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On the Fokker–Planck–Boltzmann Equation

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Abstract. We consider the Boltzmann equation perturbed by Fokker–Planck type operator. To overcome the lack of strong a priori estimates and to define a meaningful collision operator, we introduce a notion of renormalized solution which enables us to establish stability results for sequences of solutions and global existence for the Cauchy problem with large data. The proof of stability and existence combines renormalization with an analysis of a defect measure.

Contents

Introduction

We are concerned with global existence and stability of solutions of the Fokker–Planck–Boltzmann equation

$$\text{(FPB)} \qquad \frac{\partial}{\partial t} f + \xi \cdot \nabla_x f - \nu \Delta_\xi f = Q(f, f) \quad \text{in} \quad (0, \infty) \times R^N \times R^N,$$

where $N \geqq 1, x \in R^N, \xi \in R^N, \nu > 0$. Except for the additional diffusion or Fokker–Planck term, $-\nu \Delta_\xi f$, the equation FPB is the Boltzmann equation. The structure of the collision operator Q is described in Sect. II.

We shall prove that sequences of classical solutions of FPB which satisfy uniform bounds only on the physical conserved quantities converge to a renormalized solution of FPB, a notion that we define below. A straightforward consequence of this result is global existence of a renormalized solution of FPB if the initial data

$$f(0,x,\xi)=f_0(x,\xi) \quad \text{on} \quad R^N\times R^N \tag{1}$$

satisfy

$$0\leqq f_0, \quad \text{a.e. and} \quad \iint\limits_{R^N\times R^N} f_0(1+|x|^2+|\xi|^2+|\log f_0|)dx\,d\xi<\infty. \tag{2}$$

There are several reasons for treating FPB. The present study is part of a series of papers by the authors devoted to transport equations arising in the kinetic theory of gases. The primary model is the Boltzmann equation and FPB is a natural approximation both physically and mathematically. Although the Fokker–Planck term provides some mild regularizing effects which are absent in the Boltzman equation, several of the essential difficulties encountered in the study of the Boltzmann equation are present in FPB. In particular, we mention the lack of a priori estimates which are sufficiently strong to define the collision operator $Q(f,f)$ in a classical sense. Our renormalization procedure to resolve this difficulty is one of the key ingredients in our forthcoming paper [5] on the Boltzmann equation which contains results on sequential stability and global existence for the Cauchy problem with large data. In the present context of the FPB equation, renormalization is combined with an analysis of a natural defect measure in order to obtain stability and existence. Another motivation for the study of FPB stems from the fact that related equations are of physical interest in the problem of accounting for grazing collisions (see C. Cercignani [3], p. 90) and in the study of aerosols (see for instance S. K. Loyalka [13] and the references therein).

As mentioned above the main difficulty in dealing with FPB originates in the collision term which is defined as follows. If $\varphi(\xi)\in\mathscr{D}(R^N)$, then

$$Q(\varphi,\varphi)=\int\limits_{R^N} d\xi_*\int\limits_{S^{N-1}} dw\{\varphi(\xi')\varphi(\xi'_*)-\varphi(\xi)\varphi(\xi_*)\}\cdot B(\xi-\xi_*,w), \tag{3}$$

where $\xi'=\xi-(\xi-\xi_*,w)w$, $\xi'_*=\xi+(\xi-\xi_*,w)w$. The collision kernel B is a given function satisfying

$$B\geqq 0,\ B(z,w) \text{ is a function of } |z|, |(z,w)| \text{ only.} \tag{4}$$

An additional hypothesis will be imposed on B below. As is well-known, Boltzmann type equations such as FPB represent a statistical description of a gas of molecules or particles. The function f represents the density at position x, velocity ξ and time t. The collision term describes the possible collisions at position x, time t and ξ,ξ_* are the velocities of two molecules before interaction while ξ',ξ'_* are the velocities after interaction. The precise form of the collision kernel B depends upon the intermolecular potential. For inverse powers potentials, B takes the form

$$B(z,w)=b(\theta)|z|^{-\gamma} \quad \text{with} \quad \gamma=1-2(N-1)/(s-1),$$

where $s>1$ is the exponent of the potential, θ is the angle between $\xi-\xi_*$ and w so that $\cos\theta=(\xi-\xi_*,w)|\xi-\xi_*|^{-1}$. In general, b is smooth except at $\theta=\pm\frac{\pi}{2}$, where it has a singularity of the form $|\cos\theta|^{-\alpha}$ with $\alpha=s+1/s-1$ when $N=3$. As is customary in the subject, we shall impose a weak assumption of angular cut-off (see H. Grad [7], C. Cercignani [3], C. Truesdell and R. Muncaster [15]) namely that B satisfies

$$B \in L^1_{\text{loc}}(R^N \times S^{N-1}). \tag{5}$$

This clearly corresponds to a reduction of the strength of the singularity and physically means that grazing collisions are weakly represented. We note that (5) holds trivially in the classical case of hard-spheres where

$$B(z, w) = |(z, w)|.$$

In connection with the structure of Q we note that even if all difficulties concerning integrations at infinity are ignored, the only simple bound one can expect on Q is

$$|Q(\varphi, \varphi)| \leqq C |\varphi|^2_{L^1}.$$

Therefore, in order to give a meaningful interpretation of Q one might try to derive an estimate of the form

$$f \in L^2_{\text{loc}}(R^N_x \times (0, \infty); L^1(R^N_\xi)).$$

Such an estimate is not available in general. Indeed, it is not obvious that L^2 is natural for FPB.

In order to resolve this classical difficulty of defining Q, we introduce a new formulation of the equation which consists of renormalization by a suitable nonlinear transformation of the dependent variable f. As motivation for the transformation, let us first suppose that f is a smooth nonnegative solution of FPB and consider the function $\beta_\delta(t) = (1/\delta)\log(1 + \delta t)$. Notice that the composition $g_\delta = \beta_\delta(f)$ solves the following renormalized version of FPB:

$$\frac{\partial}{\partial t} g_\delta + \xi \cdot \nabla_x g_\delta - \nu \Delta_\xi g_\delta = \frac{1}{1 + \delta f} Q(f, f) + \nu \delta |\nabla_\xi g_\delta|^2 \tag{RFPB}$$

in $R^N \times R^N \times (0, \infty)$. We shall show below that, for each $\delta > 0$, the normalized interaction $(1 + \delta f)^{-1} Q(f, f)$ belongs to L^1_{loc} and $\nabla_\xi g_\delta$ belongs to L^2. In the stability and existence results for solutions of FPB mentioned above, the definition of renormalized solution requires that for all $\delta > 0$, the composite function $\beta_\delta(f)$ is a distributional solution of RFPB. The precise definition of renormalized solution is stated in Sect. II.

Our renormalization procedure is applicable to a general class of p.d.e. whose nonlinearities are not well-defined on the basis of the naturally associated a priori estimates. In this connection, we mention that renormalization is one of the tools in our analysis of large data Cauchy problem for the Boltzmann equation [5]. A second application of renormalization to linear divergence-free transport equations with bad coefficients will be given in our forthcoming paper [6]. In the context of quasilinear second order elliptic equations in L^1, Ph. Benilan suggested to the second author that ideas related to our motion of renormalization may turn out to be useful in the analysis of solutions in a spirit vaguely reminiscent of Ph. Benilan, H. Brézis and M. G. Crandall [1]. Additional applications to discrete-velocity models are discussed in Sect. V.

With regard to previous rigorous work on FPB, we are aware only of results in the small, specifically perturbations of the vacuum state. We refer the reader to K. Hamdache [8, 9] and to the references cited therein.

This paper is the first in a series devoted to a systematic study of nonlinear transport equations. In addition to the Fokker–Planck–Boltzmann equation and the Boltzmann equation we shall treat the Vlasov–Maxwell system in its classical and relativistic forms. The latter system arises in theory of collisionless plasmas for which terms of the type $Q(f, f)$ are absent. Basic questions here deal with global existence with large data and sequential weak stability.

In this general area, several extended systems arise which incorporate both electrical or electromagnetic effects and collisional effects. In this connection we mention the Vlasov–Poisson–Boltzmann system which is associated with a medium of charged and colliding particles. We shall also be concerned in a future publication with the associated existence and stability problems for VPB for solutions with large data.

A common aspect for all of these equations is the study of sequences of solutions. The study of sequences of approximate solutions is relevant to the problem of existence while the study of sequences of (exact) solutions is relevant to the problem of stability. In both settings one is presented with a list of physically natural estimates derived from the associated conservation laws. The laws for energy and entropy are the prime examples. As usual the basic conservation laws provide information on the amplitude of the solution but not on its derivatives. Consequently, the problem of passage to the limit involves further investigation. In this context we are concerned with the mechanisms of regularization and cancellation which relate to the limiting behavior of sequences of solutions. Renormalization is one of the tools which is useful in treating all of the systems above.

I. Basic Formal Conservation Laws and Estimates

In this section we recall a few basic facts concerning Boltzmann type equations and present some simple applications.

First of all, the symmetries of B such as [4] easily yield that for all $\varphi \in \mathscr{D}(R^N)$, $\psi \in C^\infty(R^N)$ (say)

$$\int_{R^N} \psi Q(\varphi, \varphi) d\xi = \frac{1}{4} \iint_{R^N \times R^N} d\xi d\xi_* \int_{S^{N-1}} dw \{\varphi(\xi')\varphi(\xi'_*) - \varphi(\xi)\varphi(\xi_*)\}\{\psi(\xi) + \psi(\xi_*) - \psi(\xi') - \psi(\xi'_*)\} B(\xi - \xi_*, w). \tag{6}$$

See [3] for details. In particular, if $\psi = a + b \cdot \xi + c|\xi|^2$, where $a, c \in R$, $b \in R^N$, then $\int_{R^N} \psi Q(\varphi, \varphi) d\xi = 0$.

