Commun. Math. Phys. 114, 599–612 (1988)

Communications in
Mathematical
Physics

Analyticity Properties of Eigenfunctions and Scattering Matrix*

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Abstract. For potentials $V=V(x)=O(|x|^{-2-\varepsilon})$ for $|x|\to\infty$, $x\in\mathbb{R}^3$, we prove that if the S-matrix of $(-\Delta, -\Delta+V)$ has an analytic extension $\tilde{S}(z)$ to a region $\mathcal{O}$ in the lower half-plane, then the family of generalized eigenfunctions of $-\Delta+V$ has an analytic extension $\tilde{\phi}(k,\omega,x)$ to $\mathcal{O}$ such that $|\tilde{\phi}(k,\omega,x)|<Ce^{b|x|}$ for $|\mathrm{Im}\,k|<b$. Consequently, the resolvent $(-\Delta+V-z^2)^{-1}$ has an analytic continuation from $\mathbb{C}^+$ to $\{k\in\mathcal{O}\,|\,|\mathrm{Im}\,k|<b\}$ as an operator $\tilde{R}(z)$ from $\mathcal{H}_b=\{f=e^{-b|x|}g\,|\,g\in L_2(\mathbb{R}^3)\}$ to $\mathcal{H}_{-b}$. Based on this, we define for potentials $W=o(e^{-2b|x|})$ resonances of $(-\Delta+V, -\Delta+V+W)$ as poles of $(1+W\tilde{R}(z))^{-1}$ and identify these resonances with poles of the analytically continued S-matrix of $(-\Delta+V, -\Delta+V+W)$.

Introduction

Analytic continuation of the scattering matrix of a two-body Schrödinger operator $-\Delta+V$ has been established for various classes of the potential V, including exponentially decaying [3] and dilation-analytic, short-range [4] potentials.

Two methods were developed to obtain a unified approach to these two classes of potentials, one [5] based on local spectral deformation techniques in momentum space, the other [9] based on an analytic family of deformations of the underlying momentum-space. These methods cover potentials of the form $V+W$, where $V=O(r^{-2-\varepsilon})$ is radial, dilation-analytic and W is exponentially decaying.

For radial potentials a different method was introduced [6]. The basic idea was that if the resolvent $(-\Delta+V-k^2)^{-1}$ can be shown to have an analytic continuation to a domain $\mathcal{O}$ in the lower half-plane as an operator from a space of exponentially decaying functions to its dual, then $-\Delta+V$ can play the role of $-\Delta$ as background for an exponentially decaying perturbation W, using analytic

* The author would like to thank the Institute for Advanced Study for its hospitality and the National Science Foundation for financial support under Grant No. DMS-8610730(1)

Fredholm theory. In particular it was shown that if the scattering matrix of $(-\Delta, -\Delta+V)$ has an analytic extension to $\mathcal{O}$, then the resolvent has such an analytic continuation.

In the present paper we extend these results to non-radial potentials. In Sect. 1 we establish the existence of an analytic extension to the upper half-plane of the generalized eigenfunctions $\psi_+(k,\omega,x)$ defined in a standard way for k real. The construction utilizes methods of [1], extended to operators of the type $e^{-ik\omega\cdot x}\Delta e^{ik\omega\cdot x}$. The proof requires that $V(x)=O(|x|^{-2-\varepsilon})$, but it is possible to extend these results to potentials $V(x)=O(|x|^{-1-\varepsilon})$ (cf. [10, 11]) and perhaps even, with modifications, to long potentials. A different method of obtaining the analytic extension of the eigenfunctions to the upper half-plane via the Green's function is indicated in [2].

In Sect. 2 we prove that $\psi(k,\omega,x)$ can be continued analytically in k to a domain $\mathcal{O}$ in the lower half-plane, provided the S-matrix $S_1(k)$ can be extended in this way (Theorem 2.2). We utilize the abstract stationary scattering theory as developed in [8]. From this we obtain analytic continuation of the resolvent $(-\Delta+V-k^2)^{-1}$, acting as an operator from an exponentially weighted L_2-space into its dual (Theorem 2.4). This allows for addition of an exponentially decaying potential W to V. As a consequence we obtain as our main result (Theorems 2.5 and 2.7) the meromorphic extension of the S-matrices $S_{12}(k)$ of $(-\Delta+V, -\Delta+V+W)$ and $S_2(k)$ of $(-\Delta, -\Delta+V+W)$. This provides a basis for the further study of resonances and resonance functions arising from a very short range potential W acting on the background of a short range smooth potential V. This will be taken up in [11].

1. Generalized Eigenfunctions

In this section we define the generalized eigenfunctions $\Psi(k,\omega,x)$ for $k\in\overline{\mathbb{C}^+}\setminus\{0\}$, $\omega\in S^2$ and $x\in\mathbb{R}^3$. We establish the analyticity properties of $\Psi(k,\omega,x)$ as functions of $k\in\overline{\mathbb{C}^+}\setminus\{0\}$ and their exponential growth as functions of $x\in\mathbb{R}^3$.

The free Hamiltonian H_0 is the selfadjoint operator defined by $H_0 f=-\Delta f$ with domain $\mathcal{D}_{H_0}=H_2(\mathbb{R}^3)$. The potential V is assumed to be multiplication by a real-valued, measurable function v on $\mathbb{R}^3$ satisfying for some $\varepsilon>0$, $R_0>0$,

(i) $v\in L_2^{\mathrm{loc}}(\mathbb{R}^3)$,

(ii) $|v(x)|<C|x|^{-2-\varepsilon}$ for $|x|>R_0$.

