the primed case is called Neveu-Schwarz.

Distinguish $\mathscr{G}_V$ and $\mathscr{G}'_V$ depending on whether $\mathscr{V}$ or $\mathscr{V}'$ is used in the construction in Eq. (4.6).

The constructions of Sect. 4 also require a vector bundle $Y \to M$ and, if necessary, a complex line bundle $L \to M$ so that the bundle U in Eq. (4.10) is oriented and spin or spin$_{\mathbb{C}}$. The required bundles are provided in the next lemma.

To state this lemma, recall from Definition 1.1 the notion of V being $\mathbb{Z}/n\cdot\mathbb{Z}$ compatible with T^*M.

Lemma 5.2. *Let M be a compact, oriented spin manifold on which S^1 acts. Let $V \to M$ be a real, oriented vector bundle to which the S^1 action has a lift. Assume that $w_2(V)=0$. For integer $n_0>1$, assume that V is $\mathbb{Z}/n_0\cdot\mathbb{Z}$ compatible with T^*M. Let $r=0$ when n_0 is odd, and let $r=1$ when n_0 is even. Then*
(1) *The line bundle*

$$\bigotimes_{0<\upsilon<n_0/2} (\det(NM(n_0;\upsilon)^*) \otimes \det(\underline{V(n_0;\upsilon)}^*))^{(r+1)\cdot\upsilon}$$

has an n_0^{th} root.
(2) *Let*

$$L \equiv \Big(\bigotimes_{0<\upsilon<n_0/2} (\det(NM(n_0;\upsilon)^*) \otimes \det(\underline{V(n_0;\upsilon)}^*))^{-1} \Big) \otimes \Big[\bigotimes_{0<\upsilon<n_0/2} (\det(NM(n_0;\upsilon)^*) \otimes \det(\underline{V(n_0;\upsilon)}^*))^{-2\cdot\upsilon} \Big]^{r/n_0},$$

and let

$$L' \equiv \Big(\bigotimes_{0<\upsilon<n_0/2} \det(NM(n_0;\upsilon)^*)^{-1} \Big) \otimes \Big[\bigotimes_{0<\upsilon<n_0/2} (\det(NM(n_0;\upsilon)^*) \otimes \det(\underline{V(n_0;\upsilon)}^*))^{-2\cdot\upsilon} \Big]^{r/n_0}.$$

Let $U \equiv TM(n_0) \oplus V(n_0;0)$ and let $U' \equiv TM(n_0) \oplus V(n_0;n_0/2)$. Then U and U' are oriented; and U with the line bundle L and U' with the line bundle L' are spin$_{\mathbb{C}}$.

This lemma is proved as Lemmas 11.3 and 11.4 in [B–T].

To summarize for future reference,

Proposition 5.3. *Let M be a compact, oriented spin manifold on which S^1 acts. Let $V \to M$ be a real, oriented vector bundle to which the S^1 action has a lift. Assume that*

$w_2(V)=0$; and for integer $n_0 \geqq 1$, assume that V is $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ compatible with T^*M. The Dirac-Ramond construction in Sect. 4 can be made with the following geometric data: There are two cases, unprimed and primed. In both, replace the manifold M in Sect. 4 with the submanifold $M(n_0) \subset M$ which is fixed under the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of S^1. In Sect. 4, use the vector bundle $NM(n_0) \to M(n_0)$ for the vector bundle E, and use $\alpha(\upsilon) \equiv \upsilon$. Use $V|_{M(n_0)} \to M(n_0)$ for the vector bundle V; in the unprimed case use $\beta(\upsilon) \equiv \upsilon/n_0$, and in the primed case, use $\beta(\upsilon) \equiv \upsilon/n_0 + 1/2$. *In the unprimed case, use the vector bundles U and L as described in Lemma 5.3 to define the* $\text{spin}_{\mathbb{C}}$ *structure. In the primed case, use the vector bundles U' and L' in Lemma 5.3 to define the* $\text{spin}_{\mathbb{C}}$ *structure. Finally, choose any auxiliary, complex vector bundle $W \to M(n_0)$. This data defines the vector bundle $\mathscr{E}_{NM(n_0),V} \to M(n_0)$ of Eq.* (4.13). *For $0 < \upsilon \leqq n_0/2$, set $\mu[\upsilon, n] \equiv n + \upsilon/n_0$ in Eq.* (4.12) *to define $\Phi_{NM(n_0)}$. Then D_t of Eq.* (4.14) *maps $C^\infty(\mathscr{E}_{NM(n_0),V} \otimes \Phi_{NM(n_0)})$ to itself and the conclusions of Propositions 4.1 and 4.2 hold.*

For a special case, consider $n=1$, so that $M(n) \equiv NM(n) \equiv M$. This is the case which is considered by Witten in [W2]. The analysis in Sect. 4 yields the following proposition as a corollary:

Proposition 5.4. *Let M be a compact, oriented Riemannian manifold. Use $E \equiv M \times \{0\}$ and any real, oriented vector bundle $V \to M$ in Proposition 4.1. Consider two cases, unprimed and primed. In the unprimed case, make no assumptions. In the primed case, assume that $w_2(TM)=0$. Then*

1) $\text{Ind}(D; \mathscr{E}^{M,Vm}, \ell_e)$ *is zero for $m<0$, and for $m \geqq 0$, it is equal to the index of the operator*

$$(d+d^*): C^\infty(\Lambda^{\text{even}}(T^*M) \otimes R_e(m)) \to C^\infty(\Lambda^{\text{even}}(T^*M) \otimes R_e(m)),$$

where $R_e(m) \to M$ is the coefficient of q^m in the following formal power series with coefficients in the real K-theory of M:

$$\bigotimes_{0<n\in\mathbb{Z}} \left(\bigoplus_{0 \leqq k \in \mathbb{Z}} q^{nk} \cdot \text{Sym}_k(T^*M) \right) \bigotimes_{0<n\in\mathbb{Z}} \left(\bigoplus_{0 \leqq k \in \mathbb{Z}} (-1)^k \cdot q^{nk} \cdot \Lambda^k(T^*M) \right).$$

When M is spin, this is the q^{2m} component of $\text{Ind}(\partial, F_E(q,V), \gamma)$ *with $F_E(q,V)$ given in Eq.* (1.5).
2) $\text{Ind}(D; \mathscr{E}_{M,Vm}, \ell_s)$ *is zero for $m<0$, and for $m \geqq 0$, it is equal to the index of the signature operator on M coupled to $R_s(m) \to M$, where $R_s(m)$ is the coefficient of q^{2m} in the following formal power series with coefficients in* $\text{Vect}(M)$:

$$\bigotimes_{0<n\in\mathbb{Z}} \left(\bigoplus_{0 \leqq k \in \mathbb{Z}} q^{nk} \cdot \text{Sym}_k(T^*M) \right) \bigotimes_{0<n\in\mathbb{Z}} \left(\bigoplus_{0 \leqq k \in \mathbb{Z}} q^{nk} \cdot \Lambda^k(T^*M) \right).$$

When M is spin, this is the q^{2m} component of $\text{Ind}(\partial, F_S(q,V), \gamma)$ *with $F_S(q,V)$ given in Eq.* (1.5).

In the primed case,

3) $\text{Ind}(D; \mathscr{E}_{M,Vm}, \ell_e)$ *is zero for $m<0$, and for $m \geqq 0$, it is equal to the q^{2m} component of* $\text{Ind}(\partial, F_D(q,V), \gamma)$ *with $F_D(q,V)$ given in Eq.* (1.5).
4) $\text{Ind}(D; \mathscr{E}_{M,Vm}, \ell_e)$ *is zero for $m<0$, and for $m \geqq 0$, it is equal to the q^{2m} component of* $\text{Ind}(\partial, F(-q,V), \gamma)$.

This section ends with the proof of Lemma 5.1.

Proof of Lemma 5.1. Since V is oriented, $w_1(V(n_0;0))=w_1(V(n_0;n_0/2))$. Since complex vector bundles are always orientable, is automatic the orientability of $V(n_0;0)$ in the case when n_0 is odd. Assume that n_0 is even. First, consider the case $n_0=2$.

Denote the S^1 action by $\varphi: S^1\times M\to M$. Since $\varphi(\pi,\cdot)$ is the identity on $M(2)$, $\varphi(\pi,\cdot)_*: V(2;0)\to V(2;0)$ is the identity map; and $\varphi(\pi,\cdot)_*: V(2;1)\to V(2;1)$ is an involution which defines the $\mathbb{Z}/2\cdot\mathbb{Z}$ action. It is convenient to fix an S^1-invariant fiber metric on V. This induces a fiber metric on $V(2;0)$ and one on $V(2;1)$. Then, $\varphi(\pi,\cdot)_*$ acts on $V(2;1)$ as a special orthogonal automorphism, A, with $A^2=1$. Thus, A is also symmetric, and so it can be diagonalized; all the eigenvalues of A are equal to -1. This means that $A\equiv -I$, with I being the identity automorphism.

Suppose that $V(2;0)$ were not orientable. With this assumption, there exists a map $q: S^1\to M(2)$ with the property that $w_1(q^*V(2;0))\neq 0$. For such q,

$$q^*(V(2;0))\approx\left(S^1\underset{\mathbb{Z}/2\cdot\mathbb{Z}}{\times}\mathbb{R}\right)\times\mathbb{R}^{k-1}, \tag{5.9}$$

where $k=\dim(V(2;0))$. Since $w_1(V(2;0))=w_1(V(2;1))$, one also has

$$q^*(V(2;1))\approx\left(S^1\underset{\mathbb{Z}/2\cdot\mathbb{Z}}{\times}\mathbb{R}\right)\times\mathbb{R}^{m-1}, \tag{5.10}$$

where $m=\dim(V(2;1))$.

The S^1 action defines from q a map, $q_1: S^1\times S^1\to M(2)$ which sends (t,s) to $q_1(t,s)\equiv\varphi(t/2,q(s))$. For fixed s,

$$q_1(\cdot,q(s))^*V(2;0)=q(s)^*\varphi_*V(2;0); \tag{5.11}$$

and therefore,

$$q_1^*V(2;0)\approx S^1\times\left(S^1\underset{\mathbb{Z}/2\cdot\mathbb{Z}}{\times}\mathbb{R}\right)\times\mathbb{R}^{m-1}. \tag{5.12}$$

The bundle $V(2;1)$ obeys

$$q_1(\cdot,q(s))^*V(2;1)=q(s)^*\varphi_*V(2;1); \tag{5.13}$$

and this means that

$$q_1{}^*V(2;1)\approx S^1\underset{\mathbb{Z}/2\cdot\mathbb{Z}}{\times}\left(\left(S^1\underset{\mathbb{Z}/2\cdot\mathbb{Z}}{\times}\mathbb{R}\right)\times\mathbb{R}^{m-1}\right), \tag{5.14}$$

where $\mathbb{Z}/2\cdot\mathbb{Z}$ acts on $(S^1\times_{\mathbb{Z}/2\cdot\mathbb{Z}}\mathbb{R})\times\mathbb{R}^{m-1}$ as multiplication by ± 1.

The $\mathbb{Z}/2\cdot\mathbb{Z}$ cohomology of $S^1\times S^1$ has generators z_1, z_2 which restrict trivially to the second S^1 and to the first S^1, respectively. The total Stiefel-Whitney class of $q_1^*V(2;0)$ can be computed from Eq. (5.12) to be

$$w(q_1^*V(2;0))=1+z_2. \tag{5.15}$$

The total Stiefel-Whitney class of $q_1^*V(2;1)$ can be computed from Eq. (5.14) to be

$$w(q_1^*V(2;1))=1+m_{\mathrm{mod}(2)}\cdot z_1+z_2+(m-1)_{\mathrm{mod}(2)}\cdot z_1\wedge z_2. \tag{5.16}$$

The total Stiefel-Whitney class of $q_1^* V$ is now computable from Eqs. (5.15, 16) as the product of the two classes:

$$w(q_1^* V) = 1 + m_{\mathrm{mod}(2)} \cdot z_1 + (m_{\mathrm{mod}(2)} + (m-1)_{\mathrm{mod}(2)}) \cdot z_1 \wedge z_2 . \tag{5.17}$$

Equation (5.17) asserts that $w_2(q_1^* V) \neq 0$. This is a contradiction, since $w_2(q_1^* V) = q_1^* w_2(V)$. The contradiction implies that $V(2; 0)$ must be orientable as claimed. For $n_0 = 2\ell > 2$, use the same arguement with the fact that $c_1(E) = 0$ for any bundle $E \to S^1 \times S^1$ to which the rotations around an S^1 lift.

6. Localization for the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ Twisted Case

Suppose that S^1 acts geometrically as a group of isometries of M, and that this action lifts to an action on the vector bundle $V \to M$. As in Sect. 5, consider for $n_0 > 0$, the fixed point set $M(n_0)$ of the natural $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of S^1. The action of S^1 on M defines an action on $M(n_0)$ which is an n_0-fold covering of an S^1 action; the n_0-*root action*. The fixed point set of the n_0-root action on $M(n_0)$ is the same as the fixed point set of the original action on M, namely, $U_i \Sigma[i]$.

Assume that the conditions of Proposition 5.3 hold so that the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ construction in said Proposition of the twisted Dirac-Ramond operator can be made using $M(n_0)$, $NM(n_0) \to M(n_0)$ and $V|_{M(n_0)}$.

For the vector bundle $W \to M(n_0)$, assume that a finite cover of the n_0-root action on $M(n_0)$ lifts to W. Give W an invariant metric and an invariant, metric compatible connection.

The n_0-root S^1 action on $M(n_0)$ has a finite cover which lifts to an action on the vector bundle $\underline{\mathscr{E}}_{NM(n_0), V} \to M(n_0)$ in both the primed and unprimed cases of Proposition 5.3; this lift commutes with the endomorphism P; there is a lift to each of the vector bundles $\underline{\mathscr{E}}_{NM(n_0), Vm}$.

The Dirac-Ramond operator D_t on $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm} \otimes \Phi_{NM(n_0)})$ is equivariant under the lifted action, which implies that the S^1 character-valued index of D_t on $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm} \otimes \Phi_{NM(n_0)})$ for the n_0-root S^1 action can be defined by mimicking the definition in Sect. 2. Indeed, decompose $\underline{\mathscr{E}}_{NM(n_0), Vm}$ into the direct sum of finite dimensional vector bundles $\oplus_h \underline{\mathscr{E}}_{NM(n_0), Vm}(h) \otimes \Phi_{NM(n_0)}$ as discussed in the Appendix. Correspondingly, the operator D_t on $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm} \otimes \Phi_{NM(n_0)})$ decomposes into a direct sum of "standard" Dirac operators, $\{\partial + A_m(h)\}$ as in Eq. (A.9). Each of the $\partial + A_m(h)$ is equivariant under the n_0-root action; and each is of the form discussed in Sect. 2 and Proposition 2.6. Therefore, each has an S^1-character-valued index which is defined as follows: Let k define a character of the induced n_0-root S^1-action on $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm}(h) \otimes \Phi_{NM(n_0)})$ (so, k is a rational number). Let $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm}(h) \otimes \Phi_{NM(n_0)}, k)$ denote the subspace of sections of $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm} \otimes \Phi_{NM(n_0)})$ on which the n_0-root S^1-action is defined by the rational number k. The involution, $\ell \equiv \ell_e$ or ℓ_s in Eqs. (4.17, 18) commutes with the n_0-root action, and so defines an involution of $C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm}(h) \otimes \Phi_{NM(n_0)}, k)$. Define

$$\begin{aligned} &\mathrm{Ind}(D; \underline{\mathscr{E}}_{NM(n_0), Vm}(h), \ell, k) \\ &\quad \equiv \dim(\ker(D_t|(ker(\ell - 1) \cap C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm}(h) \otimes \Phi_{NM(n_0)}, k))) \\ &\qquad - \dim(\ker(D_t|(ker(\ell + 1) \cap C^\infty(\underline{\mathscr{E}}_{NM(n_0), Vm}(h) \otimes \Phi_{NM(n_0)}, k))) . \end{aligned} \tag{6.1}$$

According to Proposition A.1, these indices are zero except for $h=0$. Thus, one can define

$$\operatorname{Ind}(D; \mathscr{E}_{NM(n_0),Vm}, \ell, k) \equiv \sum_h \operatorname{Ind}(D; \mathscr{E}_{E,Vm}(h), \ell, k). \tag{6.2}$$

The localization results in Sect. 2 can be used to localize the Dirac operator D_t on $C^\infty(\mathscr{E}_{NM(n_0),Vm} \otimes \Phi_{NM(n_0)})$ by localizing each D_t on each $C^\infty(\mathscr{E}_{NM(n_0),Vm}(h) \otimes \Phi_{NM(n_0)})$. Recall that Witten's description of localization as discussed in Sect. 2 is obtained by comparing the large $|s|$ behavior of the family of "standard" Dirac operators

$$D_{ts} \equiv D_t + i \cdot s \cdot e^a \cdot K_a$$

on $C^\infty(\mathscr{E}_{NM(n_0),Vm}(h) \otimes \Phi_{NM(n_0)}, k)$ with a family of differential operators on the normal bundle to the fixed point set to the geometric S^1 action. That is, the S^1-character valued index of D_t on each $C^\infty(\mathscr{E}_{NM(n_0),Vm}(h) \otimes \Phi_{NM(n_0)}, k)$ can be calculated using a Dirac operator on the normal bundle to the fixed point set; or, equivalently using the Dirac operator on the fixed point set, but coupled to a specific, finite dimensional vector bundle.

The description of these normal bundle Dirac operators (one for each pair of eigenvalues (m, h)) is facilitated by defining an operator, Q_{n_0}, on an infinite dimensional vector bundle over the fixed point set which decomposes appropriately upon restriction to an (m, h) eigenspace. This is analogous to the decomposition in the previous section of the operator D_t into a direct sum of operators $\{\partial + A_m(h)\}$. For calculational purposes, it is much more convenient to consider Q_{n_0}, the Dirac operator coupled to an infinite dimensional vector bundle, rather than a countable set of operators, each a "standard" Dirac operator coupled to a finite dimensional vector.

The justification for the manipulations of the big operator comes, ultimately, from the direct sum decomposition into "standard" Dirac operators; and then, by referral to the results for "standard" Dirac operators which are summarized in Sect. 2.

The result is Proposition 6.2, below, the analog of Proposition 4.6. This is an assertion that the S^1 character-valued index of D_t on $C^\infty(\mathscr{E}_{NM(n_0),Vm} \otimes \Phi_{NM(n_0)})$ is equal to the sum of the S^1 character-valued indices of suitable Dirac operators on the normal bundles to the components of the fixed point set of the S^1 action. The appropriate Dirac operator on the normal bundle to $\Sigma \equiv \Sigma[i]$ will be denoted by $Q_{n_0} \equiv Q_{n_0}[i]$; it is the analog of the operator in Eq. (2.21).

To write down this operator Q_{n_0} requires a digression. Recall from Sect. 2 that the normal bundle to Σ is naturally a complex vector bundle $N \to \Sigma$ which decomposes into character bundles $N = \oplus_{0<\upsilon} N_\upsilon$ under the S^1 action. Let

$$N^L \equiv \bigoplus_{0<\upsilon:\, \upsilon \in n_0 \cdot \mathbb{Z}} N_\upsilon \quad \text{and} \quad N^T \equiv \bigoplus_{0<\upsilon:\, \upsilon \notin n_0 \cdot \mathbb{Z}} N_\upsilon. \tag{6.3}$$

As a real bundle, N^L is isomorphic to the normal bundle in $M(n_0)$ of Σ. The subbundle $N^T \to \Sigma$, is, as a real bundle, isomorphic to the restriction to Σ of $NM(n_0)$, the normal bundle to $M(n_0)$ in M.

The normal bundle $NM(n_0)$ decomposes under the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ action on M according to Eq. (5.2). The restriction to Σ of $NM(n_0; \upsilon)$ is given by

$$NM(n_0; \upsilon)|_\Sigma = \bigoplus_{0<\upsilon':\, \upsilon' = \upsilon \bmod(n_0)} N_{\upsilon'} \bigoplus_{0<\upsilon':\, \upsilon' = -\upsilon \bmod(n_0)} \underline{N}_{\upsilon'}. \tag{6.4}$$

Along Σ, the S^1 action decomposes the vector bundle V into $V = V_{0R} \oplus_{0<\upsilon} V_\upsilon$, with V_{0R} real, and with each V_υ naturally complex. The vector bundle V along $M(n_0)$ decomposes under the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ action on M according to Eq. (5.6); and along Σ,

$$V(n_0;\upsilon) = \bigoplus_{0<\upsilon':\upsilon'=\upsilon \bmod (n_0)} V_{\upsilon'} \bigoplus_{0<\upsilon':\upsilon'=-\upsilon \bmod (n_0)} \underline{V}_{\upsilon'} . \tag{6.5}$$

The vector bundle $\mathcal{N}NM(n_0)$ of Eq. (5.4) restricts to the normal bundle, $N^L \to \Sigma$ in $M(n_0)$. Since $\mathcal{N}NM(n_0)$ is a bundle over N^L, and N^L is a vector bundle over Σ, the total space of $\mathcal{N}NM(n_0)$ defines a vector bundle over Σ which is the direct sum $\mathcal{N}NM(n_0) \equiv \mathcal{N}(n_0)_+ \oplus \mathcal{N}(n_0)_- \oplus \mathcal{N}(n_0)_0$, where

$$\begin{aligned}
\mathcal{N}(n_0)_+ \approx & \bigoplus_{0<n\in\mathbb{Z}} TM(\Sigma)_{n\mathbb{C}} \bigoplus_{0<\upsilon:\upsilon=0,\, n_0/2 \bmod (n_0)} \bigoplus_{0<n\in\mathbb{Z}+\upsilon/n_0} N_{\upsilon,n} \bigoplus_{0<\upsilon'<n_0/2} \\
& \left(\bigoplus_{0<\upsilon:\upsilon=\upsilon' \bmod (n_0)} \bigoplus_{0<n\in\mathbb{Z}+\upsilon/n_0} N_{\upsilon,n} \right. \\
& \left. \bigoplus_{0<\upsilon:\upsilon=-\upsilon' \bmod (n_0)} \bigoplus_{0<n\in\mathbb{Z}-\upsilon/n_0} \underline{N}_{\upsilon,-n} \right), \\
\mathcal{N}(n_0)_- \approx & \bigoplus_{0<\upsilon:\upsilon=0,\, n_0/2 \bmod (n_0)} \bigoplus_{0>n\in\mathbb{Z}+\upsilon/n_0} N_{\upsilon,n} \bigoplus_{0<\upsilon'<n_0/2} \\
& \left(\bigoplus_{0<\upsilon:\upsilon=\upsilon' \bmod (n_0)} \bigoplus_{0>n\in\mathbb{Z}+\upsilon/n_0} N_{\upsilon,n} \right. \\
& \left. \bigoplus_{0<\upsilon:\upsilon=-\upsilon' \bmod (n_0)} \bigoplus_{0>n\in\mathbb{Z}-\upsilon/n_0} \underline{N}_{\upsilon,-n} \right), \\
\mathcal{N}(n_0)_0 \approx & \bigoplus_{0<\upsilon:\upsilon=0 \bmod (n_0)} \oplus N_{\upsilon,0} ,
\end{aligned} \tag{6.6}$$

where the "subscript "$_n$"" is an indexing label of the vector bundle in question.