This immediately implies that a solution f of FPB formally satisfies the following identities:

$$\text{(conservation of mass)} \iint_{R^N \times R^N} f(x, \xi, t) dx d\xi \text{ is independent of } t, \tag{7}$$

$$\text{(conservation of momentum)} \iint_{R^N \times R^N} \xi f(x, \xi, t) dx d\xi \text{ is independent of } t, \tag{8}$$

$$\text{(increase of kinetic energy)} \frac{d}{dt} \iint_{R^N \times R^N} |\xi|^2 f(x, \xi, t) dx d\xi = (2N\nu) \iint_{R^N \times R^N} f(x, \xi, t) dx d\xi. \tag{9}$$

Observe that by (7) the right-hand side is constant.

Next, we recall another well-known identity which is based upon the remark that if we take $\psi = \log \varphi$ with $\varphi > 0$ in (6) then we obtain

$$\int_{R^N} \log \varphi Q(\varphi, \varphi) d\xi = \frac{1}{4} \iint_{R^N \times R^N} d\xi d\xi_* \int_{S^{N-1}} dw \{\varphi(\xi')\varphi(\xi'_*) - \varphi(\xi)\varphi(\xi_*)\}.$$
$$\cdot \log [\varphi(\xi)\varphi(\xi_*)/\varphi(\xi')\varphi(\xi'_*)] B(\xi - \xi_*, w) \leqq 0.$$

Therefore, if we multiply FPB by $f \log f$, recall that $f \geqq 0$ and integrate, we formally obtain

$$\frac{d}{dt} \iint_{R^N \times R^N} f \log f dx d\xi = -\nu \iint_{R^N \times R^N} |\nabla_\xi f|^2 f^{-1} dx d\xi - \frac{1}{4} \iint_{R^N \times R^N} d\xi d\xi_* \int_{S^{N-1}}$$
$$\cdot dw B(\xi - \xi_*, w) \{f(\xi')f(\xi'_*) - f(\xi)f(\xi_*)\}$$
$$\cdot \log [f(\xi')f(\xi'_*)/f(\xi)f(\xi_*)] \tag{10}$$

It is obvious that (7) and (9) provide some a priori estimates on f. In order to deduce a bound from (10), we need another estimate. This estimate is obtained by multiplying FPB by $|x|^2$ and integrating:

$$\frac{d}{dt} \iint_{R^N \times R^N} f|x|^2 dx d\xi = 2 \iint_{R^N \times R^N} (x, \xi) f dx d\xi.$$

Hence by Cauchy–Schwarz inequality

$$\frac{d}{dt} \iint_{R^N \times R^N} f|x|^2 dx d\xi \leqq 2 \left(\iint_{R^N \times R^N} f|x|^2 dx d\xi \right)^{1/2} \left(\iint_{R^N \times R^N} f|\xi|^2 dx d\xi \right)^{1/2}. \tag{11}$$

As a final remark, observe that if $g \in L^1_+(R^N \times R^N)$ satisfies

$$\iint_{R^N \times R^N} g(1 + |x|^2 + |\xi|^2) dx d\xi \leqq R \quad \text{and} \quad \iint_{R^N \times R^N} g \log g dx d\xi \leqq R \tag{12}$$

for some $R \geqq 0$, then

$$\iint_{R^N \times R^N} g|\log g| dx d\xi \leqq C_R \tag{13}$$

for some C_R depending only on R. Here we assume $g|\log g| \in L^1$ so that (12) makes sense. Indeed, one has obviously

$$\iint_{R^N \times R^N} g|\log g| dx d\xi \leqq \iint_{R^N \times R^N} g \log g dx d\xi + 2 \iint_{R^N \times R^N} 1_{(g<1)} g \log \left(\frac{1}{g} \right) dx d\xi.$$

In order to bound the second integral in the right-hand side, we split this integral in two parts. On the set where $\log(1/g) \leqq |x|^2 + |\xi|^2$ we bound the corresponding integral using (12) and obtain

$$\iint_{R^N \times R^N} g|\log g| dx d\xi \leqq 3R + 2 \iint_{R^N \times R^N} 1_{(g \leqq \exp -(|x|^2 + |\xi|^2))} g \log \frac{1}{g} dx d\xi.$$

To conclude the verification of (13), we observe that on $(0, 1)$ the function $t \log(1/t)$ is bounded for example by $C_0 \sqrt{t}$ for some $C_0 > 0$. Therefore

$$\iint_{R^N\times R^N} g|\log g|\,dx\,d\xi \leqq 3R + 2C_0 \iint_{R^N\times R^N} \exp -\tfrac{1}{2}(|x|^2+|\xi|^2)\,dx\,d\xi \leqq 3R + 2C_0(2\pi)^N.$$

Throughout the paper we shall use the following notation:

$$Q_+(f,f) = \iint_{R^N\times S^{N-1}} d\xi_*\,dw\,f'_*f'B(\xi-\xi_*,w),$$

$$Q_-(f,f) = \iint_{R^N\times S^{N-1}} d\xi_*\,dw\,ff_*B(\xi-\xi_*,w) = f\cdot Lf,$$

with

$$Lf = \int_{R^N} f(\xi_*)A(\xi-\xi_*)\,d\xi_*,\; A(z) = \int_{S^{N-1}} B(z,w)\,dw \quad \text{for} \quad z\in R^N,$$

where $f' = f(\xi')$, $f_* = f(\xi_*)$, $f'_* = f(\xi'_*)$.

II. Sequential Stability and Strategy of the Proof: Normalized Interactions, Defect Measures and Hypoellipticity

To simplify the presentation, we consider a sequence f^n of smooth nonnegative solutions of FPB. We assume for instance that $f^n \in W^{2,\infty}(R^N\times R^N\times[0,\infty))$, $f^n\to 0$ as $(x,\xi)\to\infty$ uniformly in $t\in[0,T]$ for all $T<\infty$ and that there exists a constant C_T independent of n such that

$$\iint_{R^N\times R^N} f^n(x,\xi,t)(1+|x|^2+|\xi|^2+|\log f^n|)\,dx\,d\xi \leqq C_T, \tag{14}$$

$$\int_0^T dt \iint_{R^N\times R^N} dx\,d\xi\Big\{|\nabla_\xi\sqrt{f^n}|^2 + \int_{R^N} d\xi_* \int_{S^{N-1}} dw\,B(\xi-\xi_*,w)\{f^{n'}f^{n'}_* - f^nf^n_*\}$$
$$\cdot\log\frac{f^{n'}f^{n'}_*}{f^nf^n_*} \leqq C_T \tag{15}$$

for all $T<\infty$. In view of the facts in the preceding section, these bounds are automatically satisfied provided the basic physical identities (7), (9–11) in Sect. I are justified and provided (14) holds at $t=0$. The justification of these and related identities becomes necessary only when we address the question of the existence of a solution of FPB and analyze sequences of approximate solutions. For the moment we shall assume for simplicity that (14) and (15) hold. Because of (14) we may assume by passing to a subsequence that f^n converges weakly in $L^1(R^N\times R^N\times(0,T))$ to some f for all T.

Definition. A nonnegative element f of $C([0,\infty); L^1(R^N_x\times R^N_\xi))$ is a renormalized solution of FPB if the composite function $g_\delta=\beta_\delta(f)$ satisfies RFPB in the sense of distributions, where $\beta_\delta(t) = 1/\delta\log(1+\delta t)$.

Theorem 1. *Assume that B satisfies the following mild growth condition:*

$$\left(\int_{B_R(\xi)} A(z)\,dz\right)(1+|\xi|^2)^{-1}\to 0 \quad as \quad |\xi|\to\infty, \quad for\ all \quad R<\infty. \tag{16}$$

Then $\forall p, T<\infty$, the sequence f^n converges in $L^p(0,T;L^1(R^N_x\times R^N_\xi)$ to a renormalized solution f which satisfies (14) for a.e. *$t\in(0,T)$ and (15). Furthermore, for any $\delta>0$, the* *normalized interaction terms satisfy*

$$\begin{cases} Q_-(f,f)(1+\delta f)^{-1} \in C([0,\infty); L^1(R_x^N \times B_R)), & \forall R < \infty \\ Q_+(f,f)(1+\delta f)^{-1} \in L^1((0,T) \times R_x^N \times B_R), & \forall R, T < \infty \end{cases} \tag{17}$$

and $g_\delta \in L^2((0,T) \times R_x^N; H^1(B_R))(\forall R, T < \infty)$.

Remarks: i) Many variants of this result are possible. In particular, the smoothness assumption on f^n is not necessary. In this connection we note that f^n could be a solution of an approximate equation. We shall use this fact to prove the global existence result. Finally, the method we introduce below applies to various Boltzmann type equations (see Sect. V).

ii) The above result implies that in fact f^n converges strongly to f. We shall see in Sect. V that the particular choice of β_δ which enters the definition of renormalized solution is not fundamental. Indeed, f has the following property: for all nonnegative functions β in $C^2[0,\infty)$ such that

$$\beta(0) = 0, \quad |\beta'| \leqq \frac{C}{1+t}, \quad |\beta''| \leqq \frac{C}{1+t^2},$$

then $g = \beta(f)$ solves

$$\frac{\partial g}{\partial t} + \xi \cdot \nabla_x g - \nu \Delta_\xi g = \beta'(f) Q(f,f) + \beta''(f) |\nabla_\xi f|^2 \quad \text{in} \quad \mathscr{D}'.$$

Notice that $\beta'(f)Q(f,f) \in L^1_{\text{loc}}$, and that (15) implies that $\beta''(f)|\nabla_\xi f|^2 \in L^1$. The choice $\beta = 1/\delta \log(1+\delta f)$ is merely a convenient representative of this general class.

iii) Notice that the uniform bound (15) implies that $\sqrt{f} \in L^2((0,T) \times R_x^N; H^1(R_\xi^N))$.