V is H_0-compact and hence $H_0-\varepsilon$-bounded. Thus, the Hamiltonian $H_1=H_0+V$ is selfadjoint on $\mathcal{D}_{H_1}=\mathcal{D}_{H_0}$, and $\sigma_e(H_1)=\overline{\mathbb{R}^+}$.

Two families of generalized eigenfunctions $\Psi_\pm(k,\omega,x)$ are formally defined [1] for $k>0$, $\omega\in S^2$, $x\in\mathbb{R}^3$ by

$$\Psi_\pm(k,\omega,x)=\left(1\text{-}\lim_{\varepsilon\downarrow 0}(H_1-k^2\pm i\varepsilon)^{-1}V\right)e^{ik\omega\cdot x},$$

where the limit is taken in a suitable topology.

For the purpose of studying analyticity properties it is useful to introduce the family of eigenfunctions $\Psi(k,\omega,x)$ defined formally for $k\in\overline{\mathbb{C}^+}\setminus\{0\}$ by

$$\Psi(k,\omega,x)=\lim_{\varepsilon\downarrow 0}(1-R_1((k+i\varepsilon))V)e^{ik\omega\cdot x},$$

where $R_1(k+i\varepsilon)=(H_1-(k+i\varepsilon)^2)^{-1}$.

The connection between $\Psi_\pm(k,\omega,x)$ and $\Psi(k,\omega,x)$ is given for $k>0$ by

$$\Psi_+(k,\omega,x)=\Psi(-k,-\omega,x),\qquad \Psi_-(k,\omega,x)=\Psi(k,\omega,x). \tag{1.1}$$

We shall prove that the functions $\Psi(k,\omega,x)$ exist and behave asymptotically as $e^{ik\omega\cdot x}$. For this purpose we study the functions $\Phi(k,\omega,x)$ formally defined by

$$\begin{aligned}\Phi(k,\omega,x)&=1-e^{ik\omega\cdot x}\Psi(k,\omega,x)=e^{-ik\omega\cdot x}R_{1+}(k)Ve^{ik\omega\cdot x}\\&=(1-e^{-ik\omega\cdot x}R_{0+}(k)e^{ik\omega\cdot x}V)^{-1}e^{-ik\omega\cdot x}R_{0+}(k)e^{-ik\omega\cdot x}V,\end{aligned}$$

where in a suitable topology

$$R_{i+}(k)=\lim_{\varepsilon\downarrow 0}R_i((k+i\varepsilon)),\qquad i=0,1.$$

The basic operator is $R_{0+}(k,\omega)$ defined for $k\in\overline{\mathbb{C}^+}$, $\omega\in S^2$, by

$$\begin{aligned}R_{0+}(k,\omega)&=e^{-ik\omega\cdot x}R_{0+}(k)e^{ik\omega\cdot x}=(e^{-ik\omega\cdot x}(H_0-k^2)e^{ik\omega\cdot x})_+^{-1}\\&=(T_\omega e^{-ikx_1}T_\omega^{-1}(H_0-k^2)T_\omega e^{ikx_1}T_\omega^{-1})_+^{-1}\\&=T_\omega(e^{-ikx_1}(H_0-k^2)e^{ikx_1})_+^{-1}T_\omega^{-1}=T_\omega\left(H_0-2ik\frac{\partial}{\partial x_1}\right)_+^{-1}T_\omega^{-1},\end{aligned}$$

where T_ω is the operator defined by

$$(T_\omega f)(x)=f(t_\omega^{-1}x),$$

and t_ω is any rotation which sends $(1,0,0)$ into ω.

Setting $k=\alpha+i\beta$, we have

$$H_0-2ik\frac{\partial}{\partial x_1}=e^{-i\alpha x_1}e^{\beta x_1}H_0e^{-\beta x_1}e^{i\alpha x_1}-k^2=e^{-i\alpha x_1}(H_0(i\beta)-k^2)e^{i\alpha x_1},$$

where

$$H_0(i\beta)=e^{\beta x_1}H_0e^{-\beta x_1}=H_0+2\beta\frac{\partial}{\partial x_1}-\beta^2$$

(see also Appendix 2).

The operator $H_0(i\beta)$ has as its spectrum the parabolic region $\mathscr{P}_\beta=\{z^2\,|\,|\mathrm{Im}\,z|\leqq\beta\}$. The point k^2 belongs to the boundary of $\mathscr{P}_\beta$, and for $\beta\geqq 0$ fixed, $\varepsilon>0$, $(k+i\varepsilon)^2\in\mathbb{C}\setminus\mathscr{P}_\beta$,

$$R_0(\beta,k+i\varepsilon)=(H_0(i\beta)-(k+i\varepsilon)^2)^{-1}\in\mathscr{B}(\mathscr{H}),$$

and

$$\left(H_0-2ik\frac{\partial}{\partial x_1}\right)_+^{-1}=e^{i\alpha x_1}R_{0+}(\beta,k)e^{-i\alpha x_1},$$

where

$$R_{0+}(\beta, k) = \lim_{\varepsilon \downarrow 0} R_0(\beta, k + i\varepsilon).$$

The rest of this section is devoted to a rigorous derivation of the above formalism.