The vector bundles $\mathcal{V}$ and $\mathcal{V}'$ of Eqs. (5.7, 8) restrict to vector bundles over Σ as

$$\mathcal{V}(n_0) \equiv \mathcal{V}(n_0)_+ \oplus \mathcal{V}(n_0)_- \oplus \mathcal{V}(n_0)_0$$

and

$$\mathcal{V}'(n_0) \equiv \mathcal{V}'(n_0)_+ \oplus \mathcal{V}'(n_0)_- \oplus \mathcal{V}'(n_0)_0 ,$$

where

$$\begin{aligned}
\mathcal{V}(n_0)_+ \approx & \bigoplus_{0<n\in\mathbb{Z}} V_{0,n} \bigoplus_{0<\upsilon:\upsilon=0,\, n_0/2 \bmod (n_0)} \bigoplus_{0<n\in\mathbb{Z}+\upsilon/n_0} V_{\upsilon,n} \bigoplus_{0<\upsilon'<n_0/2} \\
& \left(\bigoplus_{0<\upsilon:\upsilon=\upsilon' \bmod (n_0)} \bigoplus_{0<n\in\mathbb{Z}+\upsilon/n_0} V_{\upsilon,n} \right. \\
& \left. \bigoplus_{0<\upsilon:\upsilon=-\upsilon' \bmod (n_0)} \bigoplus_{0<n\in\mathbb{Z}-\upsilon/n_0} \underline{V}_{\upsilon,-n} \right), \\
\mathcal{V}(n_0)_- \approx & \bigoplus_{0<\upsilon:\upsilon=0,\, n_0/2 \bmod (n_0)} \bigoplus_{0>n\in\mathbb{Z}+\upsilon/n_0} V_{\upsilon,n} \bigoplus_{0<\upsilon'<n_0/2} \\
& \left(\bigoplus_{0<\upsilon:\upsilon=\upsilon' \bmod (n_0)} \bigoplus_{0>n\in\mathbb{Z}+\upsilon/n_0} V_{\upsilon,n} \right. \\
& \left. \bigoplus_{0<\upsilon:\upsilon=-\upsilon' \bmod (n_0)} \bigoplus_{0>n\in\mathbb{Z}-\upsilon/n_0} \underline{V}_{\upsilon,-n} \right), \\
\mathcal{V}(n_0)_0 \approx & V_{0R} \bigoplus_{0<\upsilon:\upsilon=0 \bmod (n_0)} \oplus V_{\upsilon,0} ;
\end{aligned} \tag{6.7}$$

and

$$\mathscr{V}'(n_0)_+ \approx \bigoplus_{0 \leq \upsilon : \upsilon = 0, n_0/2 \bmod(n_0)} \bigoplus_{0 < n \in \mathbb{Z} + \upsilon/n_0 + 1/2} V_{\upsilon, n} \bigoplus_{0 < \upsilon' < n_0/2}$$

$$\Bigg(\bigoplus_{0 < \upsilon : \upsilon = \upsilon' \bmod(n_0)} \bigoplus_{0 < n \in \mathbb{Z} + \upsilon/n_0 + 1/2} V_{\upsilon, n}$$

$$\bigoplus_{0 < \upsilon : \upsilon = -\upsilon' \bmod(n_0)} \bigoplus_{0 < n \in \mathbb{Z} - \upsilon/n_0 - 1/2} \underline{V}_{\upsilon, -n} \Bigg),$$

$$\mathscr{V}'(n_0)_- \approx \bigoplus_{0 < \upsilon : \upsilon = 0, n_0/2 \bmod(n_0)} \bigoplus_{0 > n \in \mathbb{Z} + \upsilon/n_0 + 1/2} V_{\upsilon, n} \bigoplus_{0 < \upsilon' < n_0/2}$$

$$\Bigg(\bigoplus_{0 < \upsilon : \upsilon = \upsilon' \bmod(n_0)} \bigoplus_{0 > n \in \mathbb{Z} + \upsilon/n_0 + 1/2} V_{\upsilon, n}$$

$$\bigoplus_{0 < \upsilon : \upsilon = -\upsilon' \bmod(n_0)} \bigoplus_{0 > n \in \mathbb{Z} - \upsilon/n_0 - 1/2} \underline{V}_{\upsilon, -n} \Bigg),$$

$$\mathscr{V}'(n_0)_0 \approx \bigoplus_{0 < \upsilon : \upsilon = n_0/2 \bmod(n_0)} \oplus V_{\upsilon, 0} \,. \tag{6.8}$$

The normal bundle decompositions in Eqs. (6.2–5) also induce extra structure upon the restriction to Σ of the $\mathrm{spin}_{\mathbb{C}}$ bundles $S(U; L)$ and $S(U'; L')$ given in Lemma 5.2. Consider first the bundle U. Open restriction to Σ,

$$T^*M(n_0) \otimes V(n_0; 0)_R|_\Sigma = T\Sigma^* \oplus V_{0R} \bigoplus_{0 < \upsilon : \upsilon = 0 \bmod(n_0)} N^*_\upsilon \bigoplus_{0 < \upsilon : \upsilon = 0 \bmod(n_0)} \underline{V}_\upsilon \,. \tag{6.9}$$

The $\mathrm{spin}_{\mathbb{C}}$-bundle $S(U, L)|_\Sigma \to \Sigma$ is isomorphic to

$$S(U; L)|_\Sigma = S\Big(T^*\Sigma \oplus V_{0R}; \bigotimes_{0 < \upsilon} (\det(N^*_\upsilon) \otimes \det(\underline{V}^*_\upsilon))^{-1} \Big)$$

$$\otimes \Lambda^* \Big(\bigoplus_{0 < \upsilon : \upsilon = 0 \bmod(n_0)} N^*_\upsilon \bigoplus_{0 < \upsilon : \upsilon = 0 \bmod(n_0)} \underline{V}^*_\upsilon \Big)$$

$$\bigotimes_{0 < \upsilon : n_0/2 < \upsilon < n_0 \bmod(n_0)} (\det(N^*_\upsilon) \oplus \det(\underline{V}^*_\upsilon)),$$

$$\otimes \Bigg[\bigotimes_{0 < \upsilon' < n_0/2} \Bigg(\bigotimes_{0 < \upsilon : \upsilon = \upsilon' \bmod(n_0)} (\det(N^*_\upsilon) \otimes \det(\underline{V}^*_\upsilon))^{-2 \cdot \upsilon'/n_0}$$

$$\bigotimes_{0 < \upsilon : \upsilon = -\upsilon' \bmod(n_0)} (\det(N^*_\upsilon) \otimes \det(\underline{V}^*_\upsilon))^{2 \cdot \upsilon'/n_0} \Bigg)$$

$$\bigotimes_{0 < \upsilon : \upsilon = n_0/2 \bmod(n_0)} (\det(N^*_\upsilon) \otimes \det(\underline{V}^*_\upsilon)) \Bigg]^{r/2}, \tag{6.10}$$

with $r = 0$ if n_0 is odd, and $r = 1$ if n_0 is even. Equations (5.25) and (6.4, 5) insure that the square root taken in Eq. (6.10) is well defined.

For the primed case, the following isomorphism of real bundles holds:

$$T^*M(n_0) \otimes V(n_0; n_0/2)|_\Sigma = T^*\Sigma \bigoplus_{0 < \upsilon : \upsilon = 0 \bmod(n_0)} N^*_\upsilon \bigoplus_{0 < \upsilon : \upsilon = n_0/2 \bmod(n_0)} \underline{V}^*_\upsilon \,. \tag{6.11}$$

The spin$_{\mathbb{C}}$-bundle $S(U'; L') \to M(n_0)$ decomposes over Σ as

$$S(U') = S\left(T^*\Sigma; \bigotimes_{0<\upsilon} \det(N_\upsilon^*)^{-1}\right)$$
$$\otimes V^*\left(\bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} N_\upsilon^* \bigoplus_{0<\upsilon:\upsilon=n_0/2\,\mathrm{mod}(n_0)} \underline{V}_\upsilon^*\right)$$
$$\bigotimes_{0<\upsilon:n_0/2<\upsilon<n_0\,\mathrm{mod}(n_0)} \det(N_\upsilon^*)$$
$$\otimes\left[\bigotimes_{0<\upsilon'<n_0/2}\left(\bigotimes_{0<\upsilon:\upsilon=\upsilon'\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-2\cdot\upsilon'/n_0}\right.\right.$$
$$\left.\bigotimes_{0<\upsilon:\upsilon=-\upsilon'\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{2\cdot\upsilon'/n_0}\right)$$
$$\left.\bigotimes_{0<\upsilon:\upsilon=n_0/2\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*)^{-1})\right]^{r/2}, \tag{6.12}$$

with $r=0$ if n_0 is odd, and $r=1$ if n_0 is even.

The Fermionic Fock space bundle over Σ is either $S(U; L)\otimes\underline{\mathscr{F}}_{NM(n_0)}\otimes\underline{\mathscr{G}}_V|_\Sigma$, or $S(U'; L')\otimes\underline{\mathscr{F}}_{NM(n_0)}\otimes\underline{\mathscr{G}}'_V|_\Sigma$. The isomorphisms above induce in a straightforward way, decompositions of these Fock space bundles. In the unprimed case

$$S(U; L)\otimes\underline{\mathscr{F}}_{NM(n_0)}\otimes\underline{\mathscr{G}}_V|_\Sigma$$
$$= S\left(T^*\Sigma\oplus V_{0R}; \bigotimes_{0<\upsilon} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-1}\right) \bigotimes_{0<\upsilon:n_0/2<\upsilon<n_0\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))$$
$$\otimes\left[\bigotimes_{0<\upsilon'<n_0/2}\left(\bigotimes_{0<\upsilon:\upsilon=\upsilon'\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-2\cdot\upsilon'/n_0} \bigotimes_{0<\upsilon:\upsilon=-\upsilon'\,\mathrm{mod}(n_0)}\right.\right.$$
$$\left.\left.(\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{2\cdot\upsilon'/n_0}\right) \bigotimes_{0<\upsilon:\upsilon=n_0/2\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))\right]^{r/2},$$
$$\otimes\Lambda^*\left(\mathscr{N}(n_0)_+^*\oplus\underline{\mathscr{N}(n_0)_-^*}\bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} N_{\upsilon,0}^*\right)$$
$$\otimes\Lambda^*\left(\mathscr{V}(n_0)_-^*\oplus\underline{\mathscr{V}(n_0)_+^*}\bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} \mathscr{V}_{\upsilon,0}^*\right). \tag{6.13}$$

In the primed case,

$$S(U'; L')\otimes\underline{\mathscr{F}}_{NM(n_0)}\otimes\underline{\mathscr{G}}'_V|_\Sigma$$
$$= S\left(T^*\Sigma; \bigotimes_{0<\upsilon} \det(N_\upsilon^*)^{-1}\right) \bigotimes_{0<\upsilon:n_0/2<\upsilon<n_0\,\mathrm{mod}(n_0)} \det(N_\upsilon^*)$$
$$\otimes\left[\bigotimes_{0<\upsilon'<n_0/2}\left(\bigotimes_{0<\upsilon:\upsilon=\upsilon'\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-2\cdot\upsilon'/n_0} \bigotimes_{0<\upsilon:\upsilon=-\upsilon'\,\mathrm{mod}(n_0)}\right.\right.$$
$$\left.\left.(\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{2\cdot\upsilon'/n_0}\right) \bigotimes_{0<\upsilon:\upsilon=n_0/2\,\mathrm{mod}(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*)^{-1})\right]^{r/2}$$
$$\otimes\Lambda^*\left(\mathscr{N}(n_0)_+^*\oplus\underline{\mathscr{N}(n_0)_-^*}\bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} N_{\upsilon,0}^*\right)$$
$$\otimes\Lambda^*\left(\mathscr{V}'(n_0)_-^*\oplus\underline{\mathscr{V}'(n_0)_+^*}\bigoplus_{0<\upsilon:\upsilon=n_0/2\,\mathrm{mod}(n_0)} \underline{V}_{\upsilon,0}^*\right). \tag{6.14}$$

The Bosonic Fock space bundle over Σ for the operator Q_{n_0} is

$$\underline{\mathscr{B}}_{NM(n_0)}(\Sigma) \equiv \mathrm{Sym}(\mathscr{N}NM(n_0)\oplus\underline{\mathscr{N}NM(n_0)}). \tag{6.15}$$

Let $W \to M(n_0)$ be the auxiliary, finite dimensional, complex vector bundle to which a cover of the n_0-root action lifts. Let $W(\Sigma)$ denote the restriction of W to Σ.

The operator Q_{n_0} will be defined on the space of smooth sections over Σ of the vector bundle $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes \Phi_{NM(n_0)\Sigma}$, with

$$\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \equiv \underline{\mathscr{B}}_{NM(n_0)}(\Sigma) \otimes (S(U;L) \otimes \underline{\mathscr{F}}_{NM(n_0)} \otimes \underline{\mathscr{G}}_V)|_\Sigma \otimes W(\Sigma)$$

or (6.16)

$$\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \equiv \underline{\mathscr{B}}_{NM(n_0)}(\Sigma) \otimes (S(U';L') \otimes \underline{\mathscr{F}}_{NM(n_0)} \otimes \underline{\mathscr{G}}'_V)|_\Sigma \otimes W(\Sigma),$$

and $\Phi_{NM(n_0)\Sigma}$ is the function on $\mathscr{N}$ which is given by

$$\begin{aligned}\Phi_{NM(n_0)\Sigma} = \exp &\left(-t \cdot \sum_{0<n\in\mathbb{Z}} n \cdot |x_n|^2\right) \\ &\times \exp\left(-\sum_{0<\upsilon\in n_0\cdot\mathbb{Z}} \left(\sum_{0\neq n\in\mathbb{Z}} t \cdot |n| \cdot |z_{\upsilon,n}|^2 + |s| \cdot \upsilon(j)/n_0 \cdot |z_{\upsilon,0}|^2\right)\right) \\ &\times \prod_{0<\upsilon\notin n_0\cdot\mathbb{Z}} \exp\left(-t \cdot \sum_{n\in\mathbb{Z}+\upsilon/n_0} |n| \cdot |z_{\upsilon,n}|^2\right). \qquad (6.17)\end{aligned}$$

Here, $\{z_{\upsilon,n}\}_{\upsilon>0}$ are complex fiber coordinates with respect to a local orthonormal frame for the n^{th} copy of $N_\upsilon \to \Sigma$; while $\{x_n\}$ are complex fiber coordinates with respect to a local orthonormal frame for the n^{th} copy of $T\Sigma^*_{\mathbb{C}}$.

The bundles $S(U;L) \otimes \underline{\mathscr{F}}_{NM(n_0)} \otimes \underline{\mathscr{G}}_V|_\Sigma$ and $S(U';L') \otimes \underline{\mathscr{F}}_{NM(n_0)} \otimes \underline{\mathscr{G}}'_V|_\Sigma$ have natural metrics and metric compatible covariant derivatives. The bundle $\underline{\mathscr{B}}_{NM(n_0)}(\Sigma)$ has a metric which is induced by using $\Phi_{NM(n_0)\Sigma}$ to define a Gaussian measure; or, alternately, one can introduce Boson creation and annihilation operators as in the Appendix. The natural connection on $\underline{\mathscr{B}}_{NM(n_0)}(\Sigma)$ from Eq. (6.15) is metric compatible. The finite dimensional bundle $W(\Sigma) \equiv W|_\Sigma$ has an S^1-invariant metric and an invariant metric compatible connection by assumption.

Let ∇ denote the induced covariant derivative on the space of smooth sections over Σ of the vector bundle $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \to \Sigma$. Composing with the natural Clifford multiplication map from $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes T^*\Sigma \to \underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)$ gives a Dirac operator, D_Σ which defines an endomorphism of $C^\infty(\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes \Phi_{NM(n_0)\Sigma})$.

The Dirac-Ramond operator

$$Q_{n_0} : C^\infty(\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes \Phi_{NM(n_0)\Sigma}) \to C^\infty(\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes \Phi_{NM(n_0)\Sigma})$$

is given by

$$Q_{n_0} \equiv D_\Sigma + T_{n_0}, \qquad (6.18)$$

with T_{n_0} the following covariantly constant endomorphism of $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes \Phi_{NM(n_0)\Sigma}$ [compare with Eq. (2.23)]:

$$\begin{aligned}T_{n_0} \equiv T_\Sigma + i \cdot \sum_{0<\upsilon} &\left(\sum_{0<n\in\mathbb{Z}+\upsilon/n_0} [\Gamma_{\upsilon,n} \cdot (\partial/\partial \bar{z}_{\upsilon,n} - t \cdot n \cdot z_{\upsilon,n}) + \Gamma^*_{\upsilon,n} \cdot (\partial/\partial z_{\upsilon,n} + t \cdot n \cdot \bar{z}_{\upsilon,n})]\right. \\ &+ \left.\sum_{0<n\in\mathbb{Z}-\upsilon/n_0} [\underline{\Gamma}_{\upsilon,n} \cdot (\partial/\partial z_{\upsilon,-n} - t \cdot n \cdot \bar{z}_{\upsilon,-n}) + \underline{\Gamma}^*_{\upsilon,n} \cdot (\partial/\partial \bar{z}_{\upsilon,-n} + t \cdot n \cdot z_{\upsilon,n})]\right) \\ &+ i \cdot \sum_{0<\upsilon\in n_0\cdot\mathbb{Z}} (\Gamma_{\upsilon,0} \cdot (\partial/\partial \bar{z}_{\upsilon,0} - s \cdot \upsilon/n_0 \cdot z_{\upsilon,0}) \\ &+ \Gamma^*_{\upsilon,0} \cdot (\partial/\partial z_{\upsilon,0} + s \cdot \upsilon/n_0 \cdot \bar{z}_{\upsilon,0})), \qquad (6.19)\end{aligned}$$

with the covariantly constant endomorphism T_Σ defined to be

$$T_\Sigma = \sum_{0<n\in\mathbb{Z}} [\Gamma_n \cdot (\partial/\partial \underline{x}_n - t\cdot n\cdot x_n) + \underline{\Gamma}_n^* \cdot (\partial/\partial x_n + t\cdot n\cdot \underline{x}_n)\,. \tag{6.20}$$

To explain the notation, let $\{\theta_\upsilon^j, \underline{\theta}_\upsilon^j, \theta^a\}$ be a local orthonormal basis for N^*, $\underline{N}^*$, and T^*M respectively. The creation and annihilation operators $\{\Gamma^*_{\upsilon,n}, \underline{\Gamma}^*_{\upsilon,n}, \Gamma_n^*\} \equiv \Gamma^{*j}_{\upsilon,n}$, $\underline{\Gamma}^{*j}_{\upsilon,n}, \Gamma^{*a}_\upsilon\}$ and $\{\Gamma_{\upsilon,n}, \underline{\Gamma}_{\upsilon,n}, \Gamma_n\} \equiv \{\Gamma^j_{\upsilon,n}, \underline{\Gamma}^j_{\upsilon,n}, \Gamma^a_n\}$ act, respectively, as exterior product by $\{\sqrt{2}\cdot\theta^j_{\upsilon,n}, \sqrt{2}\cdot\underline{\theta}^j_{\upsilon,-n}, \sqrt{2}\cdot\theta^a_n\}$ and interior product by $\{\sqrt{2}\cdot\underline{\theta}^j_{\upsilon,n}, \sqrt{2}\cdot\theta^j_{\upsilon,n}, \sqrt{2}\cdot\underline{\theta}^a_n\}$ on the Fock space $S(U;L)\otimes\mathscr{F}_{NM(n_0)}\otimes\underline{\mathscr{G}}_V|_\Sigma$ or $S(U';L')\otimes\underline{\mathscr{F}}_{NM(n_0)}\otimes\underline{\mathscr{G}}'_V|_\Sigma$ via the isomorphisms in Eqs. (6.13, 14).

The endomorphism P of Sect. 4 [see Eqs. (4.3, 8, 9)] acts on $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)$ and on $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)\otimes\Phi_{NM(n_0)\Sigma}$ as the covariantly constant endomorphism $P\equiv P(n_0)$ $\equiv P_B(n_0)+P_F(n_0)+P_V(n_0)$ with

$$\begin{aligned} P_B \equiv &-\sum_{0<n\in\mathbb{Z}} n\cdot(x_n\cdot\partial/\partial x_n - \underline{x}_n\cdot\partial/\partial\underline{x}_n)\\ &-\sum_\upsilon \sum_{n\in\mathbb{Z}+\upsilon/n_0} n\cdot(z_{\upsilon,n}\cdot\partial/\partial z_{\upsilon,n} - \underline{z}_{\upsilon,n}\cdot\partial/\partial\underline{z}_{\upsilon,n}), \end{aligned} \tag{6.21}$$

and

$$\begin{aligned} P_F \equiv -\tfrac{1}{2}\cdot\Big(&\sum_{0<n\in\mathbb{Z}} n\cdot\Gamma_n^*\cdot\Gamma_n + \sum_{0<\upsilon}\ \sum_{0<n\in\mathbb{Z}+\upsilon/n_0} n\cdot\Gamma^*_{\upsilon,n}\cdot\Gamma(\upsilon)_{\upsilon,n}\\ &+\sum_{0<\upsilon}\ \sum_{0<n\in\mathbb{Z}-\upsilon/n_0} n\cdot\underline{\Gamma}^*_{\upsilon,n}\cdot\underline{\Gamma}_{\upsilon,n}\Big). \end{aligned} \tag{6.22}$$

In the unprimed case,

$$P_V \equiv -\tfrac{1}{2}\cdot\Big(\sum_{0\leqq\upsilon}\ \sum_{0>n\in\mathbb{Z}+\upsilon/n_0} n\cdot\Theta_{\upsilon,n}\cdot\Theta_{\upsilon,n} + \sum_{0<\upsilon}\ \sum_{0>n\in\mathbb{Z}-\upsilon/n_0} n\cdot\underline{\Theta}^*_{\upsilon,n}\cdot\Theta_{\upsilon,n}\Big). \tag{6.23}$$

In the primed case,

$$\begin{aligned} P_V \equiv -\tfrac{1}{2}\cdot\Big(&\sum_{0\leqq\upsilon}\ \sum_{0>n\in\mathbb{Z}+\upsilon/n_0+1/2} n\cdot\Theta^*_{\upsilon,n}\cdot\Theta_{\upsilon,n}\\ &+\sum_{0<\upsilon}\ \sum_{0>n\in\mathbb{Z}-\upsilon/n_0-1/2} n\cdot\underline{\Theta}^*_{\upsilon,n}\cdot\underline{\Theta}_{\upsilon,n}\Big). \end{aligned} \tag{6.24}$$

For negative n, the creation and annihilation operators, $\{\Theta^*_{\upsilon,n}, \underline{\Theta}^*_{\upsilon,n}\} \equiv \{\Theta^{*A}_{\upsilon,n}, \underline{\Theta}^{*A}_{\upsilon,n}\}$ and $\{\Theta_{\upsilon,n}, \underline{\Theta}_{\upsilon,n}\} \equiv \{\Theta^A_{\upsilon,n}, \underline{\Theta}^A_{\upsilon,n}\}$, act as endomorphisms of the Fock spaces of Eqs. (6.13, 14). The former act as exterior multiplication by $\sqrt{2}\times$ the component $\{o^A_\upsilon, \underline{o}^A_\upsilon\}$ of an orthonormal frame for $V^*_{\upsilon,n}$ and $\underline{V}^*_{\upsilon,-n}$; the latter as interior multiplication by $\sqrt{2}\times$ the component $\{\underline{o}^A_\upsilon, o^A_\upsilon\}$ of an orthonormal frame for $\underline{V}_{\upsilon,n}$ and $V_{\upsilon-n}$.