We conclude this section by explaining briefly the strategy of the proof. First, we observe that $g_\delta^n = \beta_\delta(f^n)$ solves RFPB and that $Q_-(f^n, f^n)(1+\delta f^n)^{-1}$ is bounded in $L^\infty(0,T; L^1(R_x^N \times B_R))(\forall R, T < \infty)$. The latter fact is a consequence of the structure of Q_- and of the bounds (14). Then, integrating RFPB we deduce that the normalized positive interaction $Q_+(f^n, f^n)(1+\delta f^n)^{-1}$ is bounded in $L^1((0,T) \times R_x^N \times B_R)(\forall R, T < \infty)$.

Next, we observe that these bounds mean that $L_\nu g_\delta^n$ is bounded in $L^1((0,T) \times R_x^N \times B_R)(\forall R, T < \infty)$ where L_ν is the partial diffusive transport operator

$$L_\nu \equiv \frac{\partial}{\partial t} + \xi \cdot \nabla_x - \nu \Delta_\xi.$$

Using the fact that L_ν is hypoelliptic in the sense of L. Hörmander [11], we deduce that g_δ^n is compact in $L^1((0,T) \times R_x^N \times R_\xi^N)$. The strong convergence of f^n to f then follows by a relaxation argument.

The bounds stated in Theorem 1 are fairly straightforward. It remains to prove that g_δ solves RFPB. From the above compactness argument, we deduce the existence of a bounded nonnegative measure μ on $(0,T) \times R_x^N \times R_\xi^N$ for all $T < \infty$ such that

$$L_\nu g_\delta = (1+\delta f)^{-1} Q(f,f) + \nu\delta |\nabla_\xi g_\delta|^2 + \mu \text{ in } \mathscr{D}'.$$

The measure μ is the "defect measure" due to the weak convergence of $\nabla_\xi g_\delta^n$. The proof that μ vanishes and that g_δ is thus a solution of RFPB involves a delicate argument. Formally, it is not difficult to understand why μ should vanish. Indeed, the g_δ-equation means that f should "solve" the equation

$$L_\nu f = Q(f, f) + (1 + \delta f)\mu.$$

The statement of conservation of mass, i.e. $\iint_{R^N \times R^N} f(x, \xi, t) dx d\xi$ is independent of t, leads one to suspect that for all $T < \infty$ that

$$\int_0^T \iint_{R^N \times R^N} (1 + \delta f) d\mu \equiv 0.$$

Hence μ should vanish. A modification of this formal argument using a special set of multipliers produces the desired result that μ vanishes.

The argument sketched above is relevant to a general class of transport equations. Here attention is focused on the FPB equation for concreteness. The last section contains a discussion of extensions of the results above to equations with more general linear parts and to discrete velocity analogues.

III. Proof of Theorem 1: Sequential Weak Stability

To simplify the presentation, we shall first treat the case where $B \in L^1 \cap L^\infty(R^N; L^1(S^{N-1}))$. Then we shall discuss the modifications needed to accommodate the general case where B satisfies (16). We now follow the strategy of proof sketched at the end of Sect. II.

Step 1. We first remark that $Q_-(f^n, f^n)(1 + \delta f^n)^{-1}$ is bounded in $L^\infty(0, T; L^1(R_x^N \times R_\xi^N))$. Indeed, we have

$$\begin{aligned} Q_-(f^n, f^n)(1 + \delta f^n)^{-1} &= f^n(1 + \delta f^n)^{-1} L(f^n) \leqq \frac{1}{\delta} L(f^n) \\ &= \frac{1}{\delta} \int f^n(x, \xi_*, t) A(\xi - \xi_*) d\xi_* \end{aligned}$$

The results follow since $A \in L^1(R^N)$ and f^n is bounded in $L^\infty(0, T; L^1(R_x^N \times R_\xi^N))$ by (14).

Next, since g_δ^n solves RFPB, we deduce, at least formally by integrating over $R_x^N \times R_\xi^N \times (0, T)$ that

$$\begin{aligned} \iint_{R^N \times R^N} g_\delta^n(x, \xi, T) - g_\delta^n(x, \xi, 0) dx d\xi = \int_0^T dt \iint_{R^N \times R^N} dx d\xi [(1 + \delta f^n)^{-1} Q(f^n, f^n) \\ + \nu\delta |\nabla_\xi g_\delta^n|^2]. \end{aligned} \tag{18}$$

Using (14) and the above bound on $Q^-(f^n, f^n)(1 + \delta f^n)^{-1}$, we deduce from (18) that

$$\int_0^T dt \iint_{R^N \times R^N} dx d\xi [(1 + \delta f^n)^{-1} Q^+(f^n, f^n) + \nu\delta |\nabla_\xi g_\delta^n|^2] \leqq C_T$$

for some constant $C_T \geqq 0$ independent of n. Therefore, provided we justify (18), we have proven that $Q^+(f^n, f^n)(1+\delta f^n)^{-1}$ and $|\nabla_\xi g_\delta^n|^2$ are bounded in $L^1((0,T)\times R_x^N \times R_\xi^N)$. We mention in passing that the bound on $|\nabla_\xi g_\delta^n|^2$ also follows immediately from (15).

Justifying (18), i.e. the integration over $R^N \times R^N$, is an easy matter. Take $\varphi \in \mathscr{D}(R^N)$, $\varphi \equiv 1$ on B_1, $0 \leqq \varphi \leqq 1$ on R^N and set $\varphi_\varepsilon(\cdot) = \varphi(\varepsilon \cdot)$. Multiply RFPB by $\varphi_\varepsilon(x)\varphi_\varepsilon(\xi)$ and integrating by parts to obtain

$$\begin{aligned}
&\int_0^T dt \iint_{R^N\times R^N} dx d\xi [(1+\delta f^n)^{-1} Q^+(f^n, f^n) + \nu\delta |\nabla_\xi g_\delta^n|^2]\varphi_\varepsilon(x)\varphi_\varepsilon(\xi) \\
&\quad = \iint_{R^N\times R^N} \varphi_\varepsilon(x)\varphi_\varepsilon(\xi)\{g_\delta^n(x,\xi,T) - g_\delta^n(x,\xi,0)\} dx d\xi \\
&\qquad + \int_0^T dt \iint_{R^N\times R^N} dx d\xi [(1+\delta f^n)^{-1} Q^-(f^n, f^n)]\varphi_\varepsilon(x)\varphi_\varepsilon(\xi) \\
&\qquad + \int_0^T dt \iint_{R^N\times R^N} dx d\xi g_\delta^n(-\xi\cdot\nabla_x\varphi_\varepsilon(x)\varphi_\varepsilon(\xi) - \nu\varphi_\varepsilon(x)\Delta_\xi\varphi_\varepsilon(\xi)).
\end{aligned}$$

We conclude easily letting ε go to 0 since

$$|\xi\cdot\nabla_x\varphi_\varepsilon(x)|\varphi_\varepsilon(\xi) \leqq C_\varepsilon|\xi|, \quad |\varphi_\varepsilon(x)\Delta_\xi\varphi_\varepsilon(\xi)| \leqq C\varepsilon^2$$

for some constant $C \geqq 0$ independent of ε.

It is worth remarking that the L^1 bound on $Q^+(f^n, f^n)(1+\delta f^n)^{-1}$ can also be deduced from the bound (15). Indeed, one just has to observe that for all $K > 1$,

$$Q^\pm(f^n, f^n) \leqq KQ^\mp(f^n, f^n) + \frac{1}{\log K}\int_{R^N} d\xi_* \int_{S^{N-1}} dw B\cdot[f^{n'} f_*^{n'} - f^n f_*^n] \log\frac{f^{n'} f_*^{n'}}{f^n f_*^n}, \tag{19}$$

and the second term in the right-hand side is clearly bounded in L^1 in view of (15).

Step 2. The preceding bounds show that $L_\nu g_\delta^n$ is bounded in $L^1((0,T)\times R^N \times R^N)$. Of course, because of (14), g_δ^n is bounded in $C([0,T]; L^1(R^N\times R^N))$. At this point, we use the fact that L_ν is an hypoelliptic operator to deduce the compactness of g_δ^n in $L^1((0,T)\times R^N\times R^N)$. We shall first prove the compactness of g_δ^n in $L^1((0,T)\times B_R \times B_R)(\forall R, T < \infty)$. Combining this result with (14) easily yields the compactness of g_δ^n in $L^1((0,T)\times R^N\times R^N)$.

In order to establish local compactness of g_δ^n we consider any cut-off function $\varphi(x,\xi)$ in $\mathscr{D}(R^N\times R^N)$, and we observe that $L_\nu(\varphi g_\delta^n)$ is bounded in $L^1((0,T)\times R^N \times R^N)$ and has compact support in $[0,T]\times R^N\times R^N$ uniformly in n. This implies that φg_δ^n is compact in $L^1((0,T)\times R^N\times R^N)$. We prove this rather technical point in the Appendix. We mention here that the only fact which is required to obtain L^1 compactness is the existence of a continuous fundamental solution. The fundamental solution is actually C^∞ and an explicit formula is available, see for instance Hörmander [11].

The compactness of g_δ^n in $L^1((0,T)\times R^N\times R^N)$ yields, by classical results of measure theory, the compactness of g_δ^n in the topology of convergence in measure. We recall that φ_n converges to φ in measure if $\operatorname{meas}(|\varphi_n - \varphi| > \delta) \underset{n}{\rightarrow} 0$ for any $\delta > 0$. Since $f^n = (1/\delta)[\exp(\delta g_\delta^n) - 1]$, f^n is also compact in this topology on every set with finite measure. Next, we recall that f^n converges weakly in $L^1((0,T)\times R^N \times R^N)$ to f. Combining this information with the preceding compactness, we conclude that f^n converges in $L^1((0,T)\times R^N\times R^N)$ to f using Shur's theorem. Therefore, using the $L^\infty(0,T;L^1(R^N\times R^N))$ bound on f^n implied by (14) it follows that f^n converges in $L^p(0,T;L^1(R^N\times R^N))$ for all $1\leqq p<\infty$ and for all $T<\infty$. Hence, g_δ^n also converges in $L^p(0,T;L^1(R^N\times R^N))$ to $g_\delta=\beta_\delta(f)$ for all $1\leqq p<\infty$, $T<\infty$.