Definition. For $s \in \mathbb{R}$ we define the following spaces:

$$L^1_{2,s} = L^1_{2,s}(\mathbb{R}^3) = \left\{ f \,|\, \|f\|^2_{L^1_{2,s}} = \int_{\mathbb{R}^3} |f(x)|^2 (1 + x_1^2)^s \, dx < \infty \right\},$$

$$L_{2,s} = L_{2,s}(\mathbb{R}^3) = \left\{ f \,|\, \|f\|^2_{L_{2,s}} = \int_{\mathbb{R}^3} |f(x)|^2 (1 + |x|^2)^s \, dx < \infty \right\},$$

$$H_{2,s} = H_{2,s}(\mathbb{R}^3) = \{ f \,|\, D^\alpha f \in L_{2,s} \quad \text{for} \quad |\alpha| \leqq 2 \},$$

and

$$\|f\|^2_{H_{2,s}} = \sum_{|\alpha| \leqq 2} \|D^\alpha f\|^2_{L_{2,s}}.$$

For $b \in \mathbb{R}$,

$$\mathscr{H}^b = \left\{ f \,|\, \|f\|^2_{\mathscr{H}^b} = \int_{\mathbb{R}^3} e^{2b|x|} |f(x)|^2 \, dx < \infty \right\},$$

$$\mathscr{H}^b_2 = \{ f \,|\, D^\alpha f \in \mathscr{H}^b \quad \text{for} \quad |\alpha| \leqq 2 \},$$

and

$$\|f\|^2_{\mathscr{H}^b_2} = \sum_{|\alpha| \leqq 2} \|D^\alpha f\|^2_{\mathscr{H}^b}.$$

We set

$$\mathscr{H} = L_2(\mathbb{R}^3), \qquad \mathfrak{H} = L^2(S^2).$$

$\mathscr{H}^b_{\mathfrak{H}}$ and $\mathscr{H}^b_{2,\mathfrak{H}}$ are defined as $\mathscr{H}^b$ and $\mathscr{H}^b_2$ with f replaced by an $\mathfrak{H}$-valued function on $\mathbb{R}^3$ and $|f(x)|$ replaced by $\|f(x)\|_{\mathfrak{H}}$.

We consider for $\beta \geqq 0$ the closed operator $H_0(i\beta)$ with domain $H_2(\mathbb{R}^3)$, defined by

$$H_0(i\beta) = -\Delta + 2\beta \frac{\partial}{\partial x_1} - \beta^2,$$

whose Fourier transform is multiplication by the polynomial

$$Q(\beta) = \xi^2 + 2i\beta\xi_1 - \beta^2.$$

Lemma 1.1. *For given $s > \frac{1}{2}$, $\beta_0 > 0$, K a compact subset of $\mathbb{C} \setminus \{0\}$, there exists a constant C such that for $0 \leqq \beta \leqq \beta_0$, $z \in K$, $u \in H_2$*

$$\|u\|_{L^1_{2,-s}} \leqq C \|(H_0(i\beta) - z)u\|_{L^1_{2,s}}.$$

Proof. The lemma is proved in the same way as [1, Lemma A.1]. Note that the set of critical points $\Lambda_c(Q(\beta))$ of $Q(\beta)$ is given by

$$\Lambda_c(Q(\beta)) = \{Q(\beta)\xi \,|\, \nabla_\xi Q(\beta) = 0\} = \begin{cases} \emptyset & \text{if} \quad \beta \neq 0 \\ \{0\} & \text{if} \quad \beta = 0 \end{cases}.$$

It is easy to check that each step of the proof given in [1] is valid with $(1+|x|^2)^s$ replaced by $(1+x_1^2)^s$ and that C can be chosen independent of β, $0 \leqq \beta \leqq \beta_0$.

For the construction of generalized eigenfunctions we shall make use of the boundary values $R_{0+}(\beta, k)$, $k=\alpha+i\beta$. The existence of these boundary values is proved through a series of lemmas. In this section K denotes a compact subset of $\overline{\mathbb{C}^+}\backslash\{0\}$.

Lemma 1.2. *Let* $g \in L^1_{2,s}$, $s > \frac{1}{2}$. *For* $k \in \overline{\mathbb{C}^+}\backslash\{0\}$ *the following weak limits exist in* $H^1_{2,-s}$, *uniformly for* $k \in K$,

$$R_{0+}(\beta, k) g = \underset{\varepsilon \downarrow 0}{w\text{-lim}}\, R_0(\beta, k+i\varepsilon) g .$$

Proof. For f and g with compact support

$$(f, R_0(\beta, k+i\varepsilon) g) = (e^{\beta x_1} f, R_0(k+i\varepsilon) e^{-\beta x_1} g) \underset{\varepsilon \downarrow 0}{\longrightarrow} (e^{\beta x_1} f, R_0(k) e^{-\beta x_1} g),$$

uniformly for $k \in K \subset \mathbb{C}^+$. By Lemma A1 this also holds for $K \subset \overline{\mathbb{C}^+}\backslash\{0\}$. Then, by Lemma 1.1, the same holds for f, $g \in L^1_{2,s}$.
Thus, for $g \in L^1_{2,s}$ there exists

$$\underset{\varepsilon \downarrow 0}{w\text{-lim}}\, R_0(\beta, k+i\varepsilon) g \quad \text{in} \quad L^1_{2,-s} .$$

By Lemma 1.1, $\{R_0(\beta, k+i\varepsilon) g\}$ is also bounded in $H^1_{2,-s}$ for $\varepsilon > 0$, uniformly for $k \in K$. Hence, by the weak compactness of the unit ball in $H^1_{2,-s}$, the weak limit is attained also in $H^1_{2,-s}$, uniformly for $k \in K$.