The Lie algebra of the geometric circle action on $N\to\Sigma$ and on $V\to\Sigma$ has a lift to an action on $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)$ and on $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)\otimes\Phi_{NM(n_0)\Sigma}$ with generator $i\cdot K$, where $K\equiv K(n_0)\equiv K_B(n_0)+K_F(n_0)+K_V(n_0)+K_W$. Here, $K_B(n_0)$ acts as the covariantly constant endomorphism of $\underline{\mathscr{B}}_{NM(n_0)}(\Sigma)$ which is given by

$$K_B(n_0) \equiv -\sum_{0\leqq\upsilon}\Big(\sum_{n\in\mathbb{Z}+\upsilon/n_0} \upsilon/n_0\cdot z_{\upsilon,n}\cdot\partial/\partial z_{\upsilon,n} + \sum_{n\in\mathbb{Z}-\upsilon/n_0} \upsilon/n_0\cdot\underline{z}_{\upsilon,n}\cdot\partial/\partial\underline{z}_{\upsilon,n}\Big). \tag{6.25}$$

Both $K_F(n_0)$ and $K_V(n_0)$ are covariantly constant endomorphisms of $S(U; L)\otimes \underline{\mathscr{F}}_{NM(n_0)}\otimes \underline{\mathscr{G}}_V|_\Sigma$ or $S(U'; L')\otimes \underline{\mathscr{F}}_{NM(n_0)}\otimes \underline{\mathscr{G}}'_V|_\Sigma$. In either the primed or unprimed case,

$$\begin{aligned}K_F(n_0) \equiv & -\tfrac{1}{2}\cdot \sum_{0<\upsilon} \upsilon/n_0 \cdot \Big(\sum_{0\leqq n\in\mathbb{Z}+\upsilon/n_0} \Gamma^*_{\upsilon,n}\cdot \Gamma_{\upsilon,n} \\ & - \sum_{0<n\in\mathbb{Z}-\upsilon/n_0} \underline{\Gamma}^*_{\upsilon,n}\cdot \underline{\Gamma}_{\upsilon,n}\Big) \\ & +\tfrac{1}{2}\cdot \sum_{0\leqq \upsilon'<n_0/2}\ \sum_{0<\upsilon=\upsilon' \bmod(n_0)} \dim(N_\upsilon)\cdot \upsilon/n_0 \\ & -\tfrac{1}{2}\cdot \sum_{0<\upsilon'<n_0/2}\ \sum_{0<\upsilon=-\upsilon' \bmod(n_0)} \dim(N_\upsilon)\cdot \upsilon/n_0 \\ & -r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon/n_0\cdot \sum_{0<\upsilon=-\upsilon' \bmod(n_0)} \dim(N_\upsilon)\cdot \upsilon/n_0 \,. \end{aligned} \tag{6.26}$$

Here, again, $r=0$ if n is odd, and $r=1$ if n is even. In the unprimed case,

$$\begin{aligned}K_V(n_0) \equiv & -\tfrac{1}{2}\cdot \sum_{0<\upsilon} \upsilon/n_0 \cdot \Big(\sum_{0> n\in\mathbb{Z}+\upsilon/n_0} \Theta^*_{\upsilon,n}\cdot \Theta_{\upsilon,n} \\ & - \sum_{0\geqq n\in\mathbb{Z}-\upsilon/n_0} \underline{\Theta}^*_{\upsilon,n}\cdot \underline{\Theta}_{\upsilon,n}\Big) \\ & -\tfrac{1}{2}\cdot \sum_{0\leqq \upsilon'<n_0/2}\ \sum_{0\leqq \upsilon:\upsilon=\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \\ & +\tfrac{1}{2}\cdot \sum_{0<\upsilon'<n_0/2}\ \sum_{0\leqq \upsilon:\upsilon=-\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \\ & -r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon'/n_0\cdot \sum_{0<\upsilon=\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \\ & +r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon'/n_0\cdot \sum_{0<\upsilon=-\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \,. \end{aligned} \tag{6.27}$$

In the primed case,

$$\begin{aligned}K_V(n_0) \equiv & -\tfrac{1}{2}\cdot \sum_{0<\upsilon} \upsilon/n_0 \cdot \Big(\sum_{0> n\in\mathbb{Z}+\upsilon/n_0+1/2} \Theta^*_{\upsilon,n}\cdot \Theta_{\upsilon,n} \\ & - \sum_{0\geqq n\in\mathbb{Z}-\upsilon/n_0-1/2} \underline{\Theta}^*_{\upsilon,n}\cdot \underline{\Theta}_{\upsilon,n}\Big) \\ & +\tfrac{1}{2}\cdot \sum_{0\leqq \upsilon=n_0/2 \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \\ & -r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon/n_0\cdot \sum_{0<\upsilon=\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \\ & r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon/n_0\cdot \sum_{0<\upsilon=-\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon)\cdot \upsilon/n_0 \,. \end{aligned} \tag{6.28}$$

The endomorphism K_W of $W(\Sigma)$ is defined so that $i\cdot K_W$ is the real, skew endomorphism of $W(\Sigma)$ which generates the Lie algebra action of the n_0-root S^1 action on W. This K_W is covariantly constant, and hermitian.

As endomorphisms of $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)$ and of $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)\otimes \Phi_{NM(n_0)\Sigma}$, both P and K are hermitian, with discrete eigenvalues; their spectra do not have accumulation points. Furthermore, both commute with the endomorphism T_{n_0} of Eq. (6.19). As endomorphisms of $C^\infty(\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes\Phi_{NM(n_0)\Sigma})$, both P and K commute with Q_{n_0}. Decompose $\mathscr{E}_{NM(n_0),V}(\Sigma)$ as an orthogonal, direct sum of vector bundles $\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\to\Sigma$ on which P acts with eigenvalue m, and on which K acts with eigenvalue k. Let $L^2(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})$ and $H^1((\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})$ denote the Hilbert space completions of $C^\infty(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})$ with respect to the inner products

$$\langle\,,\rangle_{L^2}\equiv\int_\Sigma\langle\,,\rangle\quad\text{and}\quad\langle\,,\rangle_{H^1}\equiv\langle Q_{n_0}(),Q_{n_0}()\rangle_{L^2}+\langle\,,\rangle_{L^2}\,. \tag{6.29}$$

By construction, Q_{n_0} is a bounded operator from H^1 to L^2. The following summarizes Proposition A.1 in the present context.

Proposition 6.1. *Let $I\equiv\{(t,s)\in\mathbb{R}^2:|s|,t>0\}$. Let (m,k) be eigenvalues of P and K on $\mathscr{E}_{NM(n_0),V}(\Sigma)$. For $(s,t)\in I$, the operator Q_{n_0} defines a Fredholm map from $H^1(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})$ to $L^2(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})$. The kernel of the adjoint and the kernel of Q_{n_0} are both vector subspaces of $C^\infty(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})$; in fact, the same subspace. Both are empty for $m<0$.*

Define the index of Q_{n_0} on $H^1(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes_{NM(n_0)\Sigma})$ as follows: Introduce the covariantly constant automorphism $\ell\equiv\ell_e$ or ℓ_s, of $S(U)\otimes\mathscr{F}_{NM(n_0)}\otimes\mathscr{G}_V|_\Sigma$ or $S(U')\otimes\mathscr{F}_{NM(n_0)}\otimes\mathscr{G}'_V|_\Sigma$ from Eqs. (4.17, 18). This automorphism anti-commutes with Q_{n_0} and T_{n_0}, but it commutes with P and K. Extend the automorphism to $\mathscr{E}_{NM(n_0),V}(\Sigma)$. Then, define

$$\begin{aligned}&\mathrm{Ind}(Q_{n_0},\mathscr{E}_{NM(n_0),V}(\Sigma)_m,\ell,k)\\&\quad\equiv\dim(\ker(Q_{n_0})|_{\ker(\ell-1)}\cap H^1(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})\\&\qquad-\dim(\ker(Q_{n_0})|_{\ker(\ell+1)}\cap H^1(\mathscr{E}_{NM(n_0),V}(\Sigma)_m(k)\otimes\Phi_{NM(n_0)\Sigma})\,.\end{aligned} \tag{6.30}$$

An immediate consequence of Propositions 2.6 and 4.1, 4.2, and A.4 (via the discussion at the beginning of this section) is

Proposition 6.2. *Let M be a compact, oriented, even dimensional spin manifold which admits an isometric S^1 action. Let $V\to M$ be a real, oriented vector bundle with $w_2(V)=0$. Assume that the S^1 action on M lifts to an action on V. For integer $n_0>0$, let $NM(n_0)\subset M$ be fixed under the $\mathbb{Z}/n_0\cdot\mathbb{Z}$ subgroup of S^1. Let $W\to M(n_0)$ be a complex, finite dimensional vector bundle which is such that the restriction to $M(n_0)$ of a multiple of the geometric S^1 action has a lift. Make the assumptions of Proposition 5.3. As specified in Proposition 5.3, make the construction in Sect. 4 of $\mathscr{E}_{NM(n_0),V}\to M(n_0)$ and let m be an eigenvalue of P acting on $\mathscr{E}_{NM(n_0),V}\to M$. Construct the operator D_t on $C^\infty(\mathscr{E}_{NM(n_0),Vm}\otimes\Phi_{NM(n_0)})$. Let k be an eigenvalue of K on $C^\infty(\mathscr{E}_{NM(n_0),Vm}\otimes\Phi_{NM(n_0)})$ and define the index $\mathrm{Ind}(D;\mathscr{E}_{E,Vm},\ell,k)$ as in Eq. (6.1) using $\ell\equiv\ell_e$ or ℓ_s of Eqs. (4.17, 18). Let $I\equiv\{(t,s)\in\mathbb{R}:t>0$ and $s\neq0\}$. Let $\{i\}$ label the connected components in M of the fixed point set of the geometric S^1 action. For $(s,t)\in I$, and for each i, let $\mathrm{Ind}(Q_{n_0},\mathscr{E}_{NM(n_0),V}(\Sigma[i])_m,\ell,k)$ be defined by Eq. (6.30) for the i^{th} component. Then, the number $\mathrm{Ind}(Q_{n_0},\mathscr{E}_{NM(n_0),V}(\Sigma[i])_m,\ell,k)$ is constant on the connected components of I. Furthermore,*

$$\mathrm{Ind}(D;\mathscr{E}_{E,Vmq},\ell,k)=\sum_i\mathrm{Ind}(Q_{n_0},\mathscr{E}_{NM(n_0),V}(\Sigma[i])_m,\ell,k)\,.$$

For future reference, it is useful to decompose the endomorphism $\ell_{e,s}$ with respect to the decompositions of Eqs. (6.13, 14). Consider first that the volume form on M restricts to Σ where it is given in the coordinates of Eq. (2.22) as

$$\mathrm{vol}(M)|_{\Sigma} = \mathrm{vol}(\Sigma) \bigwedge_{j:0<\upsilon(j)} i/2 \cdot (dz^j \wedge d\underline{z}^j). \tag{6.31}$$

The volume form on $M(n_0)$ is defined from that on M; and due to the complex conjugate bundles in Eq. (6.4),

$$\mathrm{vol}(M(n_0))|_{\Sigma} = (-1)^{\Delta} \cdot \mathrm{vol}(\Sigma) \bigwedge_{j:0<\upsilon(j)=0\,\mathrm{mod}(n_0)} i/2 \cdot (dz^j \wedge d\underline{z}^j). \tag{6.32}$$

Here

$$\Delta \equiv \Delta(n_0, \Sigma) = \sum_{n_0/2<\upsilon<n_0} \sum_{0<\upsilon':\upsilon'=\upsilon\,\mathrm{mod}(n_0)} \dim_{\mathbb{C}}(N(\upsilon')) + o(NM(n_0, n_0/2)), \tag{6.33}$$

with $o(NM(n_0; n_0/2) \in \{0,1\}$. The value depends on whether the given orientation on $NM(n_0; n_0/2)$ agrees (0) or disagrees (1) with the complex orientation of $\bigoplus_{0<\upsilon=n_0/2\,\mathrm{mod}(n_0)} N(\upsilon)$.

From Eqs. (6.32, 33), one finds

$$\ell_s(M(n_0)) = (-1)^{\Delta} \cdot \gamma_{\Sigma} \otimes (-1)^{F_0}, \tag{6.34}$$

where γ_{Σ} is Clifford multiplication by $(i)^{\sigma(\Sigma)} \cdot d\,\mathrm{vol}_{\Sigma}$ with $\sigma(\Sigma) \equiv \dim_{\mathbb{R}}(\Sigma) \cdot (\dim_{\mathbb{R}}(\Sigma)+1)/2$, and where $(-1)^{F_0}$ measures the degree of an element in the exterior algebra

$$\Lambda^*(\mathcal{N}(n_0)^*_+ \oplus \underline{\mathcal{N}(n_0)^*_-} \bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} N^*_{\upsilon,0}).$$

To discuss ℓ_e in the unprimed case, let

$$\delta = \delta(n_0, \Sigma, V) \equiv \sum_{n_0/2<\upsilon<n_0} \sum_{0<\upsilon':\upsilon'=\upsilon\,\mathrm{mod}(n_0)} \dim_{\mathbb{C}}(V_{\upsilon'}) + o(V(n_0; n_0/2)), \tag{6.35}$$

with $o(V(n_0; n_0/2)) \in \{0,1\}$; the value depends on whether the orientation on $V(n_0; n_0/2)$ agrees (0) or disagrees (1) with the complex orientation of $\bigoplus_{0<\upsilon=n_0/2\,\mathrm{mod}(n_0)} V_{\upsilon}$.

From Eqs. (6.35), one finds

$$\ell_e(M(n_0)) = (-1)^{\Delta} \cdot (-1)^{\delta} \cdot \gamma_{\Sigma,V} \otimes (-1)^{F_0+G_0}, \tag{6.36}$$

where $\gamma_{\Sigma,V}$ is Clifford multiplication by $(i)^{\sigma(\Sigma,V)} \times$ (volume form on $T^*\Sigma \oplus V(0)_R$) with

$$\sigma(\Sigma, V) \equiv \dim_{\mathbb{R}}(T^*\Sigma \oplus V(0)_R) \cdot (\dim_{\mathbb{R}}(T^*\Sigma \oplus V(0)_R)+1)/2$$

and where $(-1)^{F_0+G_0}$ measures the degree of a form in the exterior algebra

$$\Lambda^* \left(\mathcal{N}(n_0)^*_+ \oplus \underline{\mathcal{N}(n_0)^*_-} \bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} N^*_{\upsilon,0}\right)$$
$$\times \otimes \Lambda^* \left(\mathcal{V}(n_0)^*_- \oplus \underline{\mathcal{V}(n_0)^*_+} \bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} \underline{V}^*_{\upsilon,0}\right).$$

To discuss ℓ_e in the primed case, set $\delta'(n_0, \Sigma, V) \equiv o(V(n_0; n_0/2))$. Then

$$\ell_e(M(n_0)) = (-1)^{\Delta} \cdot (-1)^{\delta'} \cdot (\gamma_s(\Sigma) \otimes (-1)^{F_0+G_0'}), \tag{6.37}$$

where $\gamma_e(\Sigma)$ is Clifford multiplication by the volume form on $T^*\Sigma \oplus V(0)_R$, and where $(-1)^{F_0+G_0'}$ measures the degree of a form in the exterior algebra

$$\Lambda^*\left(\mathcal{N}(n_0)^*_+ \oplus \underline{\mathcal{N}(n_0)}^*_- \bigoplus_{0<\upsilon:\upsilon=0\,\mathrm{mod}(n_0)} N^*_{\upsilon,0}\right)$$
$$\times \otimes \Lambda^*\left(\mathcal{V}'(n_0)^*_- \oplus \underline{\mathcal{V}'(n_0)}^*_+ \bigoplus_{0<\upsilon:\upsilon=n_0/2\,\mathrm{mod}(n_0)} \underline{V}^*_{\upsilon,0}\right).$$

7. The Shift Operator

As a vector bundle over Σ, $\mathcal{N} \equiv NM(1)$ in Eq. (6.6) admits the $\mathbb{Z}$ subgroup of automorphisms whose generator sends

$$\imath : N_{\upsilon,n} \to N_{\upsilon,n+\upsilon}. \tag{7.1}$$

The generator on $\underline{\mathcal{N}}$ is defined to commute with the complex conjugation map from $\underline{\mathcal{N}}$ to $\mathcal{N}$ so that $\imath$ defines an automorphism of the underlying real vector bundle. Note that $\imath$ commutes with the geometric S^1 action on $\mathcal{N}$.

Introduce from Sect. 6 the character decomposition of $V|_\Sigma$: $V \equiv V_{0R} \oplus_{\upsilon>0} V_\upsilon$. This induces the character decompositions in Eqs. (6.7, 8) of the vector bundles $\mathcal{V}(1)$ and $\mathcal{V}'(1)$.

The $\mathbb{Z}$-subgroup of automorphisms on $\mathcal{N}$ extends to define a subgroup of automorphisms of $\mathcal{V}(1)$ and of $\mathcal{V}'(1)$ by requiring the generator, $\imath$, to act according to the following rule: For integer $\upsilon > 0$, by the natural identification

$$\imath : V_{\upsilon,n} \to V_{\upsilon,n+\upsilon}, \tag{7.2a}$$

and to commute with complex conjugation,

$$\imath : \underline{V}_{\upsilon,n} \to \underline{V}_{\upsilon,n+\upsilon}. \tag{7.2b}$$

For a real or complex vector bundle $E \to \Sigma$, let $\det(E) \to \Sigma$ denote the line bundle $\Lambda^{\dim(E)}E$. For a real, oriented vector bundle $E \to \Sigma$ on which S^1 acts, let $E = E_{0R} \oplus_{\upsilon>0} E_\upsilon$ be the decomposition into the character subbundles. Thus, E_{0R} is the real, S^1-invariant subbundle, and $E_\upsilon \to \Sigma$ is complex with υ defining the S^1 action on E_υ. Define the line bundle

$$L(E) \equiv \prod_{\upsilon>0} \det(E)^\upsilon \to \Sigma. \tag{7.3}$$

Also, let

$$e(E) \equiv -\sum_{\upsilon>0} \upsilon^2 \cdot \dim(E_\upsilon). \tag{7.4}$$

The topological significance of this data is described in Lemma 7.5.

Proposition 7.1. *Introduce the vector bundles* $S(U,L) \otimes \mathcal{F}_{NM(1)} \otimes \underline{\mathcal{G}}_V|_\Sigma$ *and* $S(U',L') \otimes \mathcal{F}_{NM(1)} \otimes \underline{\mathcal{G}}_V|_\Sigma$ *and* $\underline{\mathcal{B}}_{NM(1)}(\Sigma)$ *of Eqs.* (6.13–15). *For a bundle* $E \to \mathcal{N}$, *let* $\imath^*(E) \to \mathcal{N}$ *denote the bundle which is pulled back by the map* $\imath$ *of* $\mathcal{N}$ *to itself. Then, there are the natural isomorphisms*

1) $\imath^*(\underline{\mathcal{B}}_{NM(1)}(\Sigma)) \approx \underline{\mathcal{B}}_{NM(1)}(\Sigma)$.
2) $\imath^*(S(U,L) \otimes \mathcal{F}_{NM(1)} \otimes \underline{\mathcal{G}}_V|_\Sigma)$
$\approx S(U,L) \otimes \mathcal{F}_{NM(1)} \otimes \underline{\mathcal{G}}_V|_\Sigma \otimes L(N^*)^{-1} \otimes L(\underline{V}^*)^{-1}$.
3) $\imath^*(S(U',L) \otimes \mathcal{F}_{NM(1)} \otimes \underline{\mathcal{G}}'_V|_\Sigma)$
$\approx S(U',L) \otimes \mathcal{F}_{NM(1)} \otimes \underline{\mathcal{G}}'_V|_\Sigma \otimes L(N^*)^{-1} \otimes L(\underline{V}^*)^{-1}$.

Proof of Proposition 7.1. Consider the first assertion. As a bundle over Σ, $\mathscr{B}_{NM(1)}(\Sigma)$ has fiber over x the set of finite polynomial functions on the fiber of $\mathscr{N}$ over x. The action of $\imath$ on $\mathscr{N}$ is a linear action, so a finite polynomial function is pulled back to a finite polynomial.

The second and third assertions of the proposition are direct consequences of the following lemma:

Lemma 7.2. *Introduce the vector bundles*

$$\underline{\mathscr{F}}(\Sigma) \equiv \Lambda^* \left(\bigoplus_{0<\upsilon} \left(\bigoplus_{0 \leqq n \in \mathbb{Z}} N^*_{\upsilon,n} \bigoplus_{0<n\in\mathbb{Z}} \underline{N}^*_{\upsilon,-n} \right)\right) \to \Sigma$$

and

$$\underline{\mathscr{G}}(\Sigma) \equiv \Lambda^* \left(\bigoplus_{0<\upsilon} \left(\bigoplus_{0 \geqq n \in \mathbb{Z}} \underline{V}^*_{\upsilon,n} \bigoplus_{0>n\in\mathbb{Z}} V^*_{\upsilon,-n} \right)\right)$$

and

$$\underline{\mathscr{G}}'(\Sigma) \equiv \Lambda^* \left(\bigoplus_{0<\upsilon} \left(\bigoplus_{0 \geqq n \in \mathbb{Z}+1/2} \underline{V}^*_{\upsilon,n} \bigoplus_{0>n\in\mathbb{Z}+1/2} V^*_{\upsilon,-n} \right)\right).$$

Then, there are natural vector bundle isomorphisms
1) $\imath^*(\underline{\mathscr{F}}(\Sigma)\otimes L(N^*)^p) \approx \underline{\mathscr{F}}(\Sigma)\otimes L(N^*)^{p-1}$.
2) $\imath^*(\underline{\mathscr{G}}(\Sigma)\otimes L(\underline{V}^*)^p) \approx \underline{\mathscr{G}}(\Sigma)\otimes L(\underline{V}^*)^{p-1}$.
3) $\imath^*(\underline{\mathscr{G}}'(\Sigma)\otimes L(\underline{V}^*)^p) \approx \underline{\mathscr{G}}'(\Sigma)\otimes L(\underline{V}^*)^{p-1}$.

Proof of Lemma 7.2. To see the assertion for $\underline{\mathscr{F}}(\Sigma)$, group the terms as

$$\underline{\mathscr{F}}(\Sigma) \equiv \bigotimes_{\upsilon,n>0} (\Lambda^* N^*_{\upsilon,n} \otimes \Lambda^* \underline{N}^*_{\upsilon,-n}) \bigotimes_{\upsilon>0} \Lambda^* N^*_{\upsilon,0}\,. \tag{7.5}$$

The pull back under $\imath$ of $\underline{\mathscr{F}}(\Sigma)$ can be readily computed to be

$$\imath^*\underline{\mathscr{F}}(\Sigma) \equiv \bigotimes_{\upsilon,n>0} (\Lambda^* N^*_{\upsilon,n+\upsilon} \otimes \Lambda^* \underline{N}^*_{\upsilon,-n+\upsilon}) \bigotimes_{\upsilon>0} \Lambda^* N^*_{\upsilon,\upsilon}\,. \tag{7.6}$$

The right-hand side above is not yet in the required form. To put it in the correct form, note that the hermitian metric on N induces a natural, $\mathbb{C}$-linear isomorphism $\Lambda^*\underline{N}^*_\upsilon \approx \Lambda^* N^*_\upsilon \otimes \det(\underline{N}^*_\upsilon)$. And, for $\alpha \geqq 0$, this last isomorphism induces a natural ($\mathbb{C}$-linear) isomorphism $\Lambda^*\underline{N}^*_{\upsilon,-\alpha} \approx \Lambda^* N^*_{\upsilon,\alpha-1} \otimes \det(\underline{N}^*_\upsilon)$. This last fact with Eq. (7.5) gives immediately the assertion for $\underline{\mathscr{F}}(\Sigma)$.

The assertions for $\underline{\mathscr{G}}(\Sigma)$ and for $\underline{\mathscr{G}}'(\Sigma)$ are proved by analogous arguments.

As a parenthetical remark, note that $\imath$ induces an automorphism of $\underline{\mathscr{F}}(\Sigma)$ only when the line bundle $L(N) \to \Sigma$ is the trivial bundle. That is, only when the first Chern class $c_1(L(N)) = 0$. Similarly, $\imath$ induces an automorphism of $\underline{\mathscr{G}}(\Sigma)$ and of $\underline{\mathscr{G}}'(\Sigma)$ only when $c_1(L(\underline{V})) = 0$. Of course, $\imath$ induces an automorphism of $\underline{\mathscr{E}}_{NM(1),V}(\Sigma)$ of Eq. (6.16) only when $c_1(L(N)) + c_1(L(\underline{V})) = 0$. The vanishing of this Chern class is implied by a global condition on the S^1 action; see Lemma 7.5 below.

The generators of the canonical S^1 action and of the geometric S^1 action define commuting endomorphisms of $\underline{\mathscr{E}}_{NM(1),V}(\Sigma)\otimes L(N^*)^p \otimes L(\underline{V}^*)^p$. The canonical S^1 action on $\underline{\mathscr{E}}_{NM(1),V}(\Sigma)\otimes L(N^*)^p \otimes L(\underline{V}^*)^p$ is generated by $-i\cdot P$, where $P = P(1)_B + P(1)_F + P(1)_V$ as defined in Eqs. (6.21–24). The geometric S^1 action is generated by $-i\cdot K$, where $K = K(1)_B + K(1)_F + K(1)_V + p\cdot(e(N^*) - e(\underline{V}^*))$, as defined in Eqs. (6.25–28).

Proposition 7.3. *The bundle isomorphism*

$$\imath^*: \underline{\mathscr{E}}_{NM(1),V}(\Sigma)\otimes L(N^*)^p\otimes L(\underline{V}^*)^p \to \underline{\mathscr{E}}_{NM(1),V}(\Sigma)\otimes L(N^*)^{p-1}\otimes L(\underline{V}^*)^{p-1}$$

obeys

1) $K\cdot \imath^* = \imath\cdot K$.
2) $P\cdot \imath^* = \imath^*\cdot P + \imath^* K + (-p+1/2)\cdot(e(N^*)-e(\underline{V}^*))$.

The topological significance of $1/2(e(N^*)-e(\underline{V}^*))$ is described in Lemma 7.5.

Proof of Proposition 7.3. Consider the separate effects of $\imath^*$ on $K(1)_{B,F,V}$ and its effects separately on $P(1)_{B,F,V}$. These are described in the next lemma from which the proposition follows as a corollary.