Step 3. We shall show that $Q(f^n,f^n)(1+\delta f^n)^{-1}$ converges in L^1 to $Q(f,f)\cdot(1+\delta f)^{-1}$ and that $Q^\pm(f,f)(1+\delta f)^{-1}$ belongs to L^1. In fact, we shall show that

$$Q^-(f,f)(1+\delta f)^{-1}\in L^\infty(0,T;L^1(R^N\times R^N)),$$

$$Q^+(f,f)(1+\delta f)^{-1}\in L^1((0,T)\times R^N\times R^N) \quad \text{for all} \quad T<\infty,$$

and that

$$Q^-(f^n,f^n)(1+\delta f^n)^{-1}\underset{n}{\rightarrow}Q^-(f,f)(1+\delta f)^{-1} \quad \text{in} \quad L^p(0,T;L^1)$$

$$\text{for all} \quad p<\infty, T<\infty, \tag{20a}$$

$$Q^+(f^n,f^n)(1+\delta f^n)^{-1}\underset{n}{\rightarrow}Q^+(f,f)(1+\delta f)^{-1} \quad \text{in} \quad L^1, \quad \text{for all} \quad T<\infty. \tag{20b}$$

The statement (20a) is easy since

$$Q^-(f^n,f^n)(1+\delta f^n)^{-1}=f^n(1+\delta f^n)^{-1}L(f^n),$$

$$L(f^n)=\int_{R^N} d\xi_* A(\xi-\xi_*)f^n(x,\xi_*,t),$$

and we assumed that $A\in L^1(R^N)$. Clearly $L(f^n)$ converges to $L(f)$ in $L^p(0,T;L^1(R^N\times R^N))(\forall p<\infty, \forall T<\infty)$ and (20a) holds.

Statement (20b) holds if we can show that $I^n\equiv Q^+(f^n,f^n)(1+\delta f^n)^{-1}$ converges in measure to $I\equiv Q^+(f,f)(1+\delta f)^{-1}$ on every set with finite measure. Indeed, granting the local convergence in measure, we observe that the sequence I^n is bounded in L^1 and satisfies

$$0\leqq I^n\leqq Kh^n+\frac{1}{\log K}e^n$$

as a consequence of (19). Here $h^n\geqq 0$ converges in L^1 while $e^n\geqq 0$ remains bounded in L^1. Therefore I^n is weakly compact in L^1. The strong L^1 convergence of I^n to I follows from general measure theory.

Next, to prove the convergence in measure of I^n to I we notice that $(1+\delta f^n)^{-1}$ converges in measure to $(1+\delta f)^{-1}$. Thus it suffices to show that $Q^+(f^n,f^n)$ converges in measure on every set with finite measure to $Q^+(f,f)$. Notice that for all $\varphi,\psi\in L^1(R^N)$ we have

$$\int_{R^N} Q^+(\varphi,\varphi)d\xi\leqq \|A\|_{L^\infty}\|\varphi\|_{L^1}^2$$

and

$$\int_{R^N} |Q^+(\varphi,\varphi)-Q^+(\psi,\psi)|d\xi\leqq \|A\|_{L^\infty}\|\varphi-\psi\|_{L^1}(\|\varphi\|_{L^1}+\|\psi\|_{L^1}).$$

Thus, it follows that for almost all (x, t),

$$\int_{R^N} |Q^+(f^n, f^n) - Q^+(f, f)| d\xi \leqq |A|_{L^\infty} \int_{R^N} |f^n - f| d\xi \int_{R^N} f^n + f d\xi,$$

We complete the proof that I^n converges to I in measure using the fact that f^n converges to f in $L^1((0, T) \times R^N \times R^N)$. This fact implies that $\int |f^n - f| d\xi$ and $\int f^n d\xi$ converge in measure on $(0, T) \times R^N_\xi$ respectively to zero and $\int f d\xi$ which is finite a.e. Thus $\int |Q^+(f^n, f^n) - Q^+(f, f)| d\xi$ converges locally in measure on $(0, T) \times R^N_x$ to zero yielding the desired result (20b).

Next, we make a preliminary passage to the limit in RFPB. Since $\nabla_\xi g^n_\delta$ converges weakly in L^2 to $\nabla_\xi g_\delta$, we deduce from general properties of weak limits that there exists a bounded nonnegative measure μ on $(0, T) \times R^N \times R^N$ such that

$$|\nabla_\xi g^n_\delta|^2 \underset{n}{\rightarrow} |\nabla_\xi g_\delta|^2 + \mu \quad \text{in} \quad \mathscr{D}'((0, T) \times R^N \times R^N).$$

Now we may pass to the limit in RFPB in the sense of distributions and deduce using the above convergences that $g_\delta = \beta_\delta(f)$ solves

$$\frac{\partial g_\delta}{\partial t} + \xi \cdot \nabla_x g_\delta - \nu \Delta_\xi g_\delta = \frac{1}{1 + \delta f} Q(f, f) + \nu\delta |\nabla_\xi g_\delta|^2 + \mu \quad \text{in} \quad \mathscr{D}'. \tag{21}$$

Of course, f satisfies (14) and (15).

Finally, we want to show that $\mu \equiv 0$. The formal argument in Sect. II indicates that μ should vanish. We are unable to justify this formal argument. Instead, we shall use a modified version of this argument which is not based upon the conservation of mass but rather on the fact that the nonlinear operator $[-\nu\Delta_\xi g_\delta - \nu\delta|\nabla_\xi g_\delta|^2]$ originates from a linear operator $-\nu\Delta_\xi f$. To this end, we first multiply the equation for g^n_δ by $\exp \theta(g^n_\delta \wedge R)$, where $0 < \theta < \delta$, $R > 0$ are fixed. Multiplying by $\varphi \in \mathscr{D}(0, T)$ and integrating over $[0, T] \times R^N \times R^N$ yields

$$-\int_0^T dt \iint_{R^N \times R^N} dx d\xi \varphi'(t) \Psi_{\theta,R}(g^n_\delta) - \int_0^T dt \varphi \iint_{R^N \times R^N} dx d\xi \frac{1}{1 + \delta f^n} Q(f^n, f^n) \Phi_{\theta,R}(g^n_\delta)$$
$$= \nu \int_0^T dt \varphi \iint_{R^N \times R^N} dx d\xi \{\delta |\nabla_\xi g^n_\delta|^2 - \theta |\nabla_\xi g^n_\delta|^2 1_{(g^n_\delta \leqq R)}\} \Phi_{\theta,R}(g^n_\delta),$$

where $\Phi_{\theta,R}(t) = \exp \theta t \wedge R$, $\Psi_{\theta,R}(t) = \int_0^t \Phi_{\theta,R}(s) ds$ for $t \geqq 0$. The integration is easily justified as in Step 1. Next, we observe that the right-hand side of the above equation is bounded by the quantity

$$\nu \sup |\varphi| \left\{ (\delta - \theta) \int_0^T dt \iint_{R^N \times R^N} dx d\xi |\nabla_\xi g^n_\delta|^2 \Phi_{\theta,R}(g^n_\delta) \right.$$
$$\left. + e^{\theta R} \int_0^T dt \iint_{R^N \times R^N} dx d\xi |\nabla_\xi g^n_\delta|^2 1_{(g^n_\delta > R)} \right\}.$$

Since $\Phi_{\theta,R}(g^n_\delta) \leqq (1 + \delta f^n)^\theta$, we can bound the first integral by

$$\int_0^T dt \iint_{R^N \times R^N} dx d\xi |\nabla_\xi f^n|^2 (1 + \delta f^n)^{-2} (1 + \delta f^n)\theta \leqq \frac{1}{\delta} \int_0^T dt \iint_{R^N \times R^N} dx d\xi |\nabla_\xi f^n|^2 (f^n)^{-1},$$

and the second integral by

$$\int_0^T dt \iint_{R^N \times R^N} dx d\xi |\nabla_\xi f^n|^2 (1+\delta f^n)^{-1} e^{-\delta R} \leqq e^{-\delta R} \frac{1}{\delta} \int_0^T dt \iint_{R^N \times R^N} dx d\xi |\nabla_\xi f^n|^2 (f^n)^{-1}.$$

In conclusion, using (15) we deduce

$$\left| -\int_0^T dt \iint_{R^N \times R^N} dx d\xi \varphi'(t) \Psi_{\theta,R}(g_\delta^n) - \int_0^T dt \varphi \iint_{R^N \times R^N} dx d\xi \frac{1}{1+\delta f^n} Q(f^n, f^n) \Phi_{\theta,R}(g_\delta^n) \right|$$
$$\leqq C \sup |\varphi| \{(\delta-\theta) + e^{(\theta-\delta)R}\}$$

for some $C \geqq 0$ independent of n, θ, R. Passing to the limit, we obtain

$$\left| -\int_0^T dt \iint_{R^N \times R^N} dx d\xi \varphi'(t) \Psi_{\theta,R}(g_\delta) - \int_0^T dt \varphi \iint_{R^N \times R^N} dx d\xi \frac{1}{1+\delta f} Q(f, f) \Phi_{\theta,R}(g_\delta) \right|$$
$$\leqq C \sup |\varphi| \{(\delta-\theta) + e^{(\theta-\delta)R}\}. \tag{22}$$

Next, we use Eq. (21) to prove that μ vanishes. Formally, we multiply (21) by $\varphi \Phi_{\theta,R}(g_\delta)$ and integrate with respect to (t, x, ξ) to obtain exactly as above

$$-\int_0^T dt \iint_{R^N \times R^N} dx d\xi \varphi' \Psi_{\theta,R}(g_\delta) - \int_0^T dt \varphi \iint_{R^N \times R^N} dx d\xi \frac{1}{1+\delta f} Q(f, f) \Phi_{\theta,R}(g_\delta) \Big|$$
$$\geqq C \sup |\varphi| \{(\delta-\theta) + e^{(\theta-\delta)R}\} + \int \varphi \Phi_{\theta,R}(g_\delta) d\mu. \tag{23}$$