Theorem 1.3. $R_{0+}(\beta, k) = \lim_{\varepsilon \downarrow 0} R_0(\beta, k+i\varepsilon)$ *in the uniform operator topology of* $\mathscr{B}(L^1_{2,s}, H_{2,-s})$, *uniformly for* $k \in K$.

Moreover, the $\mathscr{B}(L^1_{2,s}, H_{2,-s})$*-valued function of* $k=\alpha+i\beta$

$$e^{-i\alpha x_1} R_{0+}(\beta, k) e^{i\alpha x_1}$$

is analytic in $\mathbb{C}^+$ *and continuous in* $\overline{\mathbb{C}^+}\backslash\{0\}$.

Proof. By Lemma 1.2, $R_{0+}(\beta, k) := \underset{\varepsilon \downarrow 0}{w\text{-lim}}\, R_0(\beta, k+i\varepsilon)$ is well-defined as an operator in $\mathscr{B}(L^1_{2,s}, H_{2,-s})$. Note that for $u \in H_{2,-s}$ the norm $\|u\|_{2,-s}$ is equivalent to

$$\|u\| = \|u\|_{-s} + \left\| \left(-\Delta + 2\beta \frac{\partial}{\partial x_1} \right) u \right\|_{-s}$$

uniformly for $0 \leqq \beta \leqq \beta_0$, where we fix $\beta_0 > 0$ such that $K \subset \{k \mid 0 \leqq \operatorname{Im} k \leqq \beta_0\}$. Thus it suffices to show that in the uniform operator topology of $\mathscr{B}(L^1_{2,s}, L_{2,-s})$,

$$R_0(\beta, k+i\varepsilon) \underset{\varepsilon \downarrow 0}{\longrightarrow} R_{0+}(\beta, k)$$

and

$$\left(-\Delta + 2\beta \frac{\partial}{\partial x_1} \right) R_0(\beta, k+i\varepsilon) \underset{\varepsilon \downarrow 0}{\longrightarrow} \left(-\Delta + 2\beta \frac{\partial}{\partial x_1} \right) R_{0+}(\beta, k),$$

uniformly for $k \in K$.

Since

$$\left(-\Delta+2\beta\frac{\partial}{\partial x_1}\right)R_0(\beta,k+i\varepsilon)=1+(k+i\varepsilon)^2R_0(\beta,k+i\varepsilon),$$

it follows that it suffices to show that

$$R_0(\beta,k+i\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{}R_{0+}(\beta,k)$$

in the uniform operator topology of $\mathscr{B}(L^1_{2,s},L_{2,-s})$, uniformly for $k\in K$. We now proceed to show this.

First of all, the weak limit as $\varepsilon\downarrow 0$ is indeed a strong limit, which can be seen as follows. Let $g\in L^1_{2,s}$. Then $g\in L^1_{2,s'}$ for $\frac{1}{2}<s'<s$. By Lemma 1.2,

$$R_0(\beta,k+i\varepsilon)g\xrightarrow[\varepsilon\downarrow 0]{}R_{0+}(\beta,k)g,$$

weakly in $H^1_{2,-s'}$, uniformly for $k\in K$. Since $H^1_{2,s'}$ is compactly embedded in $L_{2,-s}$, we conclude that

$$R_0(\beta,k+i\varepsilon)g\xrightarrow[\varepsilon\downarrow 0]{}R_{0+}(\beta,k)g,$$

strongly in $L_{2,-s}$, uniformly for $k\in K$.

It remains to prove convergence in the uniform operator topology. This is proved along the same lines as the proof of the corresponding step in the proof of [1, Theorem 4.1], replacing $L_{2,s}$ by $L^1_{2,s}$ and $H_{2,-s'}$ by $H^1_{2,-s}$ and using the compactness of the embedding of $H^1_{2,-s'}$ in $L_{2,-s}$ for $\frac{1}{2}<s'<s$. This concludes the proof of the first part of the theorem.

For $\varepsilon>0$ the $\mathscr{B}(L^1_{2,s},H_{2,-s})$-valued function $e^{-i\alpha x_1}R_0(\beta,k+i\varepsilon)e^{i\alpha x_1}$ is analytic in $k\in\mathbb{C}^+$ and continuous in $k\in\overline{\mathbb{C}^+}\setminus\{0\}$.

It then follows from the first part of the theorem that $e^{-i\alpha x_1}R_{0+}(\beta,k)e^{i\alpha x_1}$ is analytic in $\mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+}\setminus\{0\}$ as a function of k with values in $\mathscr{B}(L^1_{2,s},H_{2,-s})$.

The theorem is proved.

Theorem 1.4. *The following limits exist in the uniform operator topology of $\mathscr{B}(L^1_{2,s},H_{2,-s})$ for $s>\frac{1}{2}$, uniformly for k in compact subsets of $\overline{\mathbb{C}^+}\setminus\Sigma$, where*

$$\Sigma=\{i\beta\,|\,-\beta^2\in\sigma_d(H)\}\cup\{0\}\cup\{\alpha\,|\,\alpha^2\in\sigma_p(H)\},$$

$$R_{1+}(\beta,k)=\lim_{\varepsilon\downarrow 0}R_1(\beta,k+i\varepsilon)=(1+R_{0+}(\beta,k)V)^{-1}R_{0+}(\beta,k).$$

Moreover, the $\mathscr{B}(L^1_{2,s},H_{2,-s})$-valued function $e^{-i\alpha x_1}R_{1+}(\beta,k)e^{i\alpha x_1}$ is meromorphic in $\mathbb{C}^+$ with poles at $\{i\beta\,|\,-\beta^2\in\sigma_d(H)\}$ and continuous in $\mathbb{C}^+\setminus\Sigma$.