Lemma 7.4. (1) *The bundle isomorphism* $\imath^*: \underline{\mathscr{B}}_{NM(1)}(\Sigma)\to\underline{\mathscr{B}}_{NM(1)}(\Sigma)$ *obeys* $K(1)_B\cdot\imath^* = \imath^* K(1)_B$ *and* $P(1)_B\cdot\imath^* = \imath^* P(1)_B + \imath^* K(1)_B$.
(2) *The bundle map* $\imath^*: \underline{\mathscr{F}}\otimes L(N^*)^p \to \underline{\mathscr{F}}\otimes L(N^*)^{p-1}$ *obeys* $K(1)_F\cdot\imath^* = \imath^* K(1)_F + e(N^*)$ *and* $P(1)_F\cdot\imath^* = \imath^* P(1)_F + \imath^* K(1)_F + 1/2e(N^*)$.
(3) *Let* $\underline{\mathscr{G}}^0$ *denote either* $\underline{\mathscr{G}}(\Sigma)$ *or* $\underline{\mathscr{G}}'(\Sigma)$. *Then, the bundle map* $\imath^*: \underline{\mathscr{G}}^0\otimes L(\underline{V}^*)^p \to \underline{\mathscr{G}}^0\otimes L(\underline{V}^*)^{p-1}$ *obeys* $K(1)_V\cdot\imath^* = \imath^* K(1)_V\cdot\imath^* = \imath^* K(1)_V - e(\underline{V}^*)$ *and* $P(1)_V\cdot\imath^* = \imath^* P(1)_V + \imath^* K(1)_V - 1/2e(\underline{V}^*)$.

Proof of Lemma 7.4. A direct computation using Eqs. (6.25–28) establishes the commutation relations of $K(1)_*$ with $\imath^*$. This is left to the reader to check. The effect of $\imath^*$ on $P(1)_*$ is more subtle. Consider first the case of $\underline{\mathscr{G}}^0(\Sigma)$. For $n\leqq 0$ and $\upsilon>0$,

$$\begin{aligned} \imath^{-1*}\underline{\Theta}_{\upsilon,n}\imath^* &= \underline{\Theta}_{\upsilon,n+\upsilon} \quad\text{and}\quad \imath^{-1*}\underline{\Theta}^*_{\upsilon,n}\imath^* = \underline{\Theta}^*_{\upsilon,n+\upsilon}, \\ \imath^{-1*}\underline{\Theta}_{\upsilon,n}\imath^* &= \underline{\Theta}_{\upsilon,n-\upsilon} \quad\text{and}\quad \imath^{-1*}\underline{\Theta}^*_{\upsilon,n}\imath^* = \underline{\Theta}^*_{\upsilon,n-\upsilon}. \end{aligned} \tag{7.7}$$

For $\upsilon>0$, and for $n\geqq-\upsilon$; this equation should be interpreted with the identity

$$\Lambda^*\underline{V}^*_{-n+\upsilon} \approx \Lambda^* V_{-n+\upsilon}\otimes\det(\underline{V}^*_\upsilon). \tag{7.8}$$

thus, for $\upsilon>0$ and $n\geqq-\upsilon$,

$$\underline{\Theta}_{\upsilon,n+\upsilon} = \Theta^*_{\upsilon,-n-\upsilon} \quad\text{and}\quad \underline{\Theta}^*_{\upsilon,n+\upsilon} = \Theta_{\upsilon,-n-\upsilon}. \tag{7.9}$$

Using these last identities, and Eq. (6.24), one computes in the unprimed case that

$$\begin{aligned} \imath^{-1*}P(1)_V\imath^* &= -\tfrac{1}{2}\cdot\Big(\sum_{0\leqq\upsilon}\ \sum_{0\geqq n\in\mathbb{Z}} n\cdot\Theta^*_{\upsilon,n-\upsilon}\cdot\Theta_{\upsilon,n-\upsilon} + \sum_{0<\upsilon}\ \sum_{0\geqq n\in\mathbb{Z}} n\cdot\underline{\Theta}^*_{\upsilon,n+\upsilon}\cdot\underline{\Theta}_{\upsilon,n+\upsilon}\Big), \\ &= P(1)_V + K(1)_V - \sum_{0<\upsilon}\ \sum_{0\leqq m<\upsilon}(m-\upsilon)\cdot\dim(\underline{V}_\upsilon) - \tfrac{1}{2}\cdot\sum_{\upsilon>0}\upsilon\cdot\dim(\underline{V}_\upsilon), \\ &= P(1)_V + K(1)_V + \tfrac{1}{2}\cdot\sum_{\upsilon>}\upsilon^2\cdot\dim(\underline{V}_\upsilon). \end{aligned} \tag{7.10}$$

where the middle line follows from the first using Eq. (7.9), Eq. (6.27) and the anti-commutation relations for the Clifford algebra.

For the $\mathscr{G}'(\Sigma)$ case, one has from Eq. (6.28)

$$\begin{aligned}\imath^{-1*}P(1)_V\imath^* &= -\tfrac{1}{2}\cdot\sum_{0\geqq n\in\mathbb{Z}+1/2}\left(\sum_{0\leqq\upsilon} n\cdot\Theta^*_{\upsilon,n-\upsilon}\cdot\Theta_{\upsilon,n-\upsilon}\right.\\ &\quad\left.+\sum_{0<\upsilon} n\cdot\underline{\Theta}^*_{\upsilon,n+\upsilon}\cdot\underline{\Theta}_{\upsilon,n+\upsilon}\right),\\ &= P(1)_V+K(1)_V-\sum_{0<\upsilon}\sum_{0\leqq m<\upsilon}(m-\upsilon-\tfrac{1}{2})\cdot\dim(\underline{V}_\upsilon),\\ &= P(1)_V+K(1)_V+\tfrac{1}{2}\cdot\sum_{\upsilon>0}\upsilon^2\cdot\dim(\underline{V}_\upsilon).\end{aligned} \tag{7.11}$$

Equation (7.11) implies the assertion for the $\mathscr{G}'(\Sigma)$ case.

Consider the case for $\mathscr{F}(\Sigma)$. The effect of $\imath$ on the Clifford algebra is summarized by

$$\begin{aligned}&\imath^{-1*}\Gamma_{\upsilon,n}\imath^*=\Gamma_{\upsilon,n-\upsilon} \quad\text{and}\quad \imath^{-1*}\Gamma^*_{\upsilon,n}\imath^*=\Gamma^*_{\upsilon,n-\upsilon},\\ &\imath^{-1*}\underline{\Gamma}_{\upsilon,n}\imath^*=\underline{\Gamma}_{\upsilon,n+\upsilon} \quad\text{and}\quad \imath^{-1*}\underline{\Gamma}^*_{\upsilon,n}\imath^*=\underline{\Gamma}^*_{\upsilon,n+\upsilon}.\end{aligned} \tag{7.12}$$

For $0\leqq n\leqq\upsilon$; the first equation should be interpreted with Eq. (7.13) below which identifies

$$\Lambda^*N^*_{\upsilon,-n}\approx\Lambda^*\underline{N}^*_{\upsilon,-n}\otimes\det(N^*_\upsilon). \tag{7.13}$$

Thus, for $0\leqq n\leqq\upsilon$,

$$\Gamma_{\upsilon,n-\upsilon}=\underline{\Gamma}_{\upsilon,-n+\upsilon} \quad\text{and}\quad \Gamma^*_{\upsilon,n-\upsilon}=\underline{\Gamma}^*_{\upsilon,-n+\upsilon}. \tag{7.14}$$

With these last three identities, one can mimic the preceding calculations to obtain from Eq. (6.22) the identity

$$\imath^{-1*}P(1)_F\imath^*=P(1)_F+K(1)_F-\tfrac{1}{2}\cdot\sum_{\upsilon>0}\upsilon^2\cdot\dim(N_\upsilon). \tag{7.15}$$

For the Bosonic part, use Eq. (6.21) to derive the identity

$$\begin{aligned}\imath^{-1*}P(1)_B\imath^* &= -\sum_{n>0} n\cdot(x_n\cdot\partial/\partial x_n-\underline{x}_n\cdot\partial/\partial\underline{x}_n)\\ &\quad+\sum_{\upsilon>0}\sum_n(n+\upsilon)\cdot(z_{\upsilon,n}\cdot\partial/\partial z_{\upsilon,n}-\underline{z}_{\upsilon,n}\cdot\partial/\partial\underline{z}_{\upsilon,n}).\end{aligned} \tag{7.16}$$

The right-hand side of Eq. (7.16) equals $P(1)_B+K(1)_B$ where $K(1)_B$ is defined in Eq. (6.25).

The numbers $e(N^*)$ and $e(\underline{V}^*)$ and the Chern classes $c_1(L(N^*))$ and $c_1(L(\underline{V}^*))$ in Eqs. (7.3, 4) have a global topological interpretation:

Lemma 7.5. *Let M be a compact Riemannian manifold on which S^1 acts isometrically and let $E\to M$ be a vector bundle to which the S^1 action lifts. Let $p_1(S^\infty\times_{S^1}E)\in H^4(S^\infty\times_{S^1}M)$ denote the first Pontrjagin class of the vector bundle $S^\infty\times_{S^1}E$ (the S^1 equivariant, first Pontrjagin class of E). Let $\Sigma\subset M$ be a connected component of the fixed point set of the S^1 action. Pull back $p_1(S^\infty\times_{S^1}E)$ to a cohomology class on $S^\infty\times_{S^1}\Sigma\equiv\mathbb{C}P^\infty\times\Sigma$. Let π, π' denote the projections to $\mathbb{C}P^\infty$ and to Σ, respectively. Then $p_1(S^\infty\times_{S^1}E)$ in $H^4(\mathbb{C}P^\infty\times\Sigma)$ is equal to $\pi'^*p_1(E)+2\cdot\pi^*u\wedge\pi'^*c_1(L(E))-e(E)\cdot\pi^*(u\wedge u)$ where u is the generator of $H^2(\mathbb{C}P^\infty)$.*

Proof of Lemma 7.5. The first Pontrjagin class of a real vector bundle V is equal to $-c_2(V\otimes\mathbb{C})$ with c_2 denoting the 2nd Chern class. In the present context, V is the restriction to $\mathbb{C}P^\infty\times\Sigma$ of $S^\infty\times_{S^1}E$; and $V\otimes\mathbb{C}\approx\oplus_v(\pi'^*E_v\otimes\pi^*H^v)$ where $H\equiv S^\infty\times_{S^1}\mathbb{C}\to\mathbb{C}P^\infty$. The assertion now follows by direct calculation.

Let L denote the complex line bundle $L(N^*)\otimes L(\underline{V}^*)$, and let $\underline{\mathscr{E}}^p$ denote $\underline{\mathscr{E}}_{NM(1),V}(\Sigma)\otimes L^p$. According to Lemmas 7.1 and 7.2, $\imath^*:\underline{\mathscr{E}}^p\to\underline{\mathscr{E}}^{p-1}$. In Sect. 6, automorphisms $\ell\equiv\ell_e$ or ℓ_s of the bundle $\underline{\mathscr{E}}^0$ were defined. Define ℓ on $\underline{\mathscr{E}}^p$ by setting $\ell(\Psi\otimes s)\equiv\ell\Psi\otimes s$. The shift $\imath^*$ does not necessarily commute with ℓ; their relationship is described by

Lemma 7.6. *Consider* $\imath^*:\underline{\mathscr{E}}^p\to\underline{\mathscr{E}}^{p-1}$ *and* $\ell_e,\ell_s:\underline{\mathscr{E}}^p\to\underline{\mathscr{E}}^p$ *as described above. Then,* $\imath^*\ell_e=(-1)^{e(N^*)-e(\underline{V}^*)}\cdot\ell_e\imath^*$ *and* $\imath\ell_s=(-1)^{e(N^*)}\cdot\ell_s\imath^*$.

Proof of Lemma 7.6. This follows from Eqs. (7.8, 13) and the fact that an integer n obeys $n=n^2\bmod(2)$.

In Sect. 6, an operator $Q_1:C^\infty(\underline{\mathscr{E}}^0\otimes\Phi_{NM(1)\Sigma})\to C^\infty(\underline{\mathscr{E}}^0\otimes\Phi_{NM(1)\Sigma})$ was introduced. Since the line bundle $L\to\Sigma$ has a natural metric and metric compatible connection, one can, by twisting in the usual way, define $Q_1:C^\infty(\underline{\mathscr{E}}^p\otimes\Phi_{NM(1)\Sigma})\to C^\infty(\underline{\mathscr{E}}^p\otimes\Phi_{NM(1)\Sigma})$. Use Eq. (6.18) but with the covariant derivative on $\underline{\mathscr{E}}^p$.

A key idea of Witten [W2] is to compare Q_1 with $\imath^{-p*}Q_1\imath^{p*}$. This will be done in Sects. 8 and 9. The comparison is complicated by the fact that $\imath^*$ does not fix $\Phi_{NM(1)\Sigma}$. But, by definition, one has

$$\imath^*:\underline{\mathscr{E}}^p\otimes\imath^{-1}\Phi_{NM(1)\Sigma}\to\underline{\mathscr{E}}^{p-1}\otimes\Phi_{NM(1)\Sigma}\,. \tag{7.17}$$

Lemma 7.7. *Define the function* $\Phi_{NM(1)\Sigma}$ *using Eq.* (6.17). *For integer p, define* $Q_1^p\equiv\imath^{-p*}Q_1\imath^{p*}$ *as an endomorphism of* $C^\infty(\underline{\mathscr{E}}^p\otimes\imath^{-p}\Phi_{NM(1)\Sigma})$. *For* $p>0$,

$$\begin{aligned}\imath^{-p}\Phi_{NM(1)\Sigma}=\exp\Big(&-t\cdot\sum_{m>0}m\cdot|x_m|^2-t\cdot\sum_{v>0}\sum_{n\neq-pv}\\&\times|n+p\cdot v|\cdot|z_{v,n}|^2-|s|\cdot p\cdot\sum_{v>0}v\cdot|z_{v,-pv}|^2\Big),\end{aligned} \tag{7.18}$$

$$\begin{aligned}Q_1^p\equiv Q_\Sigma+i\cdot\sum_v\sum_{n>0}&(\Gamma_{v,n}\cdot(\partial/\partial\underline{z}_{v,n}-t\cdot(n+pv)\cdot z_{v,n})\\&+\Gamma_{v,n}^*\cdot(\partial/\partial z_{v,n}+t\cdot(n+pv)\cdot\underline{z}_{v,n}))\\&+i\cdot\sum_v\sum_{pv\neq n>0}(\underline{\Gamma}_{v,n}\cdot(\partial/\partial z_{v,-n}-t\cdot(n-pv)\cdot\underline{z}_{v,-n})\\&+\underline{\Gamma}_{v,n}^*\cdot(\partial/\partial\underline{z}_{v,-n}+t\cdot(n-pv)\cdot z_{v,-n}))\\&+i\cdot\sum_v(\underline{\Gamma}_{v,pv}\cdot(\partial/\partial z_{v,-pv}+s\cdot pv\cdot\underline{z}_{v,-pv})\\&+\underline{\Gamma}_{v,pv}^*\cdot(\partial/\partial\underline{z}_{v,-pv}-s\cdot pv\cdot z_{v,-pv}))\\&+i\cdot\sum_v(\Gamma_{v,0}\cdot(\partial/\partial\underline{z}_{v,0}-t\cdot pv\cdot z_{v,0})\\&+\Gamma_{v,0}^*\cdot(\partial/\partial z_{v,0}+t\cdot pv\cdot\underline{z}_{v,0}))\,,\end{aligned} \tag{7.19}$$

where $Q_\Sigma\equiv D_\Sigma+T_\Sigma$ *as defined in Eq.* (6.20). *An analogous equation holds for* $p<0$.

Proof of Lemma 7.7. This is a direct calculation which is left to the reader.

For integers p and p', the function $\imath^{-p*}\Phi_{NM(1)\Sigma}$ defines a fiber metric on $\mathscr{B}_{NM(1)}(\Sigma)$ and this metric with the fiber metrics on $S(U,L)\otimes\mathscr{F}_{NM(1)}\otimes\mathscr{G}_V|_\Sigma$, on $S(U',L')\otimes\mathscr{F}_{NM(1)}\otimes\mathscr{G}'_V|_\Sigma$ and on L induce a fiber metric on $\underline{\mathscr{E}}^p$. The endomorphisms P and K of $\underline{\mathscr{E}}^p$ act symmetrically on $\underline{\mathscr{E}}^p$ and they decompose $\underline{\mathscr{E}}^p$ into a direct sum of subbundles, $\{\underline{\mathscr{E}}^p_m(k)\to\Sigma\}$ on which P and K act as multiplication by m and k, respectively.

For integers p and p', construct the operator $Q_1^{p'}$ on $C^\infty(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'}\Phi_{NM(1)\Sigma})$ as described above. The automorphisms $\ell\equiv\ell_e$ or ℓ_s of Lemma 7.6 act as involutions which commute with P and K. Hence, they restrict to involutions of $\underline{\mathscr{E}}^p_m(k)$ and of $C^\infty(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma})$ which anti-commute with $Q_1^{p'}$.

Use the fiber metric on $\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma}$ to mimic Eq. (6.29) and define the Hilbert spaces $H^1(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma})$ and $L^2(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma})$.

The following proposition summarizes this section:

Proposition 7.8. *For $t, |s|>0$ and for each pair of integers (p,p'), the operator $Q_1^{p'}$ on $C^\infty(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma})$ extends to define a Fredholm operator from $H^1(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma})$ to $L^2(\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma})$. Both the kernel and the cokernel of $Q_1^{p'}$ comprise the same vector space of smooth sections. For the involution $\ell\equiv\ell_e$, ℓ_s, define the integer* $\mathrm{Ind}(Q_1^{p'}, \underline{\mathscr{E}}^p_m, \ell, k)$ *using Eq. (6.30) with $Q_1^{p'}$ and $\underline{\mathscr{E}}^p_m(k)\otimes\imath^{-p'*}\Phi_{NM(1)\Sigma}$ replacing Q_1 and $\underline{\mathscr{E}}^0_m(k)\otimes\Phi_{NM(1)\Sigma}$. The index so defined is constant on each connected component of $((t,s):t,|s|>0)$. Also,* $\mathrm{Ind}(Q_1^0, \underline{\mathscr{E}}^p_m, \ell, k)=0$ *for $m<0$. In general*

$$\mathrm{Ind}(Q_1^{p'}, \underline{\mathscr{E}}^p_m, \ell, k)=(-1)^\mu\cdot\mathrm{Ind}(Q_1^p, \underline{\mathscr{E}}^{p-p'}_{m'}, \ell, k),$$

where $m'=m+p'k+(-pp'+p'p'/2)\cdot(e(N^*)-e(\underline{V}^*))$; and where $\mu=p'\cdot(e(N^*)-e(\underline{V}^*))$ when $\ell=\ell_e$, but $\mu=p'\cdot e(N^*)$ when $\ell=\ell_s$.

Proof of Proposition 7.8. The Fredholm property of $Q_1^{p'}$ is established in Proposition A.1. The dependence of the index on (t,s) is established in Proposition A.3. The vanishing of the index for $p'=0$ and for $m<0$ is a consequence of Proposition A.1. The final index equality summarizes Propositions 7.1, 7.3, and Lemma 7.5.

8. Deformations and the Semi-Free Case

Let M be a compact spin manifold with S^1 action. Suppose that $V\to M$ is a real, oriented vector bundle to which the S^1 action lifts. Make Proposition 5.3's assumptions and construct the vector bundle $\underline{\mathscr{E}}_{NM(1),V}\to M$; either primed or unprimed. Let m be an eigenvalue of the endomorphism P on $\underline{\mathscr{E}}_{NM(1),V}$, and let $\underline{\mathscr{E}}_{NM(1),Vm}\to M$ denote the sub-bundle on which P acts as multiplication by m. Let k be an eigenvalue of the generator, K, of the S^1 action on $C^\infty(\underline{\mathscr{E}}_{NM(1),Vm}\otimes\Phi_{NM(1)})$, and let $C^\infty(\underline{\mathscr{E}}_{NM(1),V}\otimes\Phi_{NM(1)}, k)$ denote the subset of sections on which K acts as multiplication by k. (See Sect. 6.)

Label the connected components of the fixed point set of the S^1 action by $\{\Sigma[i]\}$. For $\Sigma\equiv\Sigma[i]$, introduce the line bundle $L\to\Sigma$, where $L\equiv L(N^*)\otimes L(\underline{V}^*)$ as defined in Sect. 7. Introduce $\underline{\mathscr{E}}_{NM(1),V}(\Sigma)\to\Sigma$ as defined in Eq. (6.16). For integer p. Let $\underline{\mathscr{E}}^p_m(k)$ denote the (m,k) eigenspace for the endomorphisms P and K on $\underline{\mathscr{E}}_{NM(1),V}(\Sigma)\otimes L^p$ as defined in the preceding section. Construct the function $\Phi_{NM(1)\Sigma}$

and the operators Q_1 on $C^\infty(\mathscr{E}^p_m(k)\otimes\Phi_{NM(1)\Sigma})$, and $Q^1_1\equiv\imath^{-1*}Q^1\imath^*$ on $C^\infty(\mathscr{E}^p_m(k)\otimes\imath^{-1*}\Phi_{NM(1)\Sigma})$ as described in Sects. 6 and 7.

According to Proposition 6.1 and 7.8, these operators extend to define Fredholm operators from $H^1(\mathscr{E}^p_m(k)\otimes\Phi_{NM(1)\Sigma})$ to $L^2(\mathscr{E}^p_m(k)\otimes\Phi_{NM(1)\Sigma})$ and from $H^1(\mathscr{E}^p_m(k)\otimes\imath^{-1*}\Phi_{NM(1)\Sigma})$ to $L^2(\mathscr{E}^p_m(k)\otimes\imath^{-1*}\Phi_{NM(1)\Sigma})$, respectively. For such t and s, and for the involution $\ell\equiv\ell_e,\ell_s$, the respective indices, $\mathrm{Ind}(Q_1,\mathscr{E}^p_m,\ell,k)$ and $\mathrm{Ind}(Q^1_1,\mathscr{E}^p_m,\ell,k)$, are well defined (see Proposition 7.8.).

The purpose of this section is to relate two sets of indices, $\{\mathrm{Ind}(Q_1,\mathscr{E}^p_m,\ell,k)\}$, and $\{\mathrm{Ind}(Q^1_1,\mathscr{E}^p_m,\ell,k)\}$. This comparison comprises a crucial step in the proof of Theorem 1.3. The import of such a comparison was suggested by the discussion of Witten in [W2].

The desired comparison can be made when certain conditions on the bundle $V\to M$ are satisfied. The conditions are summarized in Definition 8.1 and Lemma 8.2, below.

For fixed i, introduce the decompositions $N=\oplus_{0<\upsilon}N_\upsilon\to\Sigma[i]$ and $V|_{\Sigma[i]}=V_{0R}\oplus_{p<\upsilon}V_\upsilon\to\Sigma[i]$. Then, introduce

$$L[i]\equiv\bigotimes_{0<\upsilon}(\det(N^*_\upsilon)\otimes\det(\underline{V}^*_\upsilon)\to\Sigma[i]\,. \tag{8.1}$$

Also, introduce the integer

$$e[i]\equiv\sum_{0<\upsilon}\upsilon^2\cdot(\dim(\underline{V}^*_\upsilon)-\dim(N^*_\upsilon))=e(N^*|_{\Sigma[i]})-e(\underline{V}^*|_{\Sigma[i]})\,. \tag{8.2}$$

Definition 8.1. Let M be a compact oriented manifold on which S^1 acts. Let $V\to M$ be a real, oriented vector bundle. The bundle V is *weakly S^1 compatible with T^*M* when the following is true:
(1) $w_2(TM)=w_2(V)=0$.
(2) Require that there exists a complex line bundle $L_0\to M$ to which the S^1-action on M has a lift, and which restricts S^1 equivariantly to each $\Sigma[i]$ as $L[i]$.
(3) For each integer $n>1$, and each component of $M(n)\supset\cup_i\Sigma[i]$, require that the restriction of L_0 to $M(n)$ has an n^{th}-root.
(4) For n as in (3), require that V be $\mathbb{Z}/n\cdot\mathbb{Z}$ compatible with T^*M as in Definition 1.1.
(5) Require that $e[i]=e[j]\equiv e$ for all i and j.

A global condition which implies the S^1-compatibility of V can be given in terms of the equivariant, first Pontrjagin class of V.

Lemma 8.2. *Let M be a compact, oriented, even dimensional spin manifold with an S^1 action. Let $V\to M$ be a real, oriented vector bundle to which the S^1 action lifts. Assume that $w_2(V)=0$. If V is strongly S^1-compatible with T^*M as defined in Definition 1.1, then V is weakly S^1-compatible with T^*M. In particular, this occurs when*

$$w_2\left(S^\infty\underset{S^1}{\times}T^*M\right)-w_2\left(S^\infty\underset{S^1}{\times}V\right)=0\in H^2\left(S^\infty\underset{S^1}{\times}M;\mathbb{Z}/2\cdot\mathbb{Z}\right)$$

and when

$$\tfrac{1}{2}\cdot p_1\left(S^\infty\underset{S^1}{\times}T^*M\right)-\tfrac{1}{2}\cdot p_1\left(S^\infty\underset{S^1}{\times}V\right)\in H^4\left(S^\infty\underset{S^1}{\times}M;\mathbb{Z}\right).$$

Proof of Lemma 8.2. In this case, Lemma 7.5 asserts that for all i, $L[i] \approx \Sigma[i] \times \mathbb{C}$, that S^1 acts with trivial character, and that $e[i] \equiv 0$.