Since $\Phi \geqq 1$, we deduce by taking $\varphi \geqq 0$ and combining (21) and (23) that

$$\int \varphi d\mu \leqq C(\sup \varphi) \{(\delta-\theta) + e^{(\theta-\delta)R}\}.$$

We conclude that μ vanishes letting $R \to \infty$ and then $\theta \to \delta$. Finally the above nonlinear multiplication by $\Phi_{\theta,R}(g_\delta)$ and the resulting integration by parts have to be justified. This can be easily done by convolution regularization. Indeed, if ρ_ε denotes a regularizing kernel ρ in (x, ξ, t),

$$\rho_\varepsilon = \frac{1}{\varepsilon^{2N+1}} \rho\left(\frac{\cdot}{\varepsilon}\right), \quad \rho \in \mathscr{D}(R^{2N+1}), \quad \rho \geqq 0, \quad \int \rho = 1,$$

we check easily that $g_\delta^\varepsilon = \rho_\varepsilon * g_\delta$ satisfies

$$\frac{\partial g_\delta^\varepsilon}{\partial t} + \xi \cdot \nabla_x g_\delta^\varepsilon - \nu \Delta_\xi g_\delta^\varepsilon = \left\{ \frac{1}{1+\delta f} Q(f, f) + \nu \delta |\nabla_\xi g_\delta^\varepsilon|^2 \right\} * \rho_\varepsilon$$
$$+ \mu * \rho_\varepsilon + r_\varepsilon \text{ in } (\alpha_\varepsilon, T) \times R^N \times R^N,$$

where $\alpha_\varepsilon \in (0, T)$, $\alpha_\varepsilon \to 0$, $r_\varepsilon = \xi \cdot \nabla_x g_\delta^\varepsilon - (\xi \cdot \nabla_x g_\delta) * \rho_\varepsilon$. Next, we observe that

$$r_\varepsilon = \int (\eta - \xi) g_\delta(y, \eta, s) \cdot \nabla_x \rho_\varepsilon(x-y, \xi-y, r-s) = g_\delta * K_\varepsilon,$$

with $K_\varepsilon = (1/(\varepsilon^{2N+1})) K(\cdot/\varepsilon)$, $K = -\xi \cdot \nabla_x \rho(x, \xi, t)$. Hence,

$$r_\varepsilon \underset{\varepsilon}{\to} \left(\int K \right) g_\delta \equiv 0 \quad \text{in} \quad L^1.$$

Then, taking $\varphi \geqq 0$, multiplying the above equation by $\Phi_{\theta,R}(g_\delta^\varepsilon)$, observing that

$\Phi_{\theta,R} \geqq 1$, we deduce for ε small enough,

$$-\int_0^T \varphi' dt \iint_{R^N \times R^N} dx d\xi \Psi_{\theta,R}(g_\delta^\varepsilon) dx d\xi$$

$$-\int_0^T \varphi dt \iint_{R^N \times R^N} dx d\xi \left\{ \frac{1}{1+\delta f} Q(f,f) \right\} * \rho_\varepsilon \Phi_{\theta,R}(g_\delta^\varepsilon)$$

$$\geqq \int \varphi \mu * \rho_\varepsilon - e^{\theta R} (\sup \varphi) |r_\varepsilon|_{L^1} + \int_0^T \varphi dt \iint_{R^N \times R^N} \delta v |\nabla_\xi g_\delta|^2 * \rho_\varepsilon \Phi_{\theta,R}(g_\delta^\varepsilon)$$

$$- v |\nabla_\xi g_\delta^\varepsilon|^2 \Phi'_{\theta,R}(g_\delta^\varepsilon) dx d\xi.$$

Letting ε go to 0, we deduce that the left-hand side of (23) is bounded from below by

$$\int \varphi d\mu + \int_0^T \varphi dt \iint_{R^N} dx d\xi \{ \delta v |\nabla_\xi g_\delta|^2 - \theta v |\nabla_\xi g_\delta|^2 1_{(g_\delta \leqq R)} \}.$$

We conclude that μ vanishes provided that the following inequalities hold:

$$|\nabla_\xi g_\delta|^2 \leqq \frac{4}{\delta} |\nabla_\xi f^{1/2}|^2 \quad \text{a.e.}, \quad |\nabla_\xi g_\delta \wedge R|^2 \leqq \frac{4}{\delta} e^{-\delta R} |\nabla_\xi f^{1/2}|^2 \quad \text{a.e.}.$$

Formally, these bounds are obvious if we recall that $g_\delta = \beta_\delta(f)$. Using a simple approximation argument, we shall justify these inequalities. (See the next section for related arguments.) At this stage, we have shown that g_δ solves RFPB and that $Q(f^n, f^n)/(1 + \delta f^n)$ converges in $L^1((0,T) \times R^N \times R^N)$ for all $T < \infty$. Integrating the equations RFPB over (x, ξ, t) allows us to deduce easily that for all $T < \infty$,

$$\int_0^T dt \iint_{R^N \times R^N} |\nabla_\xi g_\delta^n|^2 dx d\xi \underset{n}{\rightarrow} \int_0^T dt \iint_{R^N \times R^N} |\nabla_\xi g_\delta|^2 dx d\xi.$$

By standard results on weak convergence, this implies that $\nabla_\xi g_\delta^n$ converges to $\nabla_\xi g_\delta$ in $L^2((0,T) \times R^N \times R^N)$. Therefore $|\nabla_\xi g_\delta^n|^2$ converges to $|\nabla_\xi g_\delta|^2$ in $L^1((0,T) \times R^N \times R^N)$. In particular, $L_v g_\delta^n$ converges in $L^1((0,T) \times R^N \times R^N)$ to $L_v g_\delta$. Let us also mention that since $L_v g_\delta \in L^1((0,T) \times R^N \times R^N)$ we have $g_\delta \in C([0,T]; L^1(R^N \times R^N))$ for all $\delta > 0$. This standard result follows for example from the argument given in the Appendix and does not require the hypoellipticity of L_v. From the fact that $g_\delta \in C([0,\infty); L^1(R^N \times R^N))$ it is straightforward to deduce that $f \in C([0,\infty); L^1(R^N \times R^N))$: one just needs to observe that a.e. $t \in [0,T]$,

$$\iint_{R^N \times R^N} dx d\xi |f(x,\xi,t) - g_\delta(x,\xi,t)| \leqq \varepsilon_\delta \iint_{R^N \times R^N} dx d\xi f(x,\xi,t) + 2 \cdot \iint_{R^N \times R^N} dx d\xi f(x,\xi,t) 1_{(f \geqq R)},$$

where $R > 0$, and $\varepsilon_\delta = \sup_{[0,R]} |1 - \beta_\delta(\lambda)\lambda^{-1}| \rightarrow 0$ as $\delta \rightarrow 0_+$. Because of (14), this implies that

$$g_\delta \rightarrow f \quad \text{in} \quad L^\infty(0,T; L^1(R^N \times R^N)) \quad \text{as} \quad \delta \rightarrow 0_+.$$

We have completed the proof of the stability result in the particular case when $B \in L^1 \cap L^\infty(R^N; L^1(S^{N-1}))$. We now explain how to modify the above arguments in the general case when B satisfies (16). First of all, in view of (19), the only modifications required in Step 1 are due to the fact that the estimates are local in ξ. We just have to explain why $L(f^n)$ is bounded in $L^\infty(0,T;L^1(R^N\times B_R))$ for all $R, T<\infty$. Indeed, we have

$$\int_{R^N} dx \int_{B_R} d\xi \int_{R^N} f^n(x,\xi_*,t)A(\xi-\xi_*)d\xi_* = \iint_{R^N\times R^N} dx d\xi_* f^n(x,\xi_*,t)C_R(\xi_*),$$

where $C_R(z)=\int_{B_R} d\xi A(\xi-z)$. Now, (16) shows that $C_R(z)$ is a continuous function over R^N which satisfies

$$C_R(z)(1+|z|^2)^{-1}\to 0 \quad \text{as} \quad |z|\to\infty,$$

Our claim follows from (14).

To modify Step 2, we consider cut-off functions ψ in $\mathscr{D}(R^N)$ and we introduce $\tilde g^n_\delta=\psi(\xi)g^n_\delta$. Obviously, $\tilde g^n_\delta$ satisfies

$$L_v\tilde g^n_\delta=\psi\frac{1}{1+\delta f^n}Q(f^n,f^n)-v(\Delta_\xi\psi)g^n_\delta-2v\nabla_\xi g^n_\delta\cdot\nabla_\xi\psi.$$

Therefore, $L_v\tilde g^n_\delta$ is bounded in L^1 and $\tilde g^n_\delta$ is compact in $L^1((0,T)\times R^N\times R^N)$ by the same argument as before. Since ψ is arbitrary, the sequence g^n_δ is compact in $L^1((0,T)\times R^N\times B_R)(\forall R, T<\infty)$. Therefore, g^n_δ converges in $L^1((0,T)\times R^N\times R^N)$ because we already know that f^n and thus g^n_δ are weakly compact in L^1. We may then argue as before.