Proof. By the 2$^{\text{nd}}$ resolvent equation,

$$R_1(\beta,k+i\varepsilon)=(1+R_0(\beta,k+i\varepsilon)V)^{-1}R_0(\beta,k+i\varepsilon).$$

Since $V\in\mathscr{C}(H_{2,-s},L_{2,s})$, by Theorem 1.3,

$$e^{-i\alpha x_1}R_0(\beta,k+i\varepsilon)Ve^{i\alpha x_1}\xrightarrow[\varepsilon\downarrow 0]{}e^{-i\alpha x_1}R_{0+}(\beta,k)e^{i\alpha x_1}:=Q_+(k)$$

in the uniform operator topology of $\mathscr{B}(H_{2,-s})$, uniformly for $k \in K$, and $Q_+(k)$ is an analytic, $\mathscr{C}(H_{2,-s})$-valued function of $k \in \mathbb{C}^+$ and continuous in $\overline{\mathbb{C}^+} \setminus \{0\}$. By Lemma A2,4, $1+Q_+(k)$ is invertible for $k \in \overline{\mathbb{C}^+} \setminus \Sigma$.

It follows from the analytic Fredholm theorem that $(1+Q_+(k))^{-1}$ is a meromorphic, $\mathscr{B}(H_{2,-s})$-valued function in $\mathbb{C}^+$ with poles at $\Sigma_i = \{i\beta \mid -\beta^2 \in \sigma_d(H)\}$, continuous in $\overline{\mathbb{C}^+} \setminus \Sigma$. Using again Theorem 1.3, we conclude the proof.

Theorem 1.5. *Set for* $k \in \overline{\mathbb{C}^+} \setminus \Sigma$,

$$\Phi(k, \omega, x) = e^{-ik\omega \cdot x} R_{1+}(k) V e^{ik\omega \cdot x}.$$

Then $\Phi(k, \omega, \cdot)$ *is for fixed* $\omega \in S^2$ *a continuous,* $H_{2,-s}$*-valued function of* $k \in \overline{\mathbb{C}^+} \setminus \Sigma$, *meromorphic in* $\mathbb{C}^+$ *with poles at most at* Σ_i.

For fixed $x \in \mathbb{R}^3$, $\omega \in S^2$, $\Phi(k, \omega, x)$ *is continuous in* $\overline{\mathbb{C}^+} \setminus \Sigma$ *and meromorphic in* $\mathbb{C}^+$ *with poles at most at* Σ_i.

Proof. Since $V \in L^1_{2,s}$ and

$$\Phi(k, \omega, x) = T_\omega e^{-i\alpha x_1} R_{1+}(\beta, k) e^{i\alpha x_1} T_\omega^{-1} V,$$

the first part of the theorem is an immediate consequence of Theorem 1.4.

It follows that for fixed $\omega \in S^2$, $f \in C_0(\mathbb{R}^3)$, the function

$$\int_{\mathbb{R}^3} \Phi(k, \omega, x) f(x)\, dx$$

is continuous in $\overline{\mathbb{C}^+} \setminus \Sigma$ and meromorphic in $\mathbb{C}^+$ with poles at most at Σ_i. Note that $\Phi(k, \omega, \cdot) \in H_{2,-s}$ implies that $\Phi(k, \omega, \cdot) \in C(\mathbb{R}^3)$. Let $\Gamma \subset \mathbb{C}^+$ be a closed curve with $\Gamma \cap \Sigma_i = \emptyset$ and containing no points of Σ_i in its interior. Then, by Fubini's theorem

$$\int_{\mathbb{R}^3} \left\{ \int_\Gamma \Phi(k, \omega, x)\, dk \right\} f(x)\, dx = \int_\Gamma \left\{ \int_{\mathbb{R}^3} \Phi(k, \omega, x) f(x)\, dx \right\} dk = 0.$$

This implies that $\int_\Gamma \Phi(k, \omega, x)\, dk = 0$ for fixed $\omega \in S^2$, $x \in \mathbb{R}^3$, and hence $\Phi(k, \omega, x)$ is analytic in $\mathbb{C}^+ \setminus \Sigma_i$. It follows from a similar argument with Γ containing a point of Σ_i that such points are at most poles of $\Phi(k, \omega, x)$.

The continuity of $\Phi(k, \omega, x)$ is proved as follows. For fixed $\omega \in S^2$, the function $(1+|x|^2)^{-s/2} \Phi(k, \omega, \cdot)$ is continuous in $k \in \overline{\mathbb{C}^+} \setminus \Sigma$ with values in H_2. Since H_2 is continuously embedded in $C(\mathbb{R}^3)$ (with the sup-norm), $(1+|x|^2)^{-s/2} \Phi(k, \omega, \cdot)$ is continuous in $k \in \overline{\mathbb{C}^+} \setminus \Sigma$ with values in $C(\mathbb{R}^3)$. Hence for every $x \in \mathbb{R}^3$, $\omega \in S^2$, the function $\Phi(k, \omega, x)$ is continuous in $k \in \overline{\mathbb{C}^+} \setminus \Sigma$.