The significance of S^1-compatibility is described below.

Proposition 8.3. *Let M be a compact, oriented manifold on which S^1 acts isometrically. Let $V \to M$ be a real, oriented vector bundle which is weakly S^1-compatible with T^*M. Let $I \equiv \{(t,s) \in \mathbb{R}^2 : t, |s| > 0\}$ and let $(t,s) \in I$. As specified in Proposition 5.3, construct $\mathscr{E}_{NM(1),V} \to M$ and the function $\Phi_{NM(1)}$. Let m be an eigenvalue of the endomorphism P on $\mathscr{E}_{NM(1),V} \to M$ as described in Sect. 4 and let k be an eigenvalue of the generator of the S^1 action, K, on $C^\infty(\mathscr{E}_{NM(1),Vm} \otimes \Phi_{NM(1)})$ as described in Sect. 6. Let $\ell \equiv \ell_e$ or ℓ_s of Eqs.* (6, 34, 36, 37). *For each component, $\Sigma[i]$, of the fixed point set of the S^1 action, introduce the integers* $\mathrm{Ind}(Q_1, \mathscr{E}^p_m, \ell, k)$ *and* $\mathrm{Ind}(Q^1_1, \mathscr{E}^p_m, \ell, k)$ *of Proposition 7.8. Then,*

$$\sum_i \mathrm{Ind}(Q_1, \mathscr{E}^p_m, \ell, k) = \sum_i \mathrm{Ind}(Q^1_1, \mathscr{E}^p_m, \ell, k).$$

The proof of this proposition occupies this and the next section. The comparison of indices for the two sets of operators is accomplished in both the semi-free and the general case by constructing a 1-parameter family of index problems which interpolates between the two in question.

The construction of this family of index problems starts with the following considerations: Pick $c \in (0, 1/4)$. *Let* $\{\upsilon\}$ denote the set of characters for the S^1 action on the normal bundle in M to a component, $\Sigma \equiv \Sigma[i]$ of the fixed point set. Define for each $\alpha \in [0,1]$ and for each υ and integer $n \neq 0$, the number $\mu[\upsilon, n] \equiv n + \alpha \cdot \upsilon$. For $n = 0$ define $\mu[\upsilon, 0] \equiv (c + \alpha) \cdot \upsilon$.

Let

$$\Omega \equiv \Omega[i] \equiv \{\alpha \in (0,1] : \alpha \neq -c \text{ and } \alpha \cdot \upsilon \in \mathbb{Z} \text{ for some } \upsilon\}. \tag{8.3}$$

Note that always, $1 \in \Omega$.

For $\alpha \in [0,1] \backslash \Omega$, the data $\{\mu[\upsilon, n]\}$ satisfies Conditions (1)–(3) in Proposition A.1. For $\alpha \in [0,1] \backslash \Omega$, use the data $\{\mu[\upsilon, n]\}$ in Eq. (4.12) to define a function $\Phi[\alpha]$ on $\mathscr{N}$.

Using the data $\{\mu[\upsilon, n]\}$, define the operator $Q[\alpha]$ on $C^\infty(\mathscr{E}^p_m(k) \otimes \Phi[\alpha])$ as in Eqs. (4.13, 14). Note that for $\alpha \equiv 0$ and for $c = s/t$, one has $Q[0] \equiv Q_1$ on $C^\infty(\mathscr{E}^p_m(k) \otimes \Phi_{NM(1)})$ as defined in Sect. 7 and considered in Proposition 7.8. As in the Appendix, introduce $L^2(\mathscr{E}^p_m(k) \otimes \Phi[\alpha])$ *and* $H^1(\mathscr{E}^p_m(k) \otimes \Phi[\alpha])$, the two Hilbert space completions of $C^\infty(\mathscr{E}^p_m(k) \otimes \Phi[\alpha])$.

According to Proposition A.1 and A.3, the following is true:

Lemma 8.4. *Let $I \subset [0,1] \backslash \Omega$ be a connected subset. Pick $c \in (0, 1/4)$ and $\{\mu(n,j)\}$ as above. Then, the operator $Q[\alpha] : H^1(\mathscr{E}^p_m(k) \otimes \Phi[\alpha]) \to L^2(\mathscr{E}^p_m(k) \otimes \Phi[\alpha])$ is Fredholm. The integer* $\mathrm{Ind}(Q[\alpha], \mathscr{E}^p_m, \ell, k)$, *as defined in Eq.* (*A.28*) *with $\ell = \ell_e$ or ℓ_s is independent of $\alpha \in I$.*

The behaviour of $Q[\alpha]$ near points in Ω must now be considered. The simplest case occurs when the following assumption is made: Require for each connected component $\Sigma[i]$ of the fixed point set that $\{\upsilon[i] \equiv \upsilon \equiv 1\}$ is obeyed for the characters of the S^1 action on the normal bundle $N \to \Sigma[i]$. This case will occur if the geometric S^1 action is semi-free. The remainder of this section treats the case $\{\upsilon[i] \equiv \upsilon \equiv 1\}$. The general case is considered in the next section.

Proof of Proposition 8.3 in the Semi-Free Case. When the S^1 action is semi-free, then $\{\upsilon[i] \equiv \upsilon \equiv 1\}$. Assume that such is the case.

With $s>0$, each set $\Omega[i]$ of Eq. (8.3) contains only the point $\{1\}$. Let $\alpha \equiv 1-\delta$, and observe that

$$\begin{aligned}\Phi[1-\delta] = \exp\Big(-t\cdot \sum_{m>0} \cdot |x_m|^2 - \Big(\sum_{0<n\in\mathbb{Z}} t\cdot |n+1-\delta|\cdot |z_{1,n}|^2 \\ + t\cdot(1-\delta+c)\cdot|z_{1,0}|2 + \sum_{0<n\in\mathbb{Z}} t\cdot|n-1+\delta|\cdot z_{1,-n}|^2\Big)\Big). \qquad (8.4)\end{aligned}$$

Introduce the shift operator $\imath^*$ from Sect. 7, and observe that

$$\begin{aligned}\imath^*\Phi[1-\delta] = \exp\Big(-t\cdot \sum_{m>0} m\cdot |x_m|^2 - \Big(\sum_{1<n\in\mathbb{Z}} t\cdot |n-\delta|\cdot |z_{1,n}|^2 \\ + t\cdot(1-\delta+c)\cdot|z_{1,1}|^2 + t\cdot\delta\cdot|z_{1,0}|^2 \\ + \sum_{0<n\in\mathbb{Z}} t\cdot|n+\delta|\cdot |z_{1,-n}|^2\Big)\Big). \qquad (8.5)\end{aligned}$$

The operator $\imath^* Q[1-\delta]\imath^{-1*}$ is given by

$$\begin{aligned}\imath^* Q[1-\delta]\imath^{-1*} \equiv T_\Sigma + i\cdot \sum_{1<n\in\mathbb{Z}} [\Gamma_{1,n}\cdot(\partial/\partial \underline{z}_{1,n} - t\cdot(n-\delta)\cdot z_{1,n}) \\ + \Gamma^*_{1,n}\cdot(\partial/\partial z_{1,n} + t\cdot(n-\delta)\cdot \underline{z}_{1,n})] \\ \times i\cdot[\Gamma_{1,1}\cdot(\partial/\partial \underline{z}_{1,1} - (t\cdot(1-\delta)+s)\cdot z_{1,1}) \\ + \Gamma^*_{1,1}\cdot(\partial/\partial z_{1,1} + (t\cdot(1-\delta)+s)\cdot \underline{z}_{1,1})] \\ + i\cdot \sum_{0<n\in\mathbb{Z}} [\underline{\Gamma}_{1,n}(\partial/\partial z_{1,-n} - t\cdot(n-\delta)\cdot \underline{z}_{1,-n}) \\ + \underline{\Gamma}^*_{1,n}\cdot(\partial/\partial \underline{z}_{1,-n} + t\cdot(n-\delta)\cdot z_{1,n})] \\ + i\cdot[\Gamma_{1,0}\cdot(\partial/\partial \underline{z}_{1,0} + t\cdot\delta\cdot z_{1,0}) \\ + \Gamma^*_{1,0}\cdot(\partial/\partial z_{1,0} - t\cdot\delta\cdot \underline{z}_{1,0})]. \qquad (8.6)\end{aligned}$$

It is defined on $C^\infty(\imath^*(\underline{\mathscr{E}}^p_m(k)\otimes\Phi[1-\delta]))$. For future reference, note that Lemma 7.3 asserts that

$$\imath^*(\underline{\mathscr{E}}^p_m(k)\otimes\Phi[1-\delta])) = \underline{\mathscr{E}}^{p-1}_{m'}(k)\otimes\imath^*\Phi[1-\delta],$$

where

$$m' \equiv m'[i] = m + k + (-p + \tfrac{1}{2})\cdot e[i] \qquad (8.7)$$

with $e[i]$ defined in Eq. (8.2).

Now, consider the set of numbers $\{\mu[\upsilon,n]_- \equiv n$ for $n \neq 0$ and $\mu[\upsilon,0] \equiv -c\}$. Define

$$\begin{aligned}\Phi_- \equiv \exp\Big(-t\cdot \sum_{m>0} m\cdot |x_m|^2 - \sum_{0<n\in\mathbb{Z}} t\cdot n\cdot |z_{1,n}|^2 \\ - t\cdot c\cdot |z_{1,0}|^2 - \sum_{0<n\in\mathbb{Z}} t\cdot n\cdot |z_{1,-n}|^2\Big), \qquad (8.8)\end{aligned}$$

and define the operator Q_- on $C^\infty(\underline{\mathscr{E}}_{m'}^{p-1}(k)\otimes\Phi_-)$ by using the set $\{\mu[\upsilon,n]_-\}$ in Eq. (4.14).

Another direct application of Propositions A.1 and A.3 gives

Lemma 8.5. *Fix $\delta>0$, but much less than 1, and fix $c\in(0,1/4)$. Introduce the operators $\imath^*Q[1-\delta]\imath^{-1*}$ on $C^\infty(\underline{\mathscr{E}}_{m'}^{p-1}(k)\otimes\imath^*\Phi[1-\delta])$ and Q_- on $C^\infty(\underline{\mathscr{E}}_{m'}^{p-1}(k)\otimes\Phi_1)$ as above. These operators extend as Fredholm operators from the respective H^1 spaces to the respective L^2 spaces as defined in Proposition A.1. Define the two integers $\operatorname{Ind}(\imath^*Q[1-\delta]\imath^{-1*}, \underline{\mathscr{E}}_{m'}^{p-1}(k, \ell, k)$ and $\operatorname{Ind}(Q_-, \underline{\mathscr{E}}_{m'}^{p-1}, \ell, k)$ using $\ell=\ell_e$ or ℓ_s in Eq. (A.28). Then, these two integers agree.*

Define a set of numbers $\{\mu[\upsilon,n]_+\equiv n$ for $n\neq 0$ and $\mu[\upsilon,0]\equiv c\}$. Note that $\{\mu[\upsilon,n]_+\}$ and $\{\mu[\upsilon,n]_-\}$ differ in the sign of $\mu[\upsilon,0]_\pm$. Use the data $\{\mu[\upsilon,n]_+\}$ in Eq. (4.12) to define a function, Φ_+, on $\mathscr{N}$. Use the data in Eq. (4.14) to define the operator Q_+ on $C^\infty(\underline{\mathscr{E}}_{m'}^{p-1}(k)\otimes\Phi_+)$. Note that with $c=s/t$, one has $\Phi_+\equiv\Phi_{NM(1)\Sigma}$, and $Q_+\equiv Q_1$ as defined in Sect. 7.

There is no continuous deformation of the data $\{\mu[\upsilon,n]_-\}$ into the data $\{\mu[\upsilon,n]_+\}$ which preserves Conditions (1)–(3) of Proposition A.1 along the whole route. Therefore, Proposition A.3 cannot be invoked to compare the indices of Q_- with Q_+. This pathology is a real one; it is precisely the difference between the cases $s<0$ and $s>0$ in Proposition 2.5. The difference between the $\pm$ cases can be analyzed only by considering behavior away from the fixed point set of the geometric S^1 action; the global topology of M must enter the discussion. Resolving this pathology requires the simultaneous consideration of all the components of the fixed point set of the S^1 action.

Let $\{\Sigma[i]\}$ denote the connected components of the fixed point set of the geometric S^1 action. Over each $\Sigma[i]$, introduce the complex line bundle $L[i]\to\Sigma[i]$ of Eq. (8.1). Let $m'[i]$ be given by Eq. (8.7). When $V\to M$ is weakly S^1-compatible with T^*M, the number $m'[i]\equiv m'$ is independent of the component $\Sigma[i]$ of the fixed point set. Over each $\Sigma[i]$ there is a vector bundle $\underline{\mathscr{E}}[i]_{m'}^p(k)\equiv\underline{\mathscr{E}}_{m'}^p(k)|_{\Sigma[i]}$ and there are the operators $Q[i]_\pm$ defined on $C^\infty(\underline{\mathscr{E}}[i]_{m'}^p(k)\otimes\Phi_\pm)$,

Lemma 8.6. *Assume that V is weakly S^1-compatible with T^*M. Pick a constant $c\in(0,1)$ and for each component $\Sigma[i]$ of the fixed point set, define the data $\{\mu[i][\upsilon,n]_\pm\}$ as above. Using this data, define for each i the function $\Phi[i]_\pm$ and the operator $Q_\pm$ on $C^\infty(\underline{\mathscr{E}}[i]_{m'}^p(k)\otimes\Phi[i]_\pm)$ as specified above. The operators $Q[i]_\pm$ have the Fredholm extensions as described in Proposition A.1. For each i, define the $\operatorname{Ind}(Q[i]_\pm, \underline{\mathscr{E}}[i]_{m'}^p, \ell, k)$ using $\ell=\ell_e$ or ℓ_s in Eq. (A.28). Then*

$$\sum_i \operatorname{Ind}(Q[i]_-,\underline{\mathscr{E}}[i]_{m'}^p,\ell,k)\equiv\sum_i \operatorname{Ind}(Q[i]_+,\underline{\mathscr{E}}[i]_{m'}^p,\ell,k).$$

This lemma will be proved shortly. Assume it for now. To complete the proof of Proposition 8.3, observe that Lemma 8.4 and Proposition 7.8 assert that

$$\sum_i \operatorname{Ind}(Q_1,\underline{\mathscr{E}}[i]_m^p,\ell,k)=\sum_i(-1)^{\mu[i]}\cdot\operatorname{Ind}(Q[i]_-,\underline{\mathscr{E}}[i]_{m+k+(-p+1/2)\cdot e}^{p-1},\ell,k),\qquad(8.9)$$

with $e\equiv e[i]$ is independent of i; and with $\mu[i]\equiv e_{\mathrm{mod}(2)}$ when $\ell\equiv\ell_e$, and with

$$\mu[i]\equiv\left(\sum_\upsilon \upsilon^2\cdot\dim(N_\upsilon^*|_{\Phi[i]})_{\mathrm{mod}(2)}\right)$$

when $\ell = \ell_s$. Meanwhile, Lemma 8.5 and Proposition 7.8 assert that

$$\sum_i \operatorname{Ind}(Q[i]_+, \mathscr{E}[i]^{p-1}_{m+k+(-p+1/2)\cdot e}, \ell, k) = \sum_i (-1)^{\mu[i]} \cdot \operatorname{Ind}(Q^1_1, \mathscr{E}^0_m, \ell, k). \qquad (8.10)$$

Proposition 8.3 follows immediately from Eqs. (8.9, 10) if $\mu[i]$ is independent of the component, $\Sigma[i]$, of the fixed point set. Such is automatically the case when $\ell = \ell_e$. When $\ell = \ell_s$, a sufficient condition for this to be the case is given in [A–H] and restated below.

Lemma 8.7. *Let M be an oriented, even dimensional spin manifold with Riemannian metric. Suppose that S^1 has an isometric action on M. Let $\{\Sigma[i]\}$ be the connected components of the fixed point set. To each $\Sigma[i]$, associate*

$$\mu[i] \equiv \left(\sum_v v^2 \cdot \dim(N^*_v|_{\Sigma[i]})_{\mathrm{mod}(2)}\right).$$

Then $\mu[i] \equiv \mu$ is independent of the index i.

Lemma 8.7 is a special case of Lemma 9.7 to which the reader is referred. This section ends with the proof of Lemma 8.6.

Proof of Lemma 8.6. The assumption that V is S^1-compatible to T^*M asserts that there is a line bundle $L_0 \to M$ to which the S^1 action lifts, and which restricts to each $\Sigma[i]$ as $L[i]$. One can construct a metric and a metric compatible connection on L_0 which restricts to $\Sigma[i]$ as the metric and connection on $L[i]$. Then, consider, as in Proposition 5.3, the vector bundle $\mathscr{E}_{NM(1),V} \otimes L^p_0 \to M$, the function $\Phi_{NM(1)}$ on $\mathscr{N}NM(1)$, and the operator

$$D_t: C^\infty(\mathscr{E}_{NM(1),V} \otimes L^p_0 \otimes \Phi_{NM(1)}) \to C^\infty(\mathscr{E}_{NM(1),V} \otimes L^p_0 \otimes \Phi_{NM(1)}).$$

The number m' is an eigenvalue of the endomorphism P on $\mathscr{E}_{NM(1),V} \otimes L^p_0 \otimes \Phi_{NM(1)}$ with eigenspace $\mathscr{E}_{NM(1),Vm'} \to M$. Indeed, in the unprimed case, $e/2 + k$ is always an integer, as can be verified from Eqs. (8.2, 6.26, 6.27). In the primed case, $e/2 + k$ can be half-integral, but in this case, P has half-integer eigenvalues. The operator D_t restricts to

$$D_t: C^\infty(\mathscr{E}_{NM(1),Vm'} \otimes L^p_0 \otimes \Phi_{NM(1)}) \to C^\infty(\mathscr{E}_{NM(1),Vm'} \otimes L^p_0 \otimes \Phi_{NM(1)}).$$

According to Proposition 4.1, D_t extends to a Fredholm operator from

$$H^1(\mathscr{E}_{NM(1),Vm'} \otimes L^p_0 \otimes \Phi_{NM(1)}) \quad \text{to} \quad L^2(\mathscr{E}_{NM(1),Vm'} \otimes L^p_0 \otimes \Phi_{NM(1)})$$

whose index is defined in Eq. (4.16) using $\ell = \ell_e$ or ℓ_s.

Since the S^1 action can be lifted to an action of a finite cover of S^1 on $\mathscr{E}_{NM(1),Vm'} \otimes L^p_0 \otimes \Phi_{NM(1)}$, the localization results in Sect. 6 apply, and allow a computation of the S^1-character valued index of D_r as defined in Eq. (6.1).

Proposition 6.2 asserts that this character valued index can be computed using either of the set of operators $\{Q[i]_\pm\}$ as long as the constant c is chosen to be equal to t/s. Indeed, according to Proposition 6.1,

$$\sum_i \operatorname{Ind}(Q[i]_-, \mathscr{E}[i]^p_{m'}, \ell, k) = \operatorname{Ind}(D, \mathscr{E}_{NM(1),Vm'}, \ell, k) = \operatorname{Ind}(Q[i]_+, \mathscr{E}[i]^p_{m'}, \ell, k). \qquad (8.11)$$

This last equality implies the assertion of Lemma 8.6.

9. Deformations: The Case of General S^1-Actions

Consider Proposition 8.3 in the case where the positive characters of the S^1 action on the normal bundle to the fixed point set are allowed to differ from 1.

To begin, fix a component $\Sigma \equiv \Sigma[i]$ of the fixed point set. Introduce the finite set $\Omega[i]$ of Eq. (8.3). Fix the constant c in Lemma 8.4. For $\alpha \in [0,1] \backslash \Omega[i]$, introduce the notation $\Phi[i,\alpha]$ and

$$Q[i,\alpha]: H^1(\underline{\mathscr{E}}[i]^p_m(k) \otimes \Phi[i,\alpha]) \to L^2(\underline{\mathscr{E}}[i]^p_m(k) \otimes \Phi[i,\alpha])$$

to distinguish the data in Lemma 8.4 at the various components of the fixed point set.

Suppose that $\alpha_0 \in \cup_i \Omega[i]$. This α_0 is a rational number; let n_0 be the minimal positive integer with the property that $p_0 \equiv n_0 \cdot \alpha_0 \in \mathbb{Z}$. If $\alpha_0 \notin \Omega[i]$, then $\Sigma[i]$ is an isolated, connected component of $M(n_0)$. Otherwise, $\Sigma[i]$ is a proper submanifold of $M(n_0)$. The following facts about n_0 are needed later:

Lemma 9.1. *Define n_0 and p_0 as in the preceding paragraph. Let Σ be a component of the fixed point set of the S^1 action on M and let $0 < \upsilon$ define a character of the S^1 action on $N|_\Sigma$. Then $p_0 \cdot \upsilon/n_0 \in \mathbb{Z}$ if and only if $\upsilon/n_0 \in \mathbb{Z}$. Also, $p_0 \cdot \upsilon/n_0 \in \mathbb{Z} + 1/2$ only if $\upsilon/n_0 \in \mathbb{Z} + 1/2$.*

Proof of Lemma 9.1. The "if" of the first assertion is obvious. For the converse, write $\upsilon = l | n_0 + \upsilon'$ with $l \in \mathbb{Z}$ and with $0 < \upsilon' < n_0$. Then $\upsilon' \cdot p_0/n_0 = \upsilon' \cdot \alpha_0 \in \mathbb{Z}$ which contradicts the fact that n_0 is the minimal positive integer which gives an integer upon multiplication by α_0. For the second assertion, write $\upsilon = l \cdot n_0/2 + \upsilon'$ with $l \in \mathbb{Z}$ and with $0 \leqq \upsilon' < n_0/2$. Then, $2 \cdot \upsilon' \cdot \alpha_0 \in \mathbb{Z}$, which gives the same contradiction unless $\upsilon' = 0$.

For $\alpha_0 \notin \Omega[i]$, the index of $Q[i,\alpha]$ is well defined and constant for all α in a neighborhood of α_0. For $\alpha \in \Omega[i]$, the index of $Q[i,\alpha]$ may jump at α_0 since $Q[i,\alpha_0]$ is not defined.

Consider this situation in greater detail: Let $\alpha \equiv \alpha_0 \cdot (1-\delta)$. Let $\Sigma \equiv \Sigma[i]$ and write $\Phi[1-\delta] \equiv \Phi[i, \alpha_0 \cdot (1-\delta)]$ and write $Q[i, \alpha_0 \cdot (1-\delta)] \equiv Q[1-\delta]$. In order to compare the indices in the two cases $\pm\delta > 0$, it is necessary to consider together the components of the fixed point set of the S^1-action on M. As in the semi-free case, this will be done by observing that the index of $Q[1-\delta]$ is equal to the index of an operator which is the localization to a component of the fixed point set of a suitable Dirac-Ramond operator which is defined on a submanifold of M on which the geometric S^1 acts with the same set of fixed points as the geometric S^1 action on M. Then, the sum, over all components of the fixed point set of the S^1 action, of the indices of $Q[1-\delta]$ for $\delta > 0$ will be seen to equal the same sum for $\delta < 0$.