With the same ideas, one can then modify Step 3. First, we show that $L(f^n)$ converges to $L(f)$ in $L^1(0,T;L^1(R^N\times B_R))(\forall T,R<\infty)$. Indeed, we just observe that

$$\int_0^T dt \iint_{R^N\times B_R} |L(f^n)-L(f)|dx d\xi \leqq \int_0^T dt \iint_{R^N\times R^N} dx d\xi_* |f^n(x,\xi_*,t)-f(x,\xi_*,t)|C_R(\xi_*),$$

and conclude easily since f^n converges in L^1 to f, $(\int|f^n||\xi|^2+|f||\xi|^2)$ is bounded and $C_R(z)(1+|z|^2)^{-1}\to 0$ as $|z|\to\infty$. Next, we show that $Q^+(f^n,f^n)(1+\delta f^n)^{-1}$ converges in $L^1((0,T)\times R^N\times B_R)$ to $Q^+(f,f)(1+\delta f)^{-1}$ for all $R,T<\infty$. To this end, we consider for any $K\geqq 1$ the following collision kernels: $B^1_K=B1_{B\leqq K}$, $B^2_K=B1_{B>K}$, and we denote by $Q^i_K, Q^{\pm,i}_K$ the associated collision operators for $i\geqq 1,2$. It is of course enough to show that, for each K, $Q^{+,1}_K(f^n,f^n)(1+\delta f^n)^{-1}$ converges in $L^1((0,T)\times R^N\times B_R)$ to $Q^{+,1}_K(f,f)(1+\delta f)^{-1}$ and that by taking K large $Q^{+,2}_K(f^n,f^n)(1+\delta f^n)^{-1}$ can be made arbitrary small in $L^1((0,T)\times R^N\times B_R)$ uniformly in n. The second fact will be proved in view of (19) and its derivation provided we show that $Q^{-,2}_K(f^n,f^n)(1+\delta f^n)^{-1}$ can be made arbitrary small in $L^1((0,T)\times R^N\times B_R)$ uniformly in n. But this is not difficult to achieve observing that for all $M\geqq 1$,

$$\int_0^T dt \int_{R^N} dx \int_{B_R} d\xi Q^{-,2}_K(f^n,f^n)(1+\delta f^n)^{-1} \leqq \frac{1}{\delta}\int_0^T dt \int_{R^N} dx \int_{R^N} d\xi_* C_R(\xi_*)1_{|\xi_*|>M}\cdot$$
$$\cdot f^n(x,\xi_*,t)+\frac{1}{\delta}\int_{B_R} d\xi \int_{B_M} d\xi_* \int_{S^{N-1}} dw B(\xi-\xi_*,w)1_{(B>K)}h^n(\xi_*),$$

where

$$h^n(\xi_*)=\int_0^T dt\int_{R^N} dx f^n(x,\xi_*,t)\underset{n}{\rightarrow} h(\xi_*)=\int_0^T dt\int_{R^N} dx f(x,\xi_*,t)$$

in $L^1(R^N)$. Then, given $\varepsilon>0$, we can choose M large enough independent of n, K such that the first term in the right-hand side is less than $\varepsilon/2$. With such an M, we can make for K large enough the second term less than $\varepsilon/2$ recalling that $B\in L^1_{\mathrm{loc}}(R^N\times S^{N-1})$. Hence $\mathrm{meas}_{B_R\times B_M\times S^{N-1}}(B>K)\to 0$ as $K\to\infty$. Since $Q_K^{+,1}(f^n,f^n)\leqq Q^+(f^n,f^n)$ and by (19) the sequence $Q^+(f^n,f^n)(1+\delta f^n)^{-1}$ is weakly compact in $L^1((0,T)\times R^N\times B_R)$, it remains only to show that $Q_K^{+,1}(f^n,f^n)$ converges in measure to $Q_K^{+,1}(f,f)$ in order to complete the proof of the convergence of $Q^+(f^n,f^n)$. But since B_K^1 is bounded the proof given in Step 3 above now applies.

We may now adapt the argument given in Step 3 and we see that (21) still holds. The fact that μ vanishes is proved exactly as in Step 3 replacing only the integration with respect to ξ over R^N by a multiplication by $\psi(\xi)\in\mathscr{D}(R^N)$ for an arbitrary cut-off function ψ and then integrating with respect to ξ. Similarly, we observe that $g_\delta\in C([0,T];L^1(R^N\times B_R))(\forall R,T<\infty)$, and we deduce from this the fact that $f\in C([0,\infty);L^1(R^N\times R^N))$. Indeed, for almost all $t,s\in(0,T)$,

$$\begin{aligned}\iint_{R^N\times R^N}|f(x,\xi,t)-f(x,\xi,s)|dxd\xi &\leqq \iint_{R^N\times B_R^c} f(x,\xi,t)+f(x,\xi,s)dxd\xi\\ &+2\sup_{(0,T)}\|f(t)-g_\delta(t)\|_{L^1(R^n\times R^N)}+\iint_{R^N\times B_R}|g_\delta(t)-g_\delta(s)|dxd\xi\end{aligned}$$

for all $R<\infty$. The first term, because of (14), may be bounded by C/R^2 for some C independent of t,s. We already showed above that the second term goes to 0 as δ goes to 0.

Remark. The preceding proof also shows that if $f^n|_{t=0}$ converges in L^1 then f^n converges in $C([0,T];L^1(R^N\times R^N))(\forall T<\infty)$.

IV. Application to Global Existence Results

Theorem 2. *Assume that B satisfies (16) and let f_0 satisfy (2). Then, there exists $f\in C([0,\infty);L^1(R^N\times R^N))$ satisfying $f|_{t=0}=f_0$, (14), (15), (17) and such that, for all $\delta>0$, $g_\delta=\beta_\delta(f)$ satisfies* RFPB *in the sense of distributions and $g_\delta\in L^2((0,T)\times R_x^N;H^1(B_R))(\forall R,T<\infty)$. In particular f is a renormalized solution of* FPB.

Proof. Truncating f_0 and regularizing the truncated function by convolution, we introduce a sequence $\tilde{f}_0^n\in\mathscr{D}(R^N\times R^N)$ such that $\tilde{f}_0^n\geqq 0$ and

$$\iint_{R^N\times R^N} dxd\xi|f_0-\tilde{f}_0^n|(1+|x|^2+|\xi|^2)\underset{n}{\rightarrow}0, \tag{24}$$

$$\iint_{R^N\times R^N} dxd\xi\tilde{f}_0^n|\log\tilde{f}_0^n|dxd\xi\leqq C \tag{25}$$

for some $C\geqq 0$ independent of n. Let

$$f_0^n=\tilde{f}_0^n+\frac{1}{n}\exp\left(-\tfrac{1}{2}|x|^2-\tfrac{1}{2}|\xi|^2\right),$$

and we notice that (24), (25) still hold with $\tilde{f}_0^n$ replaced by f_0^n.

Next, we approximate FPB by truncating conveniently the collision kernel as follows:

$$L_v f^n = \tilde{Q}_n(f^n, f^n) \quad \text{in} \quad R^N \times R^N \times (0, \infty), f^n|_{t=0} = f_0^n \quad \text{in} \quad R^N \times R^N. \tag{26}$$

Here and below, $\tilde{Q}_n$ is given by

$$\tilde{Q}_n = \left(1 + \frac{1}{n}\int |\varphi| d\xi\right)^{-1} Q_n,$$

and

$$Q_n(\varphi, \varphi) = \int_{R^N} d\xi_* \int_{S^{N-1}} dw(\varphi'\varphi'_* - \varphi\varphi_*) B_n(\xi - \xi_*, w),$$

where $B_n \in L^\infty \cap L^1(R^N \times S^{N-1})$, $B_n(z, w)$ is supported in $\{|z| \leqq n\} \times S^{N-1}$, B_n is smooth with respect to z, uniformly in w and (16) holds uniformly if we replace A by $A_n = \int_{S^{N-1}} B(z, w) dw$.

We claim there exists a unique nonnegative solution f^n of (26) which satisfies

$$\iint_{R^N \times R^N} |D^\alpha f^n|(1 + |x|^k + |\xi|^k) dx d\xi \leqq C(T, m, k) \quad \text{if} \quad t \in [0, T], \tag{27}$$

and

$$D^\alpha f^n \in L^\infty(R^N \times R^N \times (0, T)) \tag{28}$$

for all T, m, k, where D^α denotes any derivative up to order m. Existence and uniqueness can be achieved by a simple contraction type fixed point argument observing that for all $\varphi, \psi \in L^1(R^N) \cap L^\infty(R^N)$ we have

$$\int_{R^N} |Q_n(\varphi, \varphi)| \left(1 + \frac{1}{n}\int |\varphi|\right)^{-1} d\xi \leqq C_n \int_{R^N} |\varphi| d\xi,$$

$$\left\| Q_n(\varphi, \varphi) \left(1 + \frac{1}{n}\int |\varphi|\right)^{-1} d\xi \right\|_\infty \leqq C_n \|\varphi\|_{L^\infty},$$

and

$$\begin{aligned}
&\int_{R^N} |\tilde{Q}_n(\varphi, \varphi) - \tilde{Q}_n(\psi, \psi)| d\xi \\
&\leqq \int_{R^N} \left(1 + \frac{1}{n}\int_{R^N} |\varphi|\right)^{-1} |Q_n(\varphi - \psi, \varphi)| d\xi + \frac{1}{n}\int_{R^N} d\xi |Q_n(\psi, \varphi)| \left(1 + \frac{1}{n}\int_{R^N} |\varphi|\right)^{-1} \\
&\quad \cdot \left(1 + \frac{1}{n}\int_{R^N} |\psi|\right)^{-1} \int_{R^N} |\varphi - \psi| d\xi + \left(\int_{R^N} d\xi |Q_n(\psi, \varphi - \psi)|\right)\left(1 + \frac{1}{n}\int_{R^N} |\psi|\right)^{-1} \\
&\leqq C_n \int_{R^N} |\varphi - \psi| d\xi,
\end{aligned}$$

where C_n denote various constants independent of φ, ψ. These bounds also imply an $L^1 \cap L^\infty(R^N \times R^N)$ estimate uniform in t on $[0, T]$. Next, we show that

$$\sup_{[0,T]} \iint f^n(1 + |x|^k + |\xi|^k) dx d\xi < \infty \quad \text{for all} \quad k \geqq 1.$$

In fact, for $k=0$ or $k=2$, we just have to observe that the identities proven in Sect. I still apply here. For $k \geqq 3$, we multiply the equation by $|\xi|^k + |x|^k$ and integrate by parts over $R^N \times R^N$ to obtain

$$\frac{d}{dt} \iint_{R^N \times R^N} f^n(|x|^k + |\xi|^k) dx d\xi \leqq C \iint_{R^N \times R^N} f^n(|x|^{k-1}|\xi| + |\xi|^{k-2}) dx d\xi + \iint_{R^N \times R^N} |\tilde{Q}_n(f^n, f^n)| |\xi|^k d\xi.$$

Since B_n has compact support in z, one can bound easily the second term by

$$C_n \iint_{R^N \times R^N} f^n(1 + |\xi|^k) d\xi.$$

Our claim follows from Grönwall's lemma.