Definition 1.6. For $k \in \overline{\mathbb{C}^+} \setminus \Sigma$, $\omega \in S^2$, $x \in \mathbb{R}^3$,

$$\Psi(k, \omega, x) = e^{ik\omega \cdot x}(1 - \Phi(k, \omega, x)).$$

For $k > 0$, $k^2 \notin \sigma_p(H)$, $\omega \in S^2$, $x \in \mathbb{R}^3$

$$\Psi_+(k, \omega, x) = \Psi(-k, -\omega, x), \qquad \Psi_-(k, \omega, x) = \Psi(k, \omega, x).$$

2. Analytic Continuation

Under our assumptions on V the scattering operator S exists and is unitary on L_2. The generalized Fourier transforms $F_\pm$ are partial isometries with initial space

$\mathscr{H}_{ac}(H_1)$ and final space L_2, defined for $f\in C_0(\mathbb{R}^3)$, $k>0$, $\omega\in S^2$ by

$$(F_\pm f)(k,\omega)=(2\pi)^{3/2}\int_{\mathbb{R}^3} f(x)\,\bar{\Psi}_\pm(k,\omega,x)\,dx\,.$$

Letting $\mathscr{F}$ denote the Fourier-Plancherel transform and setting $\hat{S}=\mathscr{F}S\mathscr{F}^{-1}$, we have $\hat{S}F_-=F_+$, hence for $f\in C_0(\mathbb{R}^3)$,

$$\hat{S}\left[\int_{\mathbb{R}^3} f(x)\,\bar{\Psi}_-(k,\omega,x)\,dx\right]=\int_{\mathbb{R}^3} f(x)\,\bar{\Psi}_+(k,\omega,x)\,dx\,.$$

Moreover, $SH_0=H_0S$, so $\hat{S}$ is diagonalized in momentum representation, $\hat{S}=\int_0^\infty \oplus S(k)\,dk$, where for $k>0$, $f\in C_0(\mathbb{R}^3)$,

$$S(k)\int_{\mathbb{R}^3} f(x)\,\bar{\Psi}_-(k,\cdot,x)\,dx=\int_{\mathbb{R}^3} f(x)\,\bar{\Psi}_+(k,\cdot,x)\,dx\,. \tag{2.1}$$

Since $\bar{\Psi}_\pm(k,\cdot,\cdot)\in C(\mathbb{R}^3,h)$ and $S(k)$ is unitary on h, also $S(k)\,\bar{\Psi}_-(k,\cdot,\cdot)\in C(\mathbb{R}^3,h)$. Hence, (2.1) for all $f\in C_0(\mathbb{R}^3)$ implies for $k>0$, $x\in\mathbb{R}^3$,

$$S(k)\,\bar{\Psi}_-(k,\cdot,x)=\bar{\Psi}_+(k,\cdot,x)\,. \tag{2.2}$$

Using (1.1), the above result (2.2) may be expressed in terms of $\Psi(k,\omega,x)$ as follows.

Lemma 2.1. *The scattering matrix $S(k)$ satisfies $S(k)\,\Psi(-k,\cdot,x)=\Psi(k,-\cdot,x)$, $x\in\mathbb{R}^3$, $k>0$.*

Definition. Let v be a real-valued, measurable function satisfying (i) and (ii). Let $\mathscr{O}\subset\mathbb{C}^-$ be a domain with $\partial\mathscr{O}\cap\mathbb{R}^+=I$. The operator V of multiplication by v is called $\mathscr{O}$-analytic, if the scattering matrix $S(k)$ has a continuous extension $\tilde{S}(k)$ from I to $\mathscr{O}\cup I$, such that $\tilde{S}(k)$ is analytic in $\mathscr{O}$. For any set $S\subset\mathbb{C}$ we let $S_b=\{k\in S\,|\,|\mathrm{Im}\,k|<b\}$.

Theorem 2.2. *Assume that V is $\mathscr{O}$-analytic.*

1) *For fixed $x\in\mathbb{R}^3$, the function $\Psi(k,\cdot,x)$ has an $\mathfrak{H}$-valued meromorphic continuation $\tilde{\Psi}(k,\cdot,x)$ from $\mathbb{C}^+$ to $\mathscr{O}\cup I$ with poles contained in $\Sigma\cup\bar{\Sigma}$, defined for $k\in(\mathscr{O}\cup I)\backslash\bar{\Sigma}$, $x\in\mathbb{R}^3$, by*

$$\tilde{\Psi}(k,\cdot,x)=R\tilde{S}(k)\,\Psi(-k,\cdot,x)\,, \tag{2.3}$$

where R is the reflection operator in $\mathfrak{H}$, $(R\sigma)(\omega)=\sigma(-\omega)$.