To begin, introduce the "fractional" shift

$$\imath(n_0, p_0): \bigoplus_{m>0} (T\Sigma_m \otimes \mathbb{C}) \bigoplus_{n \in \mathbb{Z}} N_{\upsilon,n} \to \bigoplus_{m>0} (T\Sigma_m \otimes \mathbb{C}) \bigoplus_{n \in \mathbb{Z} + p_0 \cdot \upsilon/n_0} N_{\upsilon,n} \tag{9.1}$$

with restriction

$$\imath(n_0, p_0): N_{\upsilon,n} \to N_{\upsilon, n + p_0 \cdot \upsilon/n_0}. \tag{9.2}$$

Also, define

$$\imath(n_0, p_0): \bigoplus_{m>0} V_{0,m} \bigoplus_{n \in \mathbb{Z}} V_{\upsilon,n} \to \bigoplus_{m>0} V_{0,m} \bigoplus_{n \in \mathbb{Z} + p_0 \cdot \upsilon/n_0} V_{\upsilon,n} \tag{9.3}$$

with restriction

$$\imath(n_0, p_0): V_{\upsilon,n} \to V_{\upsilon,n+p_0\upsilon/n_0}. \tag{9.4}$$

Finally, define

$$\imath(n_0, p_0): \bigoplus_{0<m \in \mathbb{Z}+1/2} V_{0,m} \bigoplus_{n \in \mathbb{Z}} V_{\upsilon,n+1/2} \to \bigoplus_{0<m \in \mathbb{Z}+1/2} 0,m \bigoplus_{n \in \mathbb{Z} + p_0 \cdot \upsilon/n_0 + 1/2} V_{\upsilon,n} \tag{9.5}$$

with restriction

$$\imath(n_0, p_0): V_{\upsilon,n} \to V_{\upsilon,n+p_0\upsilon/n_0}. \tag{9.6}$$

There is no expectation that the fractional shift induces a bundle automorphism of $\underline{\mathscr{E}}^0(\Sigma) \equiv \underline{\mathscr{E}}_{NM(1),V}(\Sigma)$, or even an automorphism up to tensoring with a line bundle. However, note that $\imath(n_0, p_0)^* \underline{\mathscr{E}}_{NM(1),V}(\Sigma)$ looks like $\underline{\mathscr{E}}_{NM(n_0),V}(\Sigma)$ as defined in Eq. (6.16). To be precise,

Lemma 9.2 *There is a natural isomorphism* $\imath(n_0, p_0)^* \underline{\mathscr{E}}_{NM(1),V}(\Sigma) \approx \underline{\mathscr{E}}_{NM(n_0),V}(\Sigma) \otimes L(n_0, p_0)$, *with*

$$L(n_0, p_0) \equiv \bigotimes_{0<\upsilon} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{-[p_0 \cdot \upsilon/n_0]}$$
$$\bigotimes_{n_0/2 < \upsilon' < n_0} \ \bigotimes_{0<\upsilon:\, \upsilon = \upsilon' \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{-1}$$
$$\otimes \Bigg[\bigotimes_{0<\upsilon = n_0/2 \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{-1}$$
$$\bigotimes_{0<\upsilon'<n_0/2} \Bigg[\bigotimes_{0<\upsilon:\, \upsilon = \upsilon' \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{2 \cdot \upsilon'/n_0}$$
$$\bigotimes_{0<\upsilon:\, \upsilon = -\upsilon' \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{-2 \cdot \upsilon'/n_0} \Bigg] \Bigg]^{r/2},$$

in the unprimed case. Here, $r=0$ *if* n_0 *is odd, and* $r=1$ *if* n_0 *is even. The symbol* $[s]$ *denotes the greatest integer which is less than or equal to a given number s. In the primed case,*

$$L(n_0, p_0) \equiv \bigotimes_{0<\upsilon} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{-[p_0 \cdot \upsilon/n_0 + 1/2]}$$
$$\bigotimes_{1 \le k \le p_0/2} \ \bigotimes_{(k-1/2)/p_0 < \upsilon'/n_0 < k/p_0} \det(NM(n_0, \upsilon')^*)$$
$$\otimes \Bigg[\bigotimes_{0<\upsilon = n_0/2 \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))$$
$$\bigotimes_{0<\upsilon'<n_0/2} \Bigg[\bigotimes_{0<\upsilon:\, \upsilon = \upsilon' \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{2 \cdot \upsilon'/n_0}$$
$$\bigotimes_{0<\upsilon:\, \upsilon = -\upsilon' \bmod(n_0)} (\det(N_\upsilon^*) \otimes \det(\underline{V}_\upsilon^*))^{-2 \cdot \upsilon'/n_0} \Bigg] \Bigg]^{r/2}.$$

Proof of Lemma 9.2. The unprimed case follows directly from Eqs. (6.13, 14) and Proposition 7.1. In the primed cases, Eqs. (6.13, 14) and Proposition 7.1 yield directly

$$
\begin{aligned}
L(n_0, p_0) \equiv & \bigotimes_{0<\upsilon} \det(N_\upsilon^*)^{-[p_0\cdot\upsilon/n_0]} \bigotimes_{0<\upsilon} \det(\underline{V}_\upsilon^*))^{-[p_0\cdot\upsilon/n_0+1/2]} \\
& \bigotimes_{n_0/2<\upsilon'<n_0} \bigotimes_{0<\upsilon=\upsilon' \bmod(n_0)} \det(N_\upsilon^*)^{-1} \\
& \otimes \Bigg[\bigotimes_{0<\upsilon=n_0/2 \bmod(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*)^{-1}) \\
& \bigotimes_{0<\upsilon'<n_0/2} \Bigg[\bigotimes_{0<\upsilon:\upsilon=\upsilon' \bmod(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{2\cdot\upsilon'/n_0} \\
& \bigotimes_{0<\upsilon:\upsilon=-\upsilon' \bmod(n_0)} (\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-2\cdot\upsilon'/n_0} \Bigg]\Bigg]^{r/2}.
\end{aligned} \tag{9.7}
$$

The assertion follows from the preceding equation with Lemma 9.1 and Eq. (6.4.).

The generators of the canonical S^1 action and that of the n_0-root S^1 action on $\mathscr{E}_{\underline{N}M(n_0),V}(\Sigma)$ act by the endomorphisms $P(n_0)$ of Eqs. (6.21–24) and $K(n_0)$ of Eqs. (6.25–28), respectively. Define P on $\mathscr{E}_{\underline{N}M(n_0),V}(\Sigma)\otimes L(n_0,p_0)$ to send the decomposable element $\psi\otimes s$ to $P(n_0)\psi\otimes s$. Then, extend by linearity. Define K on $\mathscr{E}_{\underline{N}M(n_0),V}(\Sigma)\otimes L(n_0,p_0)$ to send the same $\psi\otimes s$ to $(K(n_0)+\kappa(n_0,p_0))\psi\otimes s$, with

$$
\begin{aligned}
\kappa(n_0,p_0) \equiv & \sum_{0<\upsilon} \upsilon/n_0\cdot[p_0\cdot\upsilon/n_0]\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& +\tfrac{1}{2}\cdot \sum_{0<\upsilon=n_0/2 \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& + \sum_{0<\upsilon'<n_0/2} \sum_{0<\upsilon:-\upsilon' \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& -r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon'/n_0\cdot\Bigg[\sum_{0<\upsilon:\upsilon' \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& - \sum_{0<\upsilon:-\upsilon' \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*))\Bigg].
\end{aligned} \tag{9.8}
$$

in the unprimed case; and with

$$
\begin{aligned}
\kappa(n_0,p_0) \equiv & \sum_{0<\upsilon} \upsilon/n_0\cdot[p_0\cdot\upsilon/n_0+\tfrac{1}{2}]\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& -\tfrac{1}{2}\cdot \sum_{0<\upsilon=n_0/2 \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& - \sum_{1\leq k\leq p_0/2} \sum_{(k-1/2)/p_0<\upsilon'/n_0<k/p_0} \Bigg(\sum_{0<\upsilon=\upsilon' \bmod(n_0)} \upsilon/n_0\cdot\dim(N_\upsilon^*) \\
& - \sum_{0<\upsilon=-\upsilon' \bmod(n_0)} \upsilon/n_0\cdot\dim(N_\upsilon^*)\Bigg) \\
& -r\cdot \sum_{0<\upsilon'<n_0/2} \upsilon'/n_0\cdot\Bigg[\sum_{0<\upsilon:\upsilon' \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*)) \\
& - \sum_{0<\upsilon:-\upsilon' \bmod(n_0)} \upsilon/n_0\cdot(\dim(N_\upsilon^*)-\dim(\underline{V}_\upsilon^*))\Bigg].
\end{aligned} \tag{9.9}
$$

in the primed case. Then, extend by linearity. The generator of the n_0-root S^1 action acts as multiplication by $\kappa(n_0,p_0)$ on $L(n_0,p_0)$.

The involutions $\ell=\ell_e$ and ℓ_s were defined on $\mathscr{E}_{NM(1),V}(\Sigma)$ and on $\mathscr{E}_{NM(n_0),V}(\Sigma)$ in Eqs. (6.23–37). Extend the definition of ℓ to $\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)$ by requiring the action to be linear, and to send a decomposable element $\psi\otimes s$ to $\ell\psi\otimes s$.

Lemma 9.3. *The induced isomorphism*

$$\imath(n_0,p_0)^*:\mathscr{E}_{NM(1),V}(\Sigma)\to\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)$$

has the following properties:
1) $K\cdot\imath(n_0,p_0)^*=\imath(n_0,p_0)^*\cdot K$.
2) $P\cdot\imath(n_0,p_0)^*=\imath(n_0,p_0)^*\cdot P+\imath(n_0,p_0)^*\cdot(p_0\cdot K+\varepsilon)$,
where ε is defined as follows: In the unprimed case,

$$\begin{aligned}\varepsilon\equiv\tfrac{1}{2}\cdot\sum_{0<\upsilon}&(\dim(N^*_\upsilon)-\dim(\underline{V}^*_\upsilon))\cdot([p_0\cdot\upsilon/n_0]\cdot([p_0\cdot\upsilon/n_0]+1)\\&-p_0\cdot\upsilon/n_0-2\cdot p_0\cdot\upsilon/n_0\cdot[p_0\cdot\upsilon/n_0]);\end{aligned}$$

and in the primed case,

$$\begin{aligned}\varepsilon\equiv\tfrac{1}{2}\cdot\sum_{0<\upsilon}&\dim(N^*_\upsilon)\cdot([p_0\cdot\upsilon/n_0]\cdot([p_0\cdot\upsilon/n_0]+1)-p_0\cdot\upsilon/n_0\\&-2\cdot p_0\cdot\upsilon/n_0\cdot[p_0\cdot\upsilon/n_0])\\&-\tfrac{1}{2}\cdot\sum_{0<\upsilon}\dim(\underline{V}^*_\upsilon)\cdot([p_0\cdot\upsilon/n_0+\tfrac{1}{2}]^2\\&-2\cdot p_0\cdot\upsilon/n_0\cdot[p_0\upsilon/n_0+\tfrac{1}{2}]).\end{aligned}$$

3) $\ell_e(M(n_0))\cdot\imath(n_0,p_0)^*=(-1)^\mu\cdot\imath(n_0,p_0)^*\cdot\ell_e(M)$ with

$$\mu\equiv\left(\sum_{0<\upsilon}[p_0\cdot\upsilon/n_0]\cdot(\dim(N^*_\upsilon)-\dim(\underline{V}^*_\upsilon))+\Delta(n_0,\Sigma)+\delta(n_0,\Sigma,V)\right)_{\mathrm{mod}(2)}$$

in the unprimed case; with Δ as defined in Eq. (6.33) and δ as defined in Eq. (6.35). In the primed case,

$$\begin{aligned}\mu=\Big(\sum_{0<\upsilon}&[p_0\cdot\upsilon/n_0]\cdot\dim(N^*_\upsilon)-\sum_{0<\upsilon}[p_0\cdot\upsilon/n_0+\tfrac{1}{2}]\\&\times\dim(\underline{V}^*_\upsilon)+\Delta(n_0,\Sigma)+\delta'(n_0,\Sigma,V)\Big)_{\mathrm{mod}(2)}\end{aligned}$$

with δ' as defined prior to Eq. (6.37).
4) $\ell_s(M(n_0))\cdot\imath(n_0,p_0)^*=(-1)^\mu\cdot\imath(n_0,p_0)^*\cdot\ell_s(M)$ with

$$\mu=\left(\sum_{0<\upsilon}[p_0\cdot\upsilon/n_0]\cdot\dim(N^*_\upsilon)+\Delta(n_0,\Sigma)\right)_{\mathrm{mod}(2)}.$$

Proof of Lemma 9.3. This is a straightforward calculation which mimics the proof of Lemma 7.4. Use Eqs. (7.8, 9), Eqs. (7.13, 14) and Eqs. (6.31–33).

The effect of $\imath(n_0,p_0)^*$ on the function $\Phi[1-\delta]$, and on the Dirac operator $Q[1-\delta]$ can be calculated in a straightforward way. To summarize the result, let $L\to\Sigma$ be as in Eq. (8.1) and let m, k be eigenvectors of the endomorphisms P and K on $\mathscr{E}_{NM(1),V}(\Sigma)\otimes L^p$ and let $(\mathscr{E}_{NM(1),V}(\Sigma)_m\otimes L^p)(k)$ be the eigenspace bundle over Σ. According to the preceding lemma,

$$\imath(n_0,p_0)^*(\mathscr{E}_{NM(1),V}(\Sigma)_m\otimes L^p)(k)$$

is the subbundle of

$$\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p$$

on which K acts with eigenvalue k, and on which P acts with an eigenvalue $m'(m,k,n_0,p_0)$. Let

$$(\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p)_{m'}(k)$$

denote

$$\imath(n_0,p_0)^*(\mathscr{E}_{NM(1),V}(\Sigma)_m(k)\otimes L^p).$$

Lemma 9.4. *There exists $\delta_0>0$ with the following significance: Let m, k be eigenvalues of the endomorphisms P and K, respectively, on $\mathscr{E}_{NM(1),V}(\Sigma)\otimes L^p$. Let $m' \equiv m+p_0\cdot k+\varepsilon$ be the eigenvalue of P on*

$$\imath(n_0,p_0)^*(\mathscr{E}_{NM(1),V}(\Sigma)_m(k)\otimes L^p)\equiv(\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p)_{m'}(k)$$

as given in Lemma 9.3. Fix $c\in(0,1/4]$ and $\delta\in(-\delta_0,\delta_0)\backslash\{0\}$ and $t>0$.

(1) *The operator $\imath(n_0,p_0)^*Q[1-\delta]\imath(n_0,p_0)^{-1*}$ defines a Fredholm operator from*

$$H^1((\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p)_{m'}(k)\otimes\imath(n_0,p_0)^*\Phi[1-\delta])$$

to

$$L^2((\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p(_{m'}(k)\otimes\imath(n_0,p_0)^*\Phi[1-\delta]).$$

With sign$(s)=$sign(δ), *introduce the function $\Phi_{NM(n_0)\Sigma}$ of Eq.* (6.17) *and the Dirac operator Q_{n_0} on*

$$C^\infty((\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p)_{m'}(k)\otimes\Phi_{NM(n_0)\Sigma})$$

of Eq. (6.18). *Let $\ell=\ell_e$ or ℓ_s be as defined in Eqs.* (6.31–33), *and define μ as in Assertion* 3) *or* 4) *of Lemma* 9.3.

(2) *Then*

$$\begin{aligned}
&\mathrm{Ind}(Q[1-\delta],\mathscr{E}^p_m\otimes\ell,k)\\
&\quad=(-1)^\mu\cdot\mathrm{Ind}(\imath(n_0,p_0)^*Q[1-\delta]\imath(n_0,p_0)^{-1*},\ (\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p)_{m'},\ell,k)\\
&\quad=(-1)^\mu\cdot\mathrm{Ind}(Q_{n_0},(\mathscr{E}_{NM(n_0),V}(\Sigma)\otimes L(n_0,p_0)\otimes L^p)_{m'},\ell,k).
\end{aligned}$$

Proof of Lemma 9.4. Assertion 1) is a calculation which is left to the reader. Assertion 2) is a direct consequence of Proposition A.1, A.3, and 6.1.

Lemma 9.4 exhibits the difference between the $\delta>0$ and the $\delta<0$ indices of $Q[1-\delta]$ in terms of the $s>0$ and $s<0$ indices of Q_{n_0}. This difference might not be zero, but under certain conditions, the sum of the changes over all the components of the fixed point set for the n_0-root action on $M(n_0)$ will vanish. Proposition 6.2 makes this notion precise, the next lemma summarizes.

Lemma 9.5. *Let $\{\Sigma[i]\}$ be the connected components of the fixed point set of the n_0-root S^1 action on a connected component of $M(n_0)$. Require that V be $\mathbb{Z}/n_0\cdot\mathbb{Z}$ compatible with T^*M in the sense of Definition 1.1. Require that $w_2(T^*M)=w_2(V)=0$. For each $\Sigma=\Sigma[i]$, introduce the line bundle $L(n_0,p_0)\,[i]$ of Lemma 9.2.*

Introduce the number $\varepsilon[i]$, with $\varepsilon \equiv \varepsilon[i]$ defined in Lemma 9.2, and for $\ell = \ell_e$ or ℓ_s, introduce $\mu[i] \equiv \mu$ as defined in Lemma 9.3.
1) *Require that $L(n_0, p_0)[i]$ be the restriction to $\Sigma[i]$ of a line bundle over $M(n_0)$ to which a cover of the n_0-root action lifts. Require that this lift be compatible with the S^1 action over $\Sigma[i]$ as defined in Eqs.* (9.8, 9).
2) *Require that $\varepsilon[i]$ be independent of i.*
3) *Require that $\mu[i]$ be independent of i.*
Then, for all δ small and positive,

$$\sum_i \mathrm{Ind}(Q[1-\delta], \mathscr{E}[i]^p_m, \ell, k) = \sum_i \mathrm{Ind}(Q[1+\delta], \mathscr{E}[i]^p_m, \ell, k).$$

It still remains to determine the circumstances under which the requirements of Lemma 9.5 are fulfilled.

Lemma 9.6. *Let $\{\Sigma[i]\}$ be the connected components of the fixed point set of the n_0-root S^1 action on a connected component of $M(n_0)$. For each $\Sigma = \Sigma[i]$, introduce the line bundle $L(n_0, p_0)[i]$ of Lemma 9.2. Introduce the number $\varepsilon[i]$, with $\varepsilon \equiv \varepsilon[i]$ defined in Lemma 9.3. Introduce the line bundles $L[i] \to \Sigma[i]$ of Eq.* (8.1) *and the numbers $e[i]$ of Eq.* (8.2). *Then, Conditions* 1) *and* 2) *of Lemma 9.5 are satisfied when V is weakly S^1-compatible with T^*M in the sense of Definition 8.2.*

This lemma will be proved shortly. Consider the obvious example $V \equiv TM$. In the unprimed case,

$$L(n_0, p_0)[i] \equiv \Sigma[i] \times \mathbb{C}, \tag{9.10}$$

and in the primed case

$$L(n_0, p_0)[i] \equiv \bigotimes_{1 \leq k \leq p_0/2} \; \bigotimes_{(k-1/2)/p_0 < v'/n_0 < k/p_0} \det(NM(n_0, v')^*)|_{\Sigma[i]}. \tag{9.11}$$

In both cases, the bundles extend from the fixed point sets $\{\Sigma[i]\}$ of the n_0-root action to $M(n_0)$ with a compatible lift of a cover of the n_0-root S^1-action.

In the unprimed case, one has $\varepsilon[i] \equiv 0$. In the primed case, one has

$$\varepsilon[i] = \tfrac{1}{2} \cdot \sum_{0 < v' \leq n_0/2} \dim(NM(n_0; v')) \cdot ([p_0 \cdot v'/n_0] - p_0 \cdot v'/n_0). \tag{9.12}$$

which is clearly independent of i.

Consider the question of the $\{\mu[i]\}$. The following lemma generalizes a result in [A–H] (see also Lemma 9.3 in [B–T]):

Lemma 9.7. *Let M be an oriented manifold on which S^1 acts isometrically. Let $M(n_0)$ be a connected component of the fixed point set of the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of S^1. Let $\{\Sigma[i]\}$ be the connected components of the fixed point set of the n_0-root S^1 action on a connected component of $M(n_0)$. For $\ell = \ell_s$, ℓ_e and for each $\Sigma \equiv \Sigma[i]$, introduce $\mu[i] \equiv \mu$ as defined in Lemma 9.3.*
1) *Let $\ell = \ell_s$. If $w_2(T^*M) = 0$, then $\mu[i]$ is independent of i.*
2) *Let $\ell = \ell_e$. If $w_2(T^*M) = w_2(V)$, then $\mu[i]$ is independent of i.*

This lemma will be proved at the end of the section. Observe that Lemmas 9.4–7 complete the proof of Proposition 8.3 in the general case.

Proof of Lemma 9.6. Consider first $\varepsilon\equiv\varepsilon[i]$. In the unprimed case, consider separately different values of $p_0\cdot\upsilon \bmod(n_0)$ in the defining sum. This yields

$$\begin{aligned}\varepsilon\equiv\tfrac{1}{2}\cdot\sum_{0\leqq\upsilon'<n_0}\ \sum_{0<\upsilon:\,p_0\upsilon=\upsilon'\bmod(n_0)}&(\dim(N_\upsilon^*)\\ &-\dim(\underline{V}_\upsilon^*)\cdot(-(p_0\upsilon/n_0)^2+\upsilon'\cdot(n_0-\upsilon')/n_0^2),\\ =p_0\cdot e[i]/n_0+\tfrac{1}{2}\cdot\sum_{0<\upsilon'<n_0/2}&(\dim(N(n_0;\upsilon')^*\\ &-\dim(\underline{V(n_0;\upsilon')^*})\cdot\omega(\upsilon')\cdot(n_0-\omega(\upsilon'))/n_0^2\\ &+\tfrac{1}{16}\cdot(\dim_{\mathbb{R}}(N(n_0;n_0/2)_R^*)-\dim_{\mathbb{R}}(\underline{V(n_0;n_0/2)_R^*})),\end{aligned}\tag{9.13}$$

where $\omega(\upsilon')\equiv p_0\cdot\upsilon'-[p_0\upsilon']\in(0,n_0)$. The last two terms are evidently independent of the index i, and the first term is independent of the index i when the conditions of weak S^1-compatibility in Definition 8.1 hold.

The primed case for $\varepsilon[i]$ is handled in a similar way. The result is an expression which is identical to that in Eq. (9.13) but for the addition of the term

$$\begin{aligned}&-\tfrac{1}{2}\cdot\sum_{0\leqq k<p_0}\ \sum_{k/p_0<\upsilon'<(k+1/2)/p_0}\dim(\underline{V(n_0;\upsilon')^*}\cdot\omega(\upsilon')/n_0\\ &-\tfrac{1}{2}\cdot\sum_{0<k\leqq p_0}\ \sum_{(k-1/2)/p_0<\upsilon'<k/p_0}\dim(\underline{V(n_0;\upsilon')^*}\cdot(n_0-\omega(\upsilon'))/n_0\\ &-\tfrac{1}{4}\cdot\dim_{\mathbb{R}}(\underline{V(n_0;n_0/2)_R^*}).\end{aligned}\tag{9.14}$$

Consider now the line bundle question. To facilitate the analysis, write $[p_0\upsilon/n_0]\equiv p_0\cdot\upsilon/n_0-\omega(\upsilon)/n_0$. Note that for $\upsilon=n_0/2(\bmod(n_0))$, $\omega(\upsilon)/n_0=1/2$. For the unprimed case, one obtains the following formal expression for $L(n_0,p_0)$:

$$\begin{aligned}L(n_0,p_0)=L[i]^{-p_0/n_0}&\bigotimes_{0<\upsilon:\,\upsilon\neq n_0/2\bmod(n_0)}(\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{\omega(\upsilon)/n_0}\\ &\bigotimes_{0<\upsilon:\,n_0/2<\upsilon<n_0\bmod(n_0)}(\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-1}\\ &\bigotimes_{0<\upsilon'<n_0/2}\Bigg[\bigotimes_{0<\upsilon:\,\upsilon=\upsilon'\bmod(n_0)}(\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{\upsilon'/n_0}\\ &\bigotimes_{0<\upsilon:\,\upsilon=-\upsilon'\bmod(n_0)}(\det(N_\upsilon^*)\otimes\det(\underline{V}_\upsilon^*))^{-\upsilon'/n_0}\Bigg]^r,\end{aligned}\tag{9.15}$$

with $r=0$ if n_0 is odd, and with $r=1$ if n_0 is even. Observe that when $0<\upsilon'<n_0$ and when $\upsilon=\upsilon'\bmod(n_0)$, then $\omega(\upsilon)=\omega(\upsilon')$. Also, $\omega(n_0-\upsilon')=n_0-\omega(\upsilon')$. These facts imply that, formally,

$$\begin{aligned}L(n_0,p_0)=L[i]^{-p_0/n_0}\otimes\Bigg[\bigotimes_{0<\upsilon'<n_0/2}&(\det(N(n_0;\upsilon')^*)\\ &\otimes\det(\underline{V(n_0;\upsilon')^*}))^{\omega(\upsilon')+\delta\cdot\upsilon'}\Bigg]^{1/n_0}.\end{aligned}\tag{9.16}$$

For the primed case, observe that $[p_0\cdot\upsilon/n_0+1/2]=p_0\upsilon/n_0-\omega(\upsilon)/n_0$ provided that $k/p_0\leqq\omega(\upsilon)/n_0<(k+1/2)/p_0$ for some integer $k\in[0,p_0)$. If $(k-1/2)/p_0\leqq\omega(\upsilon)/p_0<k/p_0$ for $k\in(0,p_0]$, then $[p_0\cdot\upsilon/n_0+1/2]=p_0\upsilon/n_0+(n_0-\omega(\upsilon))/n_0$. These obser-

vations imply for the primed case that formally,

$$L(n_0, p_0) = L[i]^{-p_0/n_0} \otimes \left[\bigotimes_{0<v'<n_0/2} (\det(N(n_0; v')^*) \otimes \det(\underline{V(n_0; v')}^*))^{\omega(v')+r\cdot v'} \right]^{1/n_0}$$

$$\bigotimes_{1 \leqq k \leqq p_0/2} \quad \bigotimes_{(k-1/2)/p_0 < v'/n_0 < k/p_0} \det(NM(n_0, v')^*). \tag{9.17}$$

Caution. Equations (9.16, 17) are formal only; they make honest sense only when the bundles in question have the appropriate roots.