Next, we establish an a priori estimate on derivatives of f^n in $L^\infty(0, T; L^1 \cap L^\infty(R^N \times R^N))$. We begin with $\nabla_x f^n$ and we differentiate (26) with respect to x. We deduce from the resulting equation that

$$\begin{aligned}\frac{d}{dt} \iint_{R^N \times R^N} |\nabla_x f^n| dx d\xi \leqq & \iint_{R^N \times R^N} dx d\xi \{Q_n^+(|\nabla_x f^n|, f^n) + Q_n^-(|\nabla_x f^n|, f^n) \\ & + Q_n^+(f^n, |\nabla_x f^n|) + Q_n^-(f^n, |\nabla_x f^n|)\} \left(1 + \frac{1}{n} \int f_n d\xi\right)^{-1} \\ & + \iint_{R^N \times R^N} dx d\xi |Q_n(f^n, f^n)| \left(1 + \frac{1}{n} \int f^n d\xi\right)^{-1} \int |\nabla_x f^n| d\xi \\ \leqq & \, C_n \iint_{R^N \times R^N} |\nabla_x f^n| dx d\xi.\end{aligned}$$

One argues similarly for the other derivatives and for the weighted L^1 estimates.

Next, to prove Theorem 2, we observe that even though f^n is not a solution of FPB, Theorem 1 and its proof still apply to this sequence of solutions of an approximate equation and yield convergence in $C([0, T]; L^1(R^N \times R^N))(\forall T < \infty)$ to some f (up to the extraction of a subsequence) satisfying all the properties listed in Theorem 2, provided f^n satisfies (14) and (15), where B is replaced by $B_n(1 + (1/n)\int f^n d\xi)^{-1}$. In turn, these estimates follow at once provided (7), (9–11) are justified. In view of the regularity and the decay of f^n, there is no difficulty in checking (7), (9), (11). However, (10) relies upon multiplying the equation by $\log f^n$ and integrating over $[0, T] \times R^N \times R^N$. Since f^n is bounded, this integration will be justified by the following lower bound on f^n; for each $T < \infty$, there exists C such that

$$f^n \geqq \frac{1}{n} \exp\{-Ct - \tfrac{1}{2}|x - \xi t|^2 - \tfrac{1}{2}|\xi|^2\}, \quad \forall (x, \xi, t) \in R^N \times R^N \times [0, T]. \tag{29}$$

Let us denote by g^n the right-hand side of (29). In view of the choice of f_0^n, we have $f^n|_{t=0} \geqq g^n|_{t=0}$. Next, we observe that

$$\left(1 + \frac{1}{n} \int_{R^N} f^n d\xi\right)^{-1} Q_n^-(f^n, f^n) \leqq C_0 f^n \quad \text{on} \quad R^N \times R^N \times [0, T],$$

and we choose $C = C_0 + N\nu(1 + T^2)$. Now, in order to verify (29) we remark that f^n satisfies

$$L_\nu f^n + C_0 f^n \geqq 0 \quad \text{in} \quad R^N \times R^N \times [0, T].$$

Finally, we observe that

$$L_\nu g^n + C_0 g^n \leqq 0 \quad \text{in} \quad R^N \times R^N \times [0, T],$$

since a straightforward computation yields

$$L_\nu g^n + C_0 g^n = g^n\{-C + N\nu(1 + t^2) - |(1 + t^2)\xi + tx|^2\} \leqq 0.$$

This concludes the proof of Theorem 2.

V. Extensions of the Results: Mixtures, Discrete Velocity Models

We begin this section with a few observations on the results and the proofs given in the preceding section. First of all, the stability result is still valid if f^n is a sequence of solutions of FPB in the sense of Theorem 2, i.e. $\beta_\delta(f^n)$ solves RFPB provided of course we still assume (14) and (15). Indeed, we never really used the smoothness of f^n in our proof. Second, we mention that it is possible to show that the limit f satisfies the entropy inequality and therefore yields an entropy bound.

Next, it is worth explaining the role of the entropy bounds. Suppose f^n is a sequence of nonnegative smooth solutions of FPB satisfying only

$$\iint_{R^N \times R^N} f^n(x, \xi, t)(1 + |x|^2 + |\xi|^2) dx d\xi \leqq C_T, \quad \text{a.e.} \quad t \in (0, T), \quad \text{for all} \quad T < \infty. \tag{30}$$

Then, Steps 1 and 2 of our proof still apply and yield the convergence in $L^p(0, T; L^1(R^N \times R^N))$ $(\forall p < \infty, \forall T < \infty)$ of f^n to f satisfying (30). However, we are no longer able to conclude that $g_\delta = \beta_\delta(f)$ solves RFPB. Instead, we obtain the following information. The sequence $(1/1 + \delta f^n)Q^-(f^n, f^n)$ converges in $L^1((0, T) \times R^N \times B_R)$ to $(1/1 + \delta f)Q^-(f, f) \in L^\infty(0, T; L^1(R^N \times B_R))$ for all $R, T < \infty$ while

$$\frac{Q^+(f^n, f^n)}{1 + \delta f^n} \underset{n}{\rightarrow} \frac{Q^+(f, f)}{1 + \delta f} + \mu_1 \text{ in the sense of distributions,}$$

where μ_1 is nonnegative measure on $R_+ \times R^N \times R^N$ which is bounded on each $(0, T) \times R^N \times B_R$ for all $R, T < \infty$, From weak convergence, one also obtains

$$|\nabla_\xi g_\delta^n|^2 \underset{n}{\rightarrow} |\nabla_\xi g_\delta|^2 + \mu_2 \text{ in the sense of distributions,}$$

where μ_2 has the same properties as μ_1. Therefore, denoting by $\mu = \mu_1 + \mu_2$, we see that g_δ solves

$$L_\nu g_\delta = \nu\delta |\nabla_\xi g_\delta|^2 + \frac{1}{(1 + \delta f)} Q(f, f) + \mu \quad \text{in} \quad \mathscr{D}'((0, \infty) \times R^N \times R^N).$$

The formal argument given in Sect. II still applies. Since it is not clear that this formal argument can be justified, we do not even know if it is reasonable to expect that μ vanishes if we merely assume (30).

Our next observation concerns the formulation of RFPB. In the preceding sections we introduce a notion of solution of FPB consisting of writing down the equations satisfied by the family of transformed functions β_δ, $\delta > 0$. The choice of $\beta_\delta = \frac{1}{\delta}\log(1 + \delta t)$ may seem somewhat arbitrary. Indeed we could employ general nonlinear transformations β with the following properties:

$$\beta \in C^2(R), |\beta'(t)| \leqq \frac{C}{1+t}, |\beta''(t)| \leqq \frac{C}{(1+t)^2} \quad \text{on} \quad (0, \infty) \tag{31}$$

for some $C \geqq 0$. Then, one expects that $\beta(f)$ solves

$$L_\nu \beta(f) = -\nu\beta''(f)|\nabla_\xi f|^2 + \beta'(f)Q(f, f).$$

In order to make sense of the term $\beta''(f)|\nabla_\xi f|^2$ we write it in the following convenient form:

$$\beta''(f)|\nabla_\xi f|^2 = \beta''(f)(1 + f)^2|\nabla_\xi \beta_1(f)|^2.$$

One can easily check that if $\beta = \beta_\delta$ this quantity reduces to $-\delta|\nabla_\xi \beta_\delta(f)|^2$ by standard Sobolev spaces manipulations. Therefore, the above equation becomes

$$L_\nu \beta(f) = -\nu\beta''(f)(1 + f)^2|\nabla_\xi \beta_1(f)|^2 + \beta'(f)Q(f, f). \tag{32}$$

Next, we show that if RFPB holds for one $\delta > 0$, say $\delta = 1$, then (32) holds for all nonlinearities β satisfying (31). To prove this claim, we introduce $\gamma(s) = \beta(e^s - 1)$ and we observe that $\gamma \in C^2([0, \infty))$, γ' and γ'' are bounded. In view of the regularity of $\beta_1(f)$, we deduce that $g = \gamma(\beta_1(f))$ solves

$$L_\nu g = \nu\gamma'(\beta_1(f))|\nabla_\xi \beta_1(f)|^2 - \nu\gamma''(\beta_1(f))|\nabla_\xi \beta_1(f)|^2 + \gamma'(\beta_1(f))\frac{1}{1+f}Q(f, f).$$

We conclude easily by remarking that, for all $t \geqq 0$, we have

$$\gamma'(\beta_1(t))\frac{1}{1+t} = \beta'(t), \gamma''(\beta_1(t)) - \gamma'(\beta_1(t)) = \beta''(t)(1 + t)^2.$$

Finally, we conclude this section with a few examples of other equations which can be treated by our method. First, one can replace $-\nu\Delta_\xi$ by more general operators: for instance, we can add first-order terms in ∇_x or ∇_ξ and fixed external forces. We can also consider general elliptic operators in ξ or in (x, ξ) such as $-\varepsilon\Delta_x - \nu\Delta_\xi$.