2) *$\tilde{\Psi}(k,\cdot,\cdot)\in\mathscr{H}_{2,\mathfrak{H}}^{-b}$ for $|\mathrm{Im}\,k|<b$, and $\tilde{\Psi}(k,\cdot,\cdot)$ is a meromorphic, $\mathscr{H}_{2,\mathfrak{H}}^{-b}$-valued function of k in $(\mathbb{C}^+\cup I\cup\mathscr{O})_b$ with poles contained in $\Sigma\cup\bar{\Sigma}$.*

Proof. 1) Define $\tilde{\Psi}(k,\cdot,x)$ for $x\in\mathbb{R}^3$, $k\in(\mathscr{O}\cup I)\backslash\bar{\Sigma}$ by (2.3). For $k\in I\backslash\bar{\Sigma}$, by Lemma 2.1,

$$\tilde{\Psi}(k,\cdot,x)=\Psi(k,\cdot,x)\,.$$

By Theorem 1.5 and Definition 1.6, $\tilde{\Psi}(k,\cdot,x)$ is continuous on $(\mathscr{O}\cup I)\backslash\bar{\Sigma}$ and meromorphic in $\mathscr{O}$ with poles contained in $\bar{\Sigma}_i$. Using again Theorem 1.5, we conclude that for $x\in\mathbb{R}^3$, $\tilde{\Psi}(k,\cdot,x)$ is meromorphic in $\mathbb{C}^+\cup I'\cup\mathscr{O}$, where $I'=I\backslash\{\alpha\,|\,\alpha^2\in\sigma_p(H)\}$, with poles at most at $\Sigma_i\cup\bar{\Sigma}_i$. The fact that $\{\alpha\,|\,\alpha^2\in\sigma_p(H)\}$ are (simple) poles of $\tilde{\Psi}(k,\cdot,x)$ can be proved using the existence of $\lim_{z\to\alpha}(z-\alpha)R_1(z)$. This concludes the proof of 1).

2) By Theorem 1.5, for $|\alpha|\leqq 2$,

$$\int_{\mathbb{R}^3} \|D^\alpha\tilde{\Psi}(k,\cdot,x)\|_{\mathfrak{H}}^2 e^{-2b|x|}\,dx \leqq \|\tilde{S}(k)\|^2 \int_{\mathbb{R}^3} \|D^\alpha\Psi(-k,\cdot,x)\|_{\mathfrak{H}}^2\,dx\,.$$

The analyticity properties of $\tilde{\Psi}(k,\cdot,\cdot)$ as an $\mathscr{H}_{2,\mathfrak{H}}^{-b}$-valued function follows from local boundedness and weak analyticity on a dense set of $D^\alpha\tilde{\Psi}(k,\cdot,\cdot)$ for $|\alpha|\leqq 2$, and 2) is proved.

Lemma 2.3. *Assume that V is $\mathcal{O}$-analytic. The trace operator $T(k)$ defined for $k>0$ by*

$$T(k)f=(F_+f)(k,\cdot)$$

has a $\mathscr{B}(\mathscr{H}^b,\mathfrak{H})$-valued, meromorphic extension $\tilde{T}(k)$ from I to $(\mathbb{C}^+\cup I\cup\mathcal{O})_b$ with poles at most at $\Sigma\cup\bar{\Sigma}$, given by

$$\tilde{T}(k)f=(2\pi)^{-3/2}R\int_{\mathbb{R}^3} f(x)\,\tilde{\Psi}(k,\cdot,x)\,dx\,. \tag{2.4}$$

Moreover, $\tilde{T}^*(\bar{k})$, given by

$$(\tilde{T}^*(\bar{k})\sigma)(x)=(2\pi)^{-3/2}\int_{S^2}\sigma(\omega)\,\tilde{\Psi}(\bar{k},-\omega,x)\,d\omega \tag{2.5}$$

is in $\mathscr{B}(\mathfrak{H},\mathscr{H}_{2,\mathfrak{H}}^{-b})$.

Proof. It follows from Theorem 2.2, 2) that for $f\in\mathscr{H}^b$, $\tilde{T}(k)f$ as given by (2.4) is a meromorphic, $\mathfrak{H}$-valued function of $k\in(\mathbb{C}^+\cup I\cup\mathcal{O})_b$ with poles contained in $\Sigma\cup\bar{\Sigma}$. From the boundedness of $\tilde{\Psi}(k,\cdot,\cdot)\|_{\mathscr{H}_{\mathfrak{H}}^{-b}}$ on compact sets follows that $\|\tilde{T}(k)\|_{\mathscr{B}(\mathscr{H}^b,\mathfrak{H})}$ is locally bounded and hence $\tilde{T}(k)$ is a meromorphic $\mathscr{B}(\mathscr{H}^b,\mathfrak{H})$-valued function on $(\mathbb{C}^+\cup I\cup\mathcal{O})_b$. A simple calculation yields (2.5), and it follows from Theorem 1.5 that $\tilde{T}^*(\bar{k})\in\mathscr{B}(\mathfrak{H},\mathscr{H}_{2,\mathfrak{H}}^{-b})$. The lemma is proved.

Theorem 2.4. *Assume that V is $\mathcal{O}$-analytic. The resolvent $R_1(k)$ has a meromorphic, $\mathscr{B}(\mathscr{H}^b,\mathscr{H}_2^{-b})$-valued continuation $\tilde{R}_1(k)$ from $\mathbb{C}^+$ to $I\cup\mathcal{O}_b$ with poles contained in $\bar{\Sigma}$.*

Proof. We use the well-known identity, valid for $k>0$, $k^2\notin\sigma_p(H)$,

$$R_{1+}(k)=R_{1+}(-k)+\pi ikT^*(k)\,T(k)\,. \tag{2.6}$$

Using Lemma 2.3, we define the $\mathscr{B}(\mathscr{H}^b,\mathscr{H}_2^{-b})$-valued function $\tilde{R}_1(k)$ for $k\in\mathcal{O}_b\backslash\bar{\Sigma}$ by

$$\tilde{R}_1(k)=R_1(-k)+\pi ik\tilde{T}^*(\bar{k})\,\tilde{T}(k)\,. \tag{2.7}$$

Setting $\tilde{R}_1(k)=R_{1+}(k)$ for $k\in I\backslash\Sigma$, we obtain the theorem from (2.6) and Lemma 2.3.