Under the condition of the weak S^1-compatibility of V with TM^*, the bundle $L[i]$ is assumed to be the restriction to $\Sigma[i]$ of a bundle L_0 to which a cover of the n_0-root action on $M(n_0)$ lifts. The bundle L_0 is assumed to have an $n_0{}^{\text{th}}$ root, L_0^{1/n_0}; the restriction of L_0^{1/n_0} to $\Sigma[i]$ defines $L[i]^{1/n_0}$.

Equations (9.16, 17) will make honest sense provided that a $n_0{}^{\text{th}}$ root exists for the line bundle

$$\bigotimes_{0<v'<n_0} (\det(N(n_0; v')^*) \otimes \det(\underline{V(n_0; v')}^*))^{\omega(v')+r\cdot v'} \tag{9.18}$$

on $M(n_0)$. The existence of the $n_0{}^{\text{th}}$ root is guaranteed by Lemma 5.2.

The proof of Lemma 9.7 requires Lemma 9.3 of [B–T] as an auxiliary lemma:

Lemma 9.8. *(Lemma 9.8 of* [B–T]*). Let M be an oriented manifold on which S^1 acts isometrically. Let $M(n_0)$ be a connected component of the fixed point set of the $\mathbb{Z}/n_0 \cdot \mathbb{Z}$ subgroup of S^1. Let $\{\Sigma[0], \Sigma[1]\}$ be distinct, connected components of the fixed point set of the n_0-root S^1 action on a connected component of $M(n_0)$. Let $V \to M$ be a real, oriented, even dimensional vector bundle to which the S^1 action on M lifts. Assume that $w_2(V) = 0$. Let $p_0 \in (0, n_0)$ be an integer which is relatively prime to n_0. Then*

$$0 = \left(\sum_{0<v\in\mathbb{Z}} (\dim(V_v|_{\Sigma[0]}) - \dim(V_v|_{\Sigma[1]}) \right) \cdot [p_0 v/n_0])_{\mathrm{mod}(2)}$$
$$+ (\delta(n_0, \Sigma(0), V) - \delta(n_0, \Sigma(1), V)_{\mathrm{mod}(2)}.$$

Here $\{V_v|_{\Sigma[i]}\}_{v\in\mathbb{Z}}$ is the character decomposition of $V \otimes \mathbb{C}|_{\Sigma[i]}$ under the lift of the S^1 action; and $\delta(\cdot)$ is defined in Eq. (6.35).

Proof of Lemma 9.7. The first assertion of the lemma follows immediately from Lemma 9.8 by replacing V by T^*M in the latter's statement. The unprimed version of the second assertion of Lemma 9.7 follows from Lemma 9.8 by replacing V by $T^* \oplus V^*$ in the latter's statement.

To prove the primed case of Assertion (2) of Lemma 9.7, consider the following facts: Let $0 \leqq v' < n_0$ be an integer. Suppose that v is a non-negative integer which equals $v' \bmod(n_0)$. Then $[p_0 v/n_0 + 1/2] = [p_0 v/n_0]$ if and only if there is an integer $k \in [0, p_0)$ with the property that $k \leqq p_0 \cdot v'/n_0 < (k+1/2)$. Therefore, for $i = 0, 1$:

$$\sum_{0<v\in\mathbb{Z}} \dim(\underline{V}_v^*|_{\Sigma(i)}) \cdot [p_0 v/n_0 + \tfrac{1}{2}] = \sum_{0<v\in\mathbb{Z}} \dim(\underline{V}_v^*|_{\Sigma(i)}) \cdot [p_0 v/n_0]$$
$$+ \sum_{0<k\leqq p_0} \; \sum_{0<v'<n_0 : (k-1/2) < p_0 v'/n_0 < k} \; \sum_{0<v : v = v' \bmod(n_0)} \dim(\underline{V}_v^*|_{\Sigma(i)})$$
$$+ \sum_{0<v : v = n_0/2 \bmod(n_0)} \dim(\underline{V}_v^*|_{\Sigma(i)}). \tag{9.19}$$

Here, an allusion to Lemma 9.3 has been made.

The last term in Eq. (9.19) is $\dim_{\mathbb{R}}(V(n_0; n_0/2))$, so is the same for $\Sigma[0]$ and for $\Sigma[1]$.

If one is only concerned with equivalences mod(2), then, mod(2), the second term on the left-hand side of Eq. (9.19) is equal to

$$\left(\sum_{n_0/2<\upsilon'<n_0}\ \sum_{0<\upsilon:\upsilon=\upsilon' \bmod(n_0)} \dim(\underline{V}_\upsilon^*|_{\Sigma(i)})\right)_{\bmod(2)}$$
$$+\left(\sum_{0<k\leqq p_0}\ \sum_{0<\upsilon'<n_0/2:(k-1/2)<p_0\upsilon'/n_0<k} \dim(V(n_0;\upsilon'))\right)_{\bmod(2)}. \qquad (9.20)$$

The second term above is the same for $\Sigma[0]$ and for $\Sigma[1]$.

Equations (9.19–20) and (6.35) imply that

$$\left(\sum_{0<\upsilon\in\mathbb{Z}} (\dim(\underline{V}_\upsilon^*|_{\Sigma(0)})-\dim(\underline{V}_\upsilon^*|_{\Sigma(1)}))\cdot[p_0\upsilon/n_0+\tfrac{1}{2}]\right.$$
$$=\left(\sum_{0<\upsilon\in\mathbb{Z}} (\dim(\underline{V}_\upsilon^*|_{\Sigma(0)})-\dim(\underline{V}_\upsilon^*|_{\Sigma(1)}))\cdot[p_0\upsilon/n_0]\right.$$
$$\left.+\delta'(n_0,\Sigma(0),V)-\delta'(n_0,\Sigma(1),V)\right)_{\bmod(2)}$$
$$\left.+\delta(n_0,\Sigma(0),V)-\delta(n_0,\Sigma(1),V)\right)_{\bmod(2)}. \qquad (9.21)$$

This last equation, plus Eq. (9.21) implies the primed version of Assertion (2) of Lemma 9.7.

10. Conclusions

The proof of Theorem 1.3 is completed in this section; due to Lemma 8.2, it is a corollary to the following proposition:

Proposition 10.1. *Let M be a compact, oriented and spin Riemannian manifold. Assume that S^1 acts isometrically on M. Let $V\to M$ be a real, oriented vector bundle to which the S^1 action lifts. Require that $w_2(V)=0$, and also, require that V be weakly S^1-compatible with T^*M in the sense of Definition 8.1. Let e denote the constant that appears in Requirement* (4) *of Definition 8.1. Construct $\mathscr{E}_{NM(1),V}$ and $\Phi_{NM(1)}$ and the operator D_t on $C^\infty(\mathscr{E}_{NM(1),V}\otimes\Phi_{NM(1)})$ in both the unprimed and primed cases, as specified in Proposition 5.2. Let $\mathscr{E}_{NM(1),Vm}$ be the subbundle of $\mathscr{E}_{NM(1),V}$ on which the canonical circle generator P acts as multiplication by m. Let k be an eigenvalue of the geometric circle generator, K, on $C^\infty(\mathscr{E}_{NM(1),Vm}\otimes\Phi_{NM(1)})$. For $\ell=\ell_e$ or ℓ_s in Eqs.* (4.17, 18), *and for the unprimed or primed cases, define* $\mathrm{Ind}(D_t,\mathscr{E}_{NM(1),Vm},\ell,k)$ *as in Eq.* (6.2). *If $e\neq 0$, then* $\mathrm{Ind}(D_t,\mathscr{E}_{NM(1),Vm},\ell,k)=0$. *If $e=0$, then* $\mathrm{Ind}(D_t,\mathscr{E}_{NM(1),Vm},\ell,k)=0$ *for all $k\neq 0$.*

Proof of Proposition 10.1. Introduce the line bundle $L_0\to M$ of Definition 8.1. Due to Propositions 6.2, 7.8, and 8.3,

$$\mathrm{Ind}(D_t,\mathscr{E}_{NM(1),Vm}\otimes L^p,\ell,k)=\pm\mathrm{Ind}(D_t,\mathscr{E}_{NM(1),Vm'}\otimes L_0^{p-p'},\ell,k), \qquad (10.1)$$

where $m'\equiv m+p'\cdot k+(-p\cdot p'+p'^2/2)\cdot e$.

Suppose first that $e=0$. Then, following Witten [W2], when $k\neq 0$, one can take $p=0$ and p' such that $m'<0$. But, for $m'<0$. Proposition 4.1 asserts that the right-hand side of Eq. (10.1) is equal to 0.

If $e<0$, take $p=0$ and $p'>2\cdot(m+k^2/e)/\sqrt{e}$. This makes $m'<0$ on the right-hand side of Eq. (10.1), and, again, the right-hand side is equal to 0. If $e>0$, take $p=p'$ and switch the roles of m and m'; then repeat the preceding argument.

Appendix. Fredholm Properties

The purpose of this appendix is to consider the detailed properties of the various operators which are constructed in Sect. 2, 3, and 4.

Part 1. Topologies

Consider topological and smooth structures on a countable direct sum of smooth, finite dimensional vector bundles. Let $\{B_n \to M\}_{n\in\Lambda}$ be a countable set of smooth, finite dimensional vector bundles over the compact manifold M. Define

$$\mathscr{C} \equiv \bigoplus_{n\in\Lambda} B_n \to M \tag{A.1}$$

to be the set of points in the countable direct sum which have only finitely many fiber coordinates non-zero. Give $\mathscr{C}$ the direct limit topology (see, e.g. [Wh]). Let X be a compact manifold. A map $f: X\to\mathscr{C}$ is declared to be C^k $(0\leqq k\leqq\infty)$ if and only if there exists a finite subset $\lambda\subset\Lambda$ such that f factors through the finite dimensional sub-bundle $\oplus_{n\in\lambda}B_n\to M$; and does so as a C^k map. Conversely, a map $f:\mathscr{C}\to X$ is C^k if and only if the restriction of f to every finite dimensional subbundle $\oplus_{n\in\lambda}B_n\to M$ is C^k. This makes the projection from $\mathscr{C}\to M$ a smooth map.

Let $\mathscr{C}$, $\mathscr{C}'\to M$ be as defined in Eq. (A.1) from countable sets $\{B_n\to M\}_{n\in\Lambda}$, $\{B'_n\to M\}_{n\in\Lambda'}$ of smooth, finite dimensional vector bundles. A map $f:\mathscr{C}\to\mathscr{C}'$ is defined to be a C^k bundle map if and only if the restriction of f to each finite dimensional subbundle of $\mathscr{C}$ defines a C^k vector bundle map into a finite dimensional subbundle of $\mathscr{C}'$.

For example, $\{B_n\oplus B_n\to M\}_{n\in\Lambda}$ defines $\mathscr{C}\oplus\mathscr{C}$; and fiber addition: $\mathscr{C}\oplus\mathscr{C}\to\mathscr{C}$ is a C^∞ bundle map. For a second example, suppose that $U\subset M$ is an open set over which each B_n admits a trivialization, $\varphi_n: B_n\to U\times\mathbb{R}^{N(n)}$. Then, $\oplus_{n\in\Lambda}\varphi_n:\mathscr{C}\to U\times_{n\in\Lambda}\mathbb{R}^{N(n)}$ defines a smooth bundle map which respects fiber addition. In this sense, $\mathscr{C}\to M$ is locally trivial; and defines a vector bundle over M with fiber $\mathbb{R}^\infty$.

For $\mathscr{C}$ and $\mathscr{C}'$ as above, define $\mathscr{C}\oplus\mathscr{C}'$ using $\{B_n\}\cup\{B_{n'}\}$ as defining data. For $\mathscr{C}\otimes\mathscr{C}'$, use $\{B_n\otimes B_{n'}\}$.

All of the infinite dimensional vector bundles over M which are constructed in this article will be defined, implicitly, in this way. All of the bundle maps between these bundles will be smooth in the sense above.

Part 2. Fredholm Properties

Introduce the vector bundle $\mathscr{B}_E\to\Sigma$ in Eq. (4.2). Note that $\mathscr{B}_E$ is locally the space of finite polynomials in the coordinates on $\mathscr{N}E$ and their complex conjugates. The function Φ_E in Eq. (4.12) defines an inner product on the vector bundle $\mathscr{B}_E\otimes\Phi_E\to\Sigma$. One can construct this inner product at $x\in M$ by using $\Phi_E|_x$ to define a Gaussian measure on $\mathscr{B}_E|_x$. Equivalently, one can define creation and annihilation operators along the fiber $\mathscr{N}E|_x$ at $x\in M$: Introduce the local coordinates

$\{x_n, z_{\upsilon,n}\} \equiv \{x_n^a, z_{\upsilon,n}^j\}$ on $\mathcal{N}E|_x$. For $0<\upsilon<r$ and $n \in \mathbb{Z}+\alpha(\upsilon)$; or for $\upsilon=r$ and $0<n\in\mathbb{Z}+\alpha(r)$, let

$$\begin{aligned} f_{\upsilon,n}^* &\equiv i\cdot(\partial/\partial z_{\upsilon,n} - t\cdot|\mu[\upsilon,n]|\cdot z_{\upsilon,n}) \quad \text{and} \quad \underline{f}_{\upsilon,n}^* \equiv i\cdot(\partial/\partial z_{\upsilon,n} - t\cdot|\mu[\upsilon,n]|\cdot \underline{z}_{\upsilon,n}); \\ f_{\upsilon,n} &\equiv i\cdot(\partial/\partial z_{\upsilon,n} + t\cdot|\mu[\upsilon,n]|\cdot \underline{z}_{\upsilon,n}) \quad \text{and} \quad \underline{f}_{\upsilon,n} \equiv i\cdot(\partial/\partial \underline{z}_{\upsilon,n} + t\cdot|\mu[\upsilon,n]|\cdot z_{\upsilon,n}). \end{aligned} \tag{A.2}$$

Here, $\mu[r,n]\equiv n$ by flat. For $\alpha=0$, introduce for $0<n\in\mathbb{Z}$,

$$\begin{aligned} a_n^* &\equiv i\cdot(\partial/\partial \underline{x}_n - t\cdot n\cdot x_n) \quad \text{and} \quad \underline{a}_n^* \equiv i\cdot(\partial/\partial x_n - t\cdot n\cdot \underline{x}_n); \\ a_n &\equiv i\cdot(\partial/\partial x_n + t\cdot n\cdot \underline{x}_n) \quad \text{and} \quad \underline{a}_n \equiv i\cdot(\partial/\partial \underline{x}_n + t\cdot n\cdot x_n). \end{aligned} \tag{A.3}$$

The function Φ_E is annihilated by each of $\{a_m, \underline{a}_m, f_{\upsilon,n}, \underline{f}_{\upsilon,n}\}$, while the fiber of $\mathcal{B}_E\otimes\Phi_E$ at x is naturally isomorphic to the vector space of finite linear combinations of functions which are obtained from Φ_E by differentiating finitely many times with the creation operators $\{a_m^*, \underline{a}_m^*, f_{\upsilon,n}^*, \underline{f}_{\upsilon,n}^*\}$. The inner product on $\mathcal{B}_E\otimes\Phi_E$ is obtained by declaring Φ_E to have norm equal to 1; and by declaring that each of $\{a_m, \underline{a}_m, f_{\upsilon,n}, \underline{f}_{\upsilon,n}\}$ and the corresponding member of $\{a_m^*, \underline{a}_m^*, f_{\upsilon,n}^*, \underline{f}_{\upsilon,n}^*\}$ are adjoint.

Introduce the vector bundle $\mathcal{E}_{E,V}$ from Eq. (4.13). A smooth fiber metric, $\langle\,,\rangle$, is defined on $\mathcal{E}_{E,V}\otimes\Phi_E$, as is a metric compatible connection.

The endomorphisms P_E of Eq. (4.3), P_F of Eq. (4.8) and P_V of Eq. (4.9) annihilate Φ_E and induce endomorphisms of $\mathcal{E}_{E,V}\otimes\Phi_E$ which are hermitian with respect to the fiber metric. They are also covariantly constant.

Define endomorphisms $\mathcal{E}_{E,V}\otimes\Phi_E$ by

$$\begin{aligned} K_E &\equiv -\sum_{0<\upsilon<r}\ \sum_{n\in\mathbb{Z}+\alpha(\upsilon)} \upsilon\cdot(z_{\upsilon,n}\cdot\partial/\partial z_{\upsilon,n} - \underline{z}_{\upsilon,n}\cdot\partial/\partial\underline{z}_{\upsilon,n}), \\ K_F &\equiv -\sum_{0<\upsilon<r}\ \sum_{0<n\in\mathbb{Z}+\alpha(r)} \upsilon\cdot\Gamma_{\upsilon,n}^*\cdot\Gamma_{\upsilon,n} \\ &\quad + \sum_{0<\upsilon<r}\ \sum_{0<n\in\mathbb{Z}-\alpha(r)} \upsilon\cdot\underline{\Gamma}_{\upsilon,n}^*\cdot\underline{\Gamma}_{\upsilon,n}, \\ K_V &\equiv -\sum_{0<\upsilon<r}\ \sum_{0>n\in\mathbb{Z}+\beta(\upsilon)} \upsilon\cdot\Theta_{\upsilon,n}^*\cdot\Theta_{\upsilon,n} \\ &\quad + \sum_{0<\upsilon<n_0/2}\ \sum_{0>n\in\mathbb{Z}-\beta(\upsilon)} \upsilon\cdot\underline{\Theta}_{\upsilon,n}^*\cdot\underline{\Theta}_{\upsilon,n}). \end{aligned}$$

These are also covariantly constant, and hermitian.

Let $P\equiv P_B+P_F+P_V$ and $K\equiv K_B+K_F+K_V$. Note that P commutes with K; and both are covariantly constant. Neither is bounded, but it is not hard to diagonalize them explicitly on each fiber. Their spectra are real, discrete, with no accumulation points. Fiberwise, the (m,k) eigenspaces of P and K define a subbundle $\mathcal{E}_{E,Vm}(k)\subset\mathcal{E}_{E,V}$. These subbundles decompose $\mathcal{E}_{E,V}\otimes\Phi_E$ orthogonally into the direct sum $\oplus_{(m,k)}(\mathcal{E}_{E,Vm}(k)\otimes\Phi_E)$.

One can also ignore K, and decompose $\mathcal{E}_{E,V}\otimes\Phi_E=\oplus_m(\mathcal{E}_{E,Vm}\otimes\Phi_E)$, where $\mathcal{E}_{E,Vm}\to\mathcal{E}_{E,V}$ is the subbundle on which P acts as multiplication by m. Thus, $\mathcal{E}_{E,Vm}\otimes\Phi_E=\oplus_k(\mathcal{E}_{E,Vm}(k)\otimes\Phi_E)$ is an orthogonal decomposition into subbundles.

Introduce the Dirac operator D_t on $C^\infty(\mathcal{E}_0(\Sigma)\otimes E\otimes\Phi_0)$ in Eq. (4.14). Observe that D_t commutes with P and with K; and so defines an endomorphism of $C^\infty(\mathcal{E}_{E,Vm}\otimes\Phi_E)$ and also one of $C^\infty(\mathcal{E}_{E,Vm}(k)\otimes\Phi_E)$.

Let $L^2(\underline{\mathscr{E}}_{E,Vm}\otimes\Phi_E)$, $L^2(\underline{\mathscr{E}}_{E,Vm}(k)\otimes\Phi_E)$ denote the L^2-completions of the space of smooth sections of $\underline{\mathscr{E}}_{E,Vm}\otimes\Phi_E$, $\underline{\mathscr{E}}_{E,Vm}(k)\otimes\Phi_E\to M$. Use the fiber metric to define the L^2-inner product in Eq. (4.15). Likewise, let $H^1(\underline{\mathscr{E}}_{E,Vm}\otimes\Phi_E)$, $H^1(\underline{\mathscr{E}}_{E,Vm}(k)\otimes\Phi_E)$ denote the completions of $C^\infty(\underline{\mathscr{E}}_{E,Vm}\otimes\Phi_E)$ and of $C^\infty(\underline{\mathscr{E}}_{E,Vm}(k)\otimes\Phi_E)$ using the H^1-norm in Eq. (4.15). By construction, D_t defines a bounded, linear operator from H^1 to L^2.

It is the purpose of this section to explore the Fredholm properties of D_t. The crucial issue is the choice of data $\{\mu[\upsilon,n]\}$.

Proposition A.1. *For each integer $\upsilon\in(0,r)$ and integer $n\in\mathbb{Z}+\alpha(\upsilon)$, specify a set of numbers $\{\mu[\upsilon,n]\}_{n\in\mathbb{Z}+\alpha(\upsilon)}$ with the following properties:*
(1) *There exists $\delta>0$, such that $|\mu[\upsilon,n]|>\delta$ for all $\upsilon\in(0,r)$.*
(2) *There exists a constant κ such that $\mu[\upsilon,n]\cdot(n-\kappa\cdot\upsilon)>\delta\cdot|\mu[\upsilon,n]|$ for all $\upsilon\in(0,r)$ and integers $n\in\mathbb{Z}+\alpha(\upsilon)$.*
(3) *The sign of $\mu[\upsilon,0]$ is the same for all υ having $\alpha(\upsilon)=0$. Distinguish Case* (I): *$\kappa=0$ and all $\alpha(\upsilon)\in(0,1]$; from Case* (II): *$\kappa\neq0$, or $\alpha(\upsilon)=0$ for some υ. In the former, set $\underline{\mathscr{E}}\equiv\underline{\mathscr{E}}_{E,Vm}$, and in the latter, set $\underline{\mathscr{E}}\equiv\underline{\mathscr{E}}_{E,Vm}(k)$.*
1) *For fixed $t>0$, D_t extends to define a Fredholm operator from $H^1(\underline{\mathscr{E}}\otimes\Phi_E)$ to $L^2(\underline{\mathscr{E}}\otimes\Phi_E)$.*
2) Coker$(D_t)\equiv$ Ker(D_t), *and this is a subspace of $C^\infty(\underline{\mathscr{E}}\otimes\Phi_E)$.*
3) *Suppose that for all $n\neq0$ and for all υ, $n\cdot\mu[\upsilon,n]>0$. Then* ker(D_t) *and* coker(D_t) *are empty for $m<0$.*

Proof of Proposition A.1. The kernel and cokernel of Q can be analyzed by using together a Wietzenbock formula and the decomposition of $\underline{\mathscr{E}}$ into its finite dimensional subbundles.

To decompose in a convenient way, introduce the (finite) set $\Omega\equiv\{(\upsilon,n\neq0):n\cdot\mu[\upsilon,n]<0\}$. Then, introduce

$$\begin{aligned}
H=2\cdot&\sum_{0<m\in\mathbb{Z}}(t^{-1}\cdot a_m^*\cdot a_m+m\cdot\Gamma_m^*\Gamma_m)\\
&+2\cdot\sum_{(\upsilon,0<n)\notin\Omega}(t^{-1}\cdot f_{\upsilon,n}^*f_{\upsilon,n}+|\mu[\upsilon,n]|\cdot\Gamma_{\upsilon,n}^*\cdot\Gamma_{\upsilon,n})\\
&+2\cdot\sum_{(\upsilon,0<n)\notin\Omega}(t^{-1}\cdot \underline{f}_{\upsilon,-n}^*\underline{f}_{\upsilon,-n}+|\mu[\upsilon,-n]|\cdot\underline{\Gamma}_{\upsilon,n}^*\cdot\underline{\Gamma}_{\upsilon,n})\\
&+2\cdot\sum_{(\upsilon,0<n)\in\Omega}(t^{-1}\cdot \underline{f}_{\upsilon,n}^*\underline{f}_{\upsilon,n}+|\mu[\upsilon,n]|\cdot\Gamma_{\upsilon,n}\cdot\Gamma_{\upsilon,n}^*)\\
&+2\cdot\sum_{(\upsilon,0<n)\in\Omega}(t^{-1}\cdot f_{\upsilon,-n}^*f_{\upsilon,-n}+|\mu[\upsilon,-n]|\cdot\underline{\Gamma}_{\upsilon,n}\cdot\underline{\Gamma}_{\upsilon,n}^*)\\
&+2\cdot\sum_{0<\upsilon<r:\alpha(\upsilon)=0}x\cdot(t^{-1}\cdot f_{\upsilon,0}^*f_{\upsilon,0}+|\mu[\upsilon,0]|\cdot\Gamma_{\upsilon,0}^*\cdot\Gamma_{\upsilon,0})\\
&+2\cdot\sum_{0<\upsilon<r:\alpha(\upsilon)=0}(1-x)\cdot(t^{-1}\cdot \underline{f}_{\upsilon,0}^*\underline{f}_{\upsilon,0}+|\mu[\upsilon,0]|\cdot\Gamma_{\upsilon,0}\cdot\Gamma_{\upsilon,0}^*)\\
&+2\cdot\sum_{0<n\in\mathbb{Z}+\alpha(r)}(t^{-1}\cdot f_{\upsilon,n}^*f_{r,n}+n\cdot\Gamma_{r,n}^*\cdot\Gamma_{r,n}),
\end{aligned}\tag{A.5}$$

where $x\equiv1$ if $\{\mu[\upsilon,0]>0\}_{0<\upsilon<r:\alpha(\upsilon)=0}$, and otherwise $x\equiv0$. The sum over $(\upsilon,0<n)$, above, is over $\upsilon\in\{0,\dots,r-1\}$ and $0<n\in\mathbb{Z}+\alpha(\upsilon)$, though subject to the relevant Ω constraint.