Next, our analysis is easily adapted to the case of mixtures, or molecules with different masses, or to the case of a dense gas and we refer to C. Cercignani [3], S. Chapman and T. G. Cowlings [4], C. Truesdell and R. Muncaster [15] for more details on these related equations. We conclude with an application to the so-called discrete velocities model. We look for nonnegative solutions f^i, $1 \leqq i \leqq m$ of

$$\frac{\partial f^i}{\partial t} + c_i \cdot \nabla f^i - \varepsilon_i \Delta f^i = Q^i(f) \quad \text{in} \quad R^N \times (0, \infty), \quad \forall 1 \leqq i \leqq m, \tag{33}$$

where $\varepsilon_i > 0$, $c_i \in R^N$ and Q^i is the operator given by

$$Q^i(f) = \sum_{j,k,l=1}^{m} A_{kl}^{ij}(f^k f^l - f^i f^j), \quad \forall 1 \leqq i \leqq m, \tag{34}$$

and $A_{kl}^{ij} \in L^\infty(R^N)$ $(\forall i,j,k,l)$ satisfy

$$A_{kl}^{ij} = A_{kl}^{ji} = A_{ij}^{kl} \quad \forall i,j,k,l; A_{kl}^{ij} \geqq 0 \quad \text{a.e.} \quad \text{in} \quad R^N. \tag{35}$$

This system of equations is nothing but a viscous approximation to a general discrete velocities model of gas dynamics. We refer the reader to H. Cabannes [2] for a presentation of these models and to L. Tartar [14], R. Illner [12], K. Hamdache [10] and the references therein for some existence results. Formally, the following identities hold:

$$\sum_i \int_{R^N} f^i(x,t)dx \text{ is independent of } t, \tag{36}$$

$$\frac{d}{dt}\sum_i \int_{R^N} f^i \log f^i dx + \varepsilon_i \sum_i \int_{R^N} |\nabla (f^i)^{1/2}|^2 dx + \frac{1}{4}\sum_{i,j,k,l}\int_{R^N} A_{kl}^{ij}\{f^k f^l - f^i f^j\}\log\left(\frac{f^k f^l}{f^i f^j}\right)dx = 0, \tag{37}$$

$$\frac{d}{dt}\sum_i \int_{R^N} f_i \psi dx = \sum_i \int_{R^N} c_i \cdot \nabla_\psi + \varepsilon_i \nabla_\psi \tag{38}$$

for any smooth test function $\psi(x)$.

The analogies of Theorem 1 and Theorem 2 hold for this class of discrete-velocity systems with straightforward adaptations and in fact simplifications of the proofs. The only technical modification is the following weighted L^1 bound on the initial data:

$$\sum_i \int_{R^N} f_0^i \psi dx \leqq C_T.$$

Here $\psi \geqq 0$, $\psi \in C^2(R^N)$ satisfies the following conditions:

$$\begin{cases} \exists C_1 \geqq 0, \quad \forall 1 \leqq i \leqq m, \quad c_i \nabla \psi + \varepsilon_i \Delta \psi \leqq C_1 \psi \quad \text{on} \quad R^N \\ \psi \to +\infty \quad \text{as} \quad |x| \to \infty \end{cases} \tag{39}$$

$$\exists C_0 > 0, \quad \exp(-C_0 \psi) \in L^1(R^N). \tag{40}$$

Assumption (39) provides the uniform integrability at infinity while (40) allows us to obtain L^1 bounds on $f^i|\log f^i|$ by a similar argument to the one made in Sect. I. Of course, the renormalized formulation of (33) is

$$\begin{aligned} \frac{\partial}{\partial t}\beta_\delta(f^i) + c_i \cdot \nabla \beta_\delta(f^i) - \varepsilon_i \Delta \beta_\delta(f^i) &= \frac{1}{1+\delta f^i} Q^i(f) + \varepsilon_i \delta |\nabla \beta_\delta(f^i)|^2 \\ &= \sum_{j,k,l=1}^{m} A_{kl}^{ij}\left\{\frac{f^k f^l}{1+\delta f^i} - \frac{f^i}{1+\delta f^i} f^j\right\} + \varepsilon_i \delta |\nabla \beta_\delta(f^i)|^2. \end{aligned} \tag{41}$$

Appendix: Compactness in L^1 for Hypoelliptic Operators

Let E be a second-order, elliptic, possibly degenerate operator on R^m with smooth coefficients. We assume that $\partial/\partial t + E$ has a continuous fundamental solution $p(t, x, y)$ satisfying

$$\sup_{t\in(0,T],y\in R^m} \int p(t,x,y)dx \leqq C(T), \quad \forall T<\infty, \tag{A.1}$$

$$p(t,x,y) \leqq C(T,h) \quad \text{if} \quad t\in[h,T], x, y\in R^m, \quad \forall 0<h<T<\infty, \tag{A.2}$$

$$\sup_{|y|\leqq M, t\in[h,T]} \int_{|x|\geqq R} p(t,x,y)dx \to 0 \quad \text{as} \quad R\to\infty, \quad \forall M<\infty, \quad \forall 0<h<T<\infty. \tag{A.3}$$

In view of the explicit fundamental solution of L_v these assumptions are met in the case considered in Sect. III (see L. Hörmander [11] for more details). Let h^n be a bounded sequence in $L^1((0,T)\times R^m)$ satisfying

$$\sup_n \int_0^T \int_{|x|\geqq R} |h^n| dx dt \to 0 \quad \text{as} \quad R\to\infty, \tag{A.4}$$

and let g_0^n be a bounded sequence in $L^1(R^m)$ satisfying

$$\sup_n \int_{|x|\geqq R} |g_0^n| dx \to 0 \quad \text{as} \quad R\to\infty. \tag{A.5}$$

We denote by g^n the solution of

$$\frac{\partial}{\partial t} g^n + E g^n = h^n \text{ in } (0,T)\times R^m, g^n|_{t=0} = g_0^n \quad \text{in} \quad R^m, \tag{A.6}$$

i.e. g^n is given by

$$g^n(t,x) = \int_0^t \int_{R^m} h^n(s,y) p(t-s,x,y) dy ds + \int_{R^m} g_0^n(y) p(t,x,y) dy \quad \text{for} \quad t\in(0,T), x\in R^m. \tag{A.7}$$

Proposition. *The sequenece g^n is compact in $L^1((0,T)\times R^m)$.*

Proof. We first show that if A is a Borel set in $(0,T)\times R^m$, then

$$\sup_n \int_A |g^n(t,x)| \to 0 \quad \text{as meas}(A)\to 0. \tag{A.8}$$

To prove this uniform continuity, we argue as follows. We first write

$$\int_A |g^n| \leqq \int_0^T \int_{R^M} 1_A(t,x) dt dx \int_0^t \int_{R^m} |h^n|(s,y) p(t-s,x,y) dy dx$$
$$+ \int_0^T \int_{R^m} 1_A(t,x) dt dx \int_{R^m} |g_0^n(y)| p(t,x,y) dy,$$

and thus (A.8) holds as soon as we have

$$\sup_{s\in[0,T], y\in R^m} q_A(s,y) \to 0, \quad \text{as meas}|A|\to 0, \tag{A.9}$$

where $q_A(s,y) = \int_s^T dt \int_{R^m} dx 1_A(t,x) p(t-s,x,y)$. But, clearly for all $\delta\in(0,T)$,

$$q_A(s,y) \leqq \int_s^{(s+\delta)\wedge T} dt\left(\int_{R^m} p(t-s,x,y)dx\right) + 1_{(s<T-\delta)}\int_{s+\delta}^{T}\int_{R^m} dtdx\, 1_A(x,t)p(t-s,x,y)$$

$$\leqq C(T)\delta + C(\delta, T) \quad \text{meas}\,(A),$$

and (A.9) is proved.

Next, we prove that

$$\sup_n \int_0^T \int_{|x|\geqq R} dtdx\,|g^n(t,x)| \to 0 \quad \text{as} \quad R\to\infty. \tag{A.10}$$

By arguments similar to the one above, we obtain for all $\delta\in(0,T)$, $M<\infty$,

$$\int_0^T \int_{|x|\geqq R} dtdx\,|g^n(t,x)| \leqq C\delta + C \sup_{|y|\leqq M, t\in(\delta,T)} \int_{|x|\geqq R} p(t,x,y)dx + C\int_0^T \int_{|y|\geqq M} dtdy\,|h^n|$$
$$+ C\int_{|y|\geqq M} |g_0^n|\,dy,$$

where C denotes various nonnegative constants independent of δ, M, R. Then (A.10) follows from (A.3), (A.4) and (A.5).

Therefore, to prove the Proposition, we just have to show that there exists a subsequence of g^n which converges a.e. on $(0,T)\times R^m$. Without loss of generality, we may assume that h^n converges tightly to some bounded measure μ on $[0,T]\times R^m$ and that g_0^n converges tightly to some bounded measure λ on R^m. Then, we have immediately

$$\int_{R^m} g_0^n(y)p(t,x,y)dy \underset{n}{\to} \int_{R^m} p(t,x,y)d\lambda(y), \quad \forall t>0, \quad \forall x\in R^m.$$

Therefore, we can assume $g_0^n\equiv 0$ without loss of generality.

Next, let $\delta>0$ and let $\varphi_\delta\in C^\infty(R)$, $0\leqq\varphi_\delta\leqq 1$, $\varphi_\delta\equiv 0$ if $s\leqq\delta$, $\varphi_\delta\equiv 1$ if $s\geqq 2\delta$. We then set

$$g_\delta^n(t,x) = \int_0^T \int_{R^m} dsdy\,h^n(s,y)\varphi_\delta(t-s)p(t-s,x,y).$$

Observe that

$$g_\delta^n(t,x) \underset{n}{\to} \iint \varphi_\delta(t-s)p(t-s,x,y)d\mu(s,y).$$

Hence, our claim will be proved if we show

$$\sup_n \int_0^T dt \int_{R^m} dx\,|g_\delta^n - g^n| \to 0 \quad \text{as} \quad \delta\to 0_+. \tag{A.11}$$

But this is not difficult since we have

$$\int_0^T dt\int_{R^m} dx\,|g_\delta^n - g^n| \leqq \int_0^T dt \int_{R^m} dx\left\{\int_{(t-2\delta)^+}^{t}\int_{R^m} dsdy\,|h^n(s,y)|\,p(t-s,x,y)\right\}$$
$$\leqq 2C_T\delta\int_0^T\int_{R^m} |h^n(s,y)|\,dxdy \leqq C\delta$$

for some C independent of n.

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