This endomorphism is covariantly constant. It is also positive semi-definite, with discrete spectrum whose first non-zero eigenvalue equals

$$\lambda_1 \equiv \inf\{1, \{|\mu[v, n]|\}_{0<v<r, n\in\mathbb{Z}+\alpha(v)}, \alpha(r)\}. \tag{A.6}$$

(It is an easy exercise to diagonalize H.)

A simple calculation shows that P and K both commute with H as endomorphisms of $\underline{\mathscr{E}}$; and, as an endomorphism of $C^\infty(\underline{\mathscr{E}})$, H commutes with D_t. Indeed, introduce the covariantly constant endomorphism T_E of Eq. (4.14). Then

$$H = t^{-1} \cdot T_E^2. \tag{A.7}$$

Decompose $\underline{\mathscr{E}}$ into the eigenspaces of H;

$$\underline{\mathscr{E}} \equiv \bigoplus_{h\in\mathrm{spec}(H)} \underline{\mathscr{E}}(h). \tag{A.8}$$

Lemma A.2. *The vector bundle $\underline{\mathscr{E}}(h)\to\Sigma$ is a finite dimensional vector bundle of the form $S^0(U)\otimes R$, where $U\to M$ is given in Eq.* (4.10), *$S^0(U)$ is the appropriate* spin *or* $\mathrm{spin}_{\mathbb{C}}$ *bundle and $R\equiv R_m(h)$ in Case* (I) *and $R\equiv R_m(k,h)$ in Case* (II).

This lemma will be proved shortly; assume its validity for now.

Since each of P, K, and H commutes with D_t, D_t induces an endomorphism of $C^\infty(\underline{\mathscr{E}}(h)\otimes\Phi_E)$.

As an endomorphism of $C^\infty(\underline{\mathscr{E}}(h)\otimes\Phi_E)$, D_t is an operator of the form

$$\partial + A, \tag{A.9}$$

where ∂ is the Dirac operator coupled to R and where $A\equiv A_m(h)$ or $A_m(k,h)$ is a covariantly constant endomorphism of $R_m(h)$ or $R_m(k,h)$, respectively.

To analyze the operator D_t one can work with the set of "ordinary" Dirac operators, $\{\partial + A\}$. However, it is convenient to manipulate these operators as one; indeed, the ability to do this is the great achievement of supersymmetry. However, the rigorous justification for the manipulations that follow stems ultimately from the decomposition of Q into its "components", $\{\partial + A\}$.

The analysis of D_t requires the Wietzenbock formula for D_t^2. Use the fact that T_E anti-commutes with $e^a\cdot\nabla_a$ to derive the following: For $\Psi\in C^\infty(\underline{\mathscr{E}}(h)\otimes\Phi_E)$,

$$\langle D_t\Psi, D_t\Psi\rangle_{L^2} = \langle e^a\cdot\nabla_a\Psi, e^a\cdot\nabla_a\Psi\rangle_{L^2} + \langle T_E\Psi, T_E\Psi\rangle_{L^2}. \tag{A.10}$$

The last term in Eq. (A.10) is $\langle\Psi, t^{-1}\cdot H\Psi\rangle_{L^2}$. Thus, for $\Psi\in C^\infty(\underline{\mathscr{E}}(h)\otimes\Phi_E)$,

$$\langle D_t\Psi, D_t\Psi\rangle_{L^2} \geqq \langle e^a\cdot\nabla_a\Psi, e^a\cdot\nabla_a\Psi\rangle_{L^2} + t\cdot h\cdot\langle\Psi,\Psi\rangle_{L^2} \geqq t\cdot h\cdot\langle\Psi,\Psi\rangle_{L^2}. \tag{A.11}$$

Equation (A.11) implies that any kernel of D_t on $H^1(\underline{\mathscr{E}}\otimes\Phi_E)$ must lie in $H^1(\underline{\mathscr{E}}(0)\otimes\Phi_E)$. According to Lemma A.1, $\underline{\mathscr{E}}(0)\to M$ is a finite dimensional vector bundle. Since the restriction of D_t to $C^\infty(\underline{\mathscr{E}}(0)\otimes\Phi_E)$ is of the form in Eq. (A.8), standard elliptic theory asserts that the kernel of D_t on $H^1(\underline{\mathscr{E}}(0)\otimes\Phi_E)$ is finite dimensional, and consists of smooth sections. Furthermore, there is a positive constant, c_0, such that the quadratic form $\langle D_t\cdot, D_t\cdot\rangle_{L^2}$ bounds $c_0\cdot\langle\Psi,\Psi\rangle_{L^2}$ on restriction to the L^2-orthogonal complement in $H^1(\underline{\mathscr{E}}(0)\otimes\Phi_E)$ to this kernel.

To summarize, for $t>0$, the kernel of D_t on $H^1(\underline{\mathscr{E}}\otimes\Phi_E)$ is finite dimensional, and it consists of smooth sections contained in $H^1(\underline{\mathscr{E}}(0)\otimes\Phi_E)$. On the complement of this kernel, $\langle D_t\cdot, D_t\cdot\rangle_{L^2}$ bounds a constant times the H^1-norm. This fact implies that D_t has closed range in $L^2(\underline{\mathscr{E}}\otimes\Phi_E)$.

For the cokernel of D_t, consider a non-zero element $\Psi \in \operatorname{coker}(D_t) \subset L^2(\mathscr{E} \otimes \Phi_E)$. Then,

$$\langle \Psi, D_t \Phi \rangle_{L^2} = 0 \tag{A.12}$$

for all $\Phi \in H^1(\mathscr{E} \otimes \Phi_E)$. Write $\Psi \equiv \sum_h \Psi_h$, with $\Psi_h \in L^2(\mathscr{E}(h) \otimes \Phi_E)$. Then, each $\Psi_h \in \operatorname{coker}(D_t)$. Dealing with Ψ_h instead of Ψ, one can invoke the standard theory of elliptic operators on sections of finite dimensional vector bundles over M to conclude that $\Psi_h \in H^1(\mathscr{E}(h) \otimes \Phi_E) \cap C^\infty(\mathscr{E}(h) \otimes \Phi_E)$ and $\Psi_h \in \ker(D_t)$. In particular, this requires $h = 0$. In conclusion, $\operatorname{coker}(D_t)$ is finite dimensional and equals $\ker(D_t)$.

Only Assertion 3 of Proposition A.1 remains unproved. To prove this last assertion, decompose the symmetric endomorphism P into positive and negative parts, $P = P_L - P_R$. Here,

$$\begin{aligned} P_R = P_F + \tfrac{1}{2} \cdot t^{-1} \cdot \Big(\sum_{0 < n \in \mathbb{Z}} a_n^* \cdot a_n + \sum_{0 < n \in \mathbb{Z} + \alpha(r)} f_{\upsilon,n}^* \cdot f_{r,n} \Big) \\ + \sum_{0 < \upsilon < r} \sum_{0 < n \in \mathbb{Z} + \alpha(\upsilon)} n \cdot |\mu[\upsilon, n]|^{-1} \cdot f_{\upsilon,n}^* \cdot f_{\upsilon,n} \\ + \sum_{0 < \upsilon < r} \sum_{0 < n \in \mathbb{Z} - \alpha(\upsilon)} n \cdot |\mu[\upsilon, -n]|^{-1} \cdot \underline{f}_{\upsilon,-n}^* \cdot \underline{f}_{\upsilon,-n} . \end{aligned} \tag{A.13}$$

Under the condition that $n \cdot \mu[\upsilon, n] > 0$ for all $n \neq 0$, there is a constant $c > 0$, which is such that

$$H \geqq c \cdot P_R \tag{A.14}$$

holds as an identity of symmetric, non-negative endomorphisms of $\mathscr{E} \otimes \Phi_E$. The verification of Eq. (A.14) can be made by comparing Eq. (A.5) with Eq. (A.13).

Proof of Lemma A.2. Introduce twelve "occupation number" endomorphisms; the first two are defined to be

$$\begin{aligned} m(R) \equiv \tfrac{1}{2} \cdot \sum_{0 < m \in \mathbb{Z}} (t^{-1} \cdot a_m^* \cdot a_m + \cdot \Gamma_m^* \Gamma_m) + \tfrac{1}{2} \cdot \sum_{0 < n \in \mathbb{Z} + \alpha(r)} (t^{-1} \cdot f_{r,n}^* f_{r,n} + n \cdot \Gamma_{r,n}^* \cdot \Gamma_{r,n}) , \\ m(L) \equiv \tfrac{1}{2} \cdot \sum_{0 < m \in \mathbb{Z}} t^{-1} \cdot \underline{a}_m^* \cdot \underline{a}_m + \tfrac{1}{2} \cdot \sum_{0 < n \in \mathbb{Z} + \alpha(r)} t^{-1} \cdot \underline{f}_{r,n}^* \underline{f}_{r,n} . \end{aligned} \tag{A.15}$$

The last ten are defined for any collection $\{\sigma[\upsilon, n] \geqq 0\}$: The third through the sixth are

$$\begin{aligned} n(R, \sigma) \equiv \tfrac{1}{2} \cdot \sum_{(\upsilon, 0 < n) \notin \Omega} \sigma[\upsilon, n] \cdot ((t \cdot |\mu[\upsilon, n]|)^{-1} \cdot f_{\upsilon,n}^* f_{\upsilon,n} + \Gamma_{\upsilon,n}^* \cdot \Gamma_{\upsilon,n}) , \\ \underline{n}(R, \sigma) \equiv \tfrac{1}{2} \cdot \sum_{(\upsilon, 0 < n) \notin \Omega} \sigma[\upsilon, -n] \cdot ((t \cdot |\mu[\upsilon, -n]|)^{-1} \cdot \underline{f}_{\upsilon,-n}^* \underline{f}_{\upsilon,-n} + \underline{\Gamma}_{\upsilon,n}^* \cdot \underline{\Gamma}_{\upsilon,n}) , \\ n(L, \sigma) \equiv \tfrac{1}{2} \cdot \sum_{(\upsilon, 0 < n) \notin \Omega} \sigma[\upsilon, -n] \cdot (t \cdot |\mu[\upsilon, -n]|)^{-1} \cdot f_{\upsilon,-n}^* f_{\upsilon,-n} , \\ \underline{n}(L, \sigma) \equiv \tfrac{1}{2} \cdot \sum_{(\upsilon, 0 < n) \notin \Omega} \sigma[\upsilon, n] \cdot (t \cdot |\mu[\upsilon, n]|)^{-1} \cdot \underline{f}_{\upsilon,n}^* \underline{f}_{\upsilon,n} . \end{aligned} \tag{A.16}$$

Here, the sums over $(\upsilon, 0 < n)$ are sums over $\upsilon \in \{1, \ldots, r-1\}$ and over $0 < n \in \mathbb{Z} + \alpha(\upsilon)$.

The seventh through the tenth are denoted by $\{n'(R, \sigma), \underline{n}'(R, \sigma), n'(L, \sigma), \underline{n}'(L, \sigma)\}$; they are given by Eq. (A.16) but with the sum restrictions in Eq. (A.16) changed to require that $(\upsilon, n) \in \Omega$. If the resulting subset of Ω is empty, set the corresponding $\{n'(R, \sigma), \underline{n}'(R, \sigma), n'(L, \sigma), \underline{n}'(L, \sigma)\}$ equal to zero.

Finally define the last two "occupation number" endomorphisms to be

$$\mathscr{n}_0(\sigma) \equiv \sum_{0<\upsilon<r:\alpha(\upsilon)=0} \sigma[\upsilon,0]\cdot((t\cdot|\mu[\upsilon,0]|)^{-1}\cdot f_{\upsilon,0}^{*}f_{\upsilon,0}+\Gamma_{\upsilon,n}^{*}\cdot\Gamma_{\upsilon,0}),$$
$$\underline{\mathscr{n}}_0(\sigma) \equiv \sum_{0<\upsilon<r:\alpha(\upsilon)=0} \sigma[\upsilon,0]\cdot(t\cdot|\mu[\upsilon,0]|)^{-1}\cdot \underline{f}_{\upsilon,0}^{*}\underline{f}_{\upsilon,0}\,. \tag{A.17}$$

Each of the twelve endomorphisms above is symmetric and each is non-negative with integer eigenvalues when $\{\sigma[\upsilon,n]\}$ are sets of non-negative integers. Furthermore, each commutes with P, K, and H; and the twelve commute amongst themselves.

By inspection, $\mathscr{E}(h)$ has finite dimensional fiber if and only if P_V from Eq. (4.9) and each of the twelve number operators from above is bounded when $\{\sigma[\upsilon,n]\}$ are bounded away from zero.

Introduce the constant κ from Assumption (2) of Proposition A.1. Introduce the variable $x\equiv 1/2\cdot(1+\operatorname{sign}(\mu[\upsilon,0]|_{\upsilon:\alpha(\upsilon)=0}))$ in Eq. (A.5). The restrictions on the set $\{\mu[\upsilon,n]\}$ imply that $x=0$ only when $\kappa\leqq 0$ and when $\{\underline{\mathscr{n}}'(R,\sigma),\ \mathscr{n}'(L,\sigma)\}$ are zero. Also, $x=1$ only when $\kappa\geqq 0$ and when $\{\mathscr{n}'(R,\sigma),\ \underline{\mathscr{n}}'(L,\sigma)\}$ are zero. Furthermore, if $\kappa=0$, then all $\{\mathscr{n}'(R,\sigma),\ \underline{\mathscr{n}}'(R,\sigma),\ \mathscr{n}'(L,\sigma),\ \underline{\mathscr{n}}'(L,\sigma)\}$ are zero.

Consider the case when $x=1$ and $\kappa\geqq 0$; the $x=0$, $\kappa\leqq 0$ case is handled by an analogous argument which is left to the reader. In the $x=1$ case, H gives a bound for

$$\mathscr{m}(R)+\mathscr{n}(R,\sigma_0)+\underline{\mathscr{n}}(R,\sigma_0)+\mathscr{n}'(L,\sigma_0)+\mathscr{n}_0(\sigma_0)\,, \tag{A.18}$$

when $\{\sigma_0[\upsilon,n]=|\mu[\upsilon,n]|\}$.

Bounds on the remaining "occupation number" endomorphisms come from studying P and K. Take $\{\sigma_1[\upsilon,n]\equiv|n|\}$ and note that P decomposes as

$$P\equiv \mathscr{m}(L)+\mathscr{n}(L,\sigma_1)+\underline{\mathscr{n}}(L,\sigma_1)+\mathscr{n}'(L,\sigma_1)+$$
$$+P_V-\mathscr{m}(R)-\mathscr{n}(R,\sigma_1)-\underline{\mathscr{n}}(R,\sigma_1)-\underline{\mathscr{n}}'(R,\sigma_1)\,, \tag{A.19}$$

where P_V is the non-negative endomorphism which is given in Eq. (4.9).

Note that in Case (I) of Proposition A.1, $\Omega=\emptyset$ and $\mathscr{n}_0(\sigma)\equiv\mathscr{n}_0(\sigma)\equiv 0$. Then, Constraint (1) on the set $\{\mu[\upsilon,n]\}$ implies via Eq. (A.18) and Eq. (A.19) that

$$\mathscr{m}(L)+\mathscr{n}(L,\sigma_1)+\underline{\mathscr{n}}(L,\sigma_1)+\mathscr{n}'(L,\sigma_1)+P_V \tag{A.20}$$

is also a bounded endomorphism of $\mathscr{E}(h)$. This last bound plus Eq. (A.18) bound all of the "occupation number" endomorphisms of $\mathscr{E}(h)$ in Eqs. (A.16–17). Such bounds imply Lemma A.2 for Case (I) of Proposition A.1.

To make further progress, note that in the general case, the bound on the endomorphism in Eq. (A.20) cannot be deduced directly. In the general case, Eq. (A.19) plus the bound on the endomorphism in Eq. (A.18), plus the fact that P acts on $\mathscr{E}(h)$ as multiplication by m give an upper and lower bound on

$$\mathscr{m}(L)+\mathscr{n}(L,\sigma_1)+\underline{\mathscr{n}}(L,\sigma_1)+P_V-\underline{\mathscr{n}}'(R,\sigma_1)\,. \tag{A.21}$$

Additional information comes from the endomorphism K. To extract it, set $\{\sigma_2[\upsilon,n]\equiv\upsilon\}$ and decompose K as

$$K\equiv\underline{\mathscr{n}}(R,\sigma_2)+\underline{\mathscr{n}}(L,\sigma_2)+\underline{\mathscr{n}}'(R,\sigma_2)+\underline{\mathscr{n}}_0(\sigma_2)+K_{V+}-\mathscr{n}(R,\sigma_2)$$
$$-\mathscr{n}'(L,\sigma_2)-\mathscr{n}(L,\sigma_2)-\mathscr{n}_0(\sigma_2)-K_{V-}\,, \tag{A.22}$$

with $K_{V\pm}$ obtained from Eq. (A.4) by taking only the first sum for K_{V-} and the second sum for K_{V+}.

Via Eq. (A.22), the bound on the endomorphism in Eq. (A.18) plus the fact that K acts on $\underline{\mathscr{E}}(h)$ with eigenvalue k give a bound on the absolute value of the following endomorphism of $\underline{\mathscr{E}}(h)$:

$$\underline{n}(L,\sigma_2)+\underline{n}_0(\sigma_2)+\underline{n}'(R,\sigma_2)+K_{V+}-n(L,\sigma_2)-K_{V-}\,. \tag{A.23}$$

Reintroduce the constant κ in the statement of Proposition A.1. Multiply the endomorphism in Eq. (A.23) by κ and add it to the endomorphism in Eq. (A.21). The result is a bound on

$$\begin{aligned}&m(L)+n(L,\sigma_1)-\kappa\cdot n(L,\sigma_2)+\kappa\cdot\underline{n}'(R,\sigma_2)-\underline{n}'(R,\sigma_1)\\&\quad+\underline{n}(L,\sigma_1)+P_V-\kappa\cdot K_{V-}+\kappa\cdot\underline{n}_0(\sigma_2)\,.\end{aligned} \tag{A.24}$$

Due to the Restriction (3) on $\{\mu[\upsilon,n]\}$ in the statement of Proposition A.1, there is a constant $\delta>0$ for which following inequalities hold:

$$\begin{aligned}n(L,\sigma_1)-\kappa\cdot n(L,\sigma_2)&\geqq n(L,\sigma\equiv\delta)\,,\\ \kappa\cdot\underline{n}'(R,\sigma_2)-\underline{n}'(R,\sigma_1)&\geqq\underline{n}'(R,\sigma\equiv\delta)\,.\end{aligned} \tag{A.25}$$

Since the characters $\{\upsilon\}$ for the S^1 action on $V|_\Sigma$ are bounded, and sincy any finite sum of endomorphisms from the set $\{\Theta^*_{\upsilon,n}\cdot\Theta_{\upsilon,n}\}$ defines a bounded operator, the constant δ can be chosen so that

$$P_V-\kappa\cdot K_{V-}>\delta\cdot P_V-\delta^{-1}\,. \tag{A.26}$$

These last two equations imply that the bound on the sum in Eq. (A.24) gives a bound on

$$n(L,\sigma\equiv\delta)+\underline{n}'(R,\sigma\equiv\delta)+\delta\cdot P_V+m(L)+\underline{n}(L,\sigma_1)+\kappa\cdot\underline{n}_0(\sigma_2)\,. \tag{A.27}$$

Each term in Eq. (A.25) is a non-negative endomorphism of $\underline{\mathscr{E}}(h)$; and hence bounded. These last bounds and those in Eq. (A.18) imply that $\underline{\mathscr{E}}(h)$ is a finite dimensional vector bundle as claimed.

Suppose that M is even dimensional, so that $S^0(U)\otimes\mathscr{F}_E\otimes\mathscr{G}_V$ admits the covariantly constant involution $\ell\equiv\ell_e$ or ℓ_s as described in Eqs. (4.17, 18). This involution extends to an involution of $\underline{\mathscr{E}}_{E,V}\otimes\Phi_E$ which is covariantly constant and which commutes with P and K (and H) and which anti-commutes with Clifford multiplication. The involution ℓ extends to define an involution of $C^\infty(\underline{\mathscr{E}}_{E,V}\otimes\Phi_E)$ which anti-commutes with D_t.

Distinguish Cases (I) and (II) of Proposition A.1. In the former case, let $\underline{\mathscr{E}}\equiv\underline{\mathscr{E}}_{E,Vm}$; and in the latter, let $\underline{\mathscr{E}}\equiv\underline{\mathscr{E}}_{E,Vm}(h)$.

Define the index of D_t on $C^\infty(\underline{\mathscr{E}}\otimes\Phi_E)$ by

$$\operatorname{Ind}(D_t,\underline{\mathscr{E}},\ell)\equiv\dim\ker(D_t|_{\ker(\ell-1)})-\dim\ker(D_t|_{\ker(\ell+1)})\,, \tag{A.28}$$

with $\ker(\ell+1)\equiv(\ell-1)\cdot H^1(\underline{\mathscr{E}}\otimes\Phi_E)$ and $\ker(\ell-1)\equiv(\ell+1)\cdot H^1(\underline{\mathscr{E}}\otimes\Phi_E)$.

Proposition A.3. *Suppose that a set of numbers $\{\mu[\upsilon,n]\}$ has been specified with the properties required for Proposition A.1. Let (m,k) be a pair of eigenvalues of P and K acting on $\underline{\mathscr{E}}_{E,V}$. Let $\underline{\mathscr{E}}\equiv\underline{\mathscr{E}}_{E,Vm}$ for Case* (I), *and let $\underline{\mathscr{E}}\equiv\underline{\mathscr{E}}_{E,Vm}(h)$ for Case* (II). *Let $\ell=\ell_e$ or ℓ_s as in Eqs.* (4.17, 18).

(1) $\operatorname{Ind}(D_t,\underline{\mathscr{E}},\ell)$ *is independent of the choice of $t>0$.*

(2) *If $n\cdot\mu[\upsilon,n]>0$ for all $\upsilon\in(0,r)$ and $0\neq n\in\mathbb{Z}+\alpha(\upsilon)$, then* $\operatorname{Ind}(D_t,\underline{\mathscr{E}},\ell)=0$ *for $m<0$.*

(3) *Suppose that* $\{\mu[\upsilon, n](s): s \in [0,1]\}$ *is a continuous deformation of the data which defines* Φ_E *in Eq.* (4.12) *and* D_t *in Eq.* (4.14). *Allow the metric on TM and the connections on each* $\{E[\upsilon]\}$ $[\upsilon]\}_{\upsilon \in (0,r]}$ *and on each* $\{V[\upsilon]\}_{\upsilon \in [0,r]}$ *to vary continuously with s also. Require that for each s, this data obey the conditions required by Proposition A.1; allow the constant* $\kappa = \kappa(s)$ *to vary continuously with* $s \in [0,1]$. *Then,* $\mathrm{Ind}(D_t, \mathscr{E}, \ell)$ *of Eq.* (*A*.28) *is independent of the choice of* $s \in [0,1]$.

Proof of Proposition A.3. The second assertion follows from Assertion 3) of Proposition A.1. The first and the third assertions are standard consequences of the stability of the index for Fredholm operators. Indeed, the kernel and cokernel of D_t on $H^1(\mathscr{E} \otimes \Phi_E)$ reside as a sub-vector space in the space of sections of a finite dimensional sub-bundle of $\mathscr{E}(h) \otimes \Phi_E$; so the standard theorems apply.

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