Theorem 2.5. *Assume that V is $\mathcal{O}$-analytic. Let W be a symmetric operator of the form*

$$W=e^{-b|x|}Qe^{-b|x|},\ Q\in\mathscr{C}(H_2(\mathbb{R}^3),L_2(\mathbb{R}^3))\,.$$

Let H_2 be the selfadjoint operator on $\mathscr{D}_{H_2}=\mathscr{D}_{H_1}=\mathscr{D}_{H_0}$ defined by

$$H_2=H_1+W=H_0+V+W,$$

and

$$R_2(k)=(H_2-k^2)^{-1} \quad \textit{for} \quad k^2\in\varrho(H_2).$$

Let $S_{12}(k)$ be the scattering matrix of the pair (H_1,H_2) associated with the spectral representation of $H_{1,\mathrm{ac}}$ defined by $T(k)$ (cf. [8]).

1) *$R_2(k)$ has a $\mathscr{B}(\mathscr{H}^b,\mathscr{H}_2^{-b})$-valued, meromorphic continuation $\tilde{R}_2(k)$ from $\mathbb{C}^+$ to $(I\cup\mathcal{O}_b)\backslash\bar{\Sigma}$, given by*

$$\tilde{R}_2(k)=\tilde{R}_1(k)(1+W\tilde{R}_1(k))^{-1}. \tag{2.8}$$

2) *$S_{12}(k)$ has a meromorphic extension $\tilde{S}_{12}(k)$ from I to $\mathcal{O}_b\backslash\bar{\Sigma}_i$ with the same poles as $\tilde{R}_2(k)$, given by*

$$\tilde{S}_{12}(k)=1-\pi ik\,\tilde{T}(k)(W-W\tilde{R}_2(k)W)\,\tilde{T}^*(\bar{k}). \tag{2.9}$$

Proof. 1) By Theorem 2.4, $WR_1(k)$ has a $\mathscr{C}(\mathscr{H}^b)$-valued analytic continuation $W\tilde{R}_1(k)$ from $\mathbb{C}^+$ to $(I\cup\mathcal{O}_b)\backslash\bar{\Sigma}$. By the analytic Fredholm theorem, $(1-W\tilde{R}_1(k))^{-1}$ is meromorphic in $(I\cup\mathcal{O}_b)\backslash\bar{\Sigma}$. Using the 2nd resolvent equation, we obtain 1).

2) Using the representation of $H_{1,\mathrm{ac}}$ as k^2 on $L_2(\mathbb{R}^+,\mathfrak{H};k^2\,dk)$ defined by $T(k)$, the scattering matrix $S_{12}(k)$ is given for $k>0$ (cf. [8]) by

$$S_{12}(k)=1-\pi ik\,T(k)(W-WR_{2+}(k)W)\,T^*(k). \tag{2.10}$$

It follows from (2.10), 1) and Lemma 2.3, that $S_{12}(k)$ has a meromorphic extension $\tilde{S}_{12}(k)$ to $\mathcal{O}_b\backslash\bar{\Sigma}$ with poles at most at the poles of $\tilde{R}_2(k)$. The fact that the poles of $\tilde{S}_{12}(k)$ and $\tilde{R}_2(k)$ coincide follows from the next lemma.

Lemma 2.6. *For $k\in\mathcal{O}_b\backslash\bar{\Sigma}_i$, $\mathcal{N}(\tilde{S}_{12}^{-1}(k))$ and $\mathcal{N}(1+W\tilde{R}_1(k))$ are isomorphic via the maps*

$$\mathcal{N}(1+W\tilde{R}_1(k))\ni\Omega\to\sigma=\tilde{T}(k)\Omega\in\mathcal{N}(\tilde{S}_{12}^{-1}(k))$$

with the inverse $Z(k)$ defined by

$$\Omega=Z(k)\sigma=-\pi ik(1-WR_2(-k))\,W\tilde{T}^*(\bar{k})\sigma.$$

Proof. 1) Let $\Omega\in\mathcal{N}(1+W\tilde{R}_1(k)$. Then $\sigma\neq 0$, since otherwise by (2.7) $\Omega\in\mathcal{N}(1+WR_1(-k))$ implying $R_1(-k)\Omega\in\mathcal{N}(H_2-k^2)$, a contradiction. Using the expression for $\tilde{S}_{12}^{-1}$ obtained from (2.8) on replacing i by $-i$ and $\tilde{R}_2(k)$ by $R_2(-k)$, we get in view of (2.7),

$$\begin{aligned}\tilde{S}_{12}^{-1}(k)\sigma&=(1+\pi ik\,\tilde{T}(k)(1-WR_2(-k))\,W\tilde{T}^*(\bar{k})\,\tilde{T}(k)\Omega\\&=\tilde{T}(k)\Omega+\tilde{T}(k)(1-WR_2(-k))\\&\quad\times[(1+W\tilde{R}_1(k))-(1+WR_1(-k))]\,\Omega=0.\end{aligned}$$

2) Assume that $\sigma\in\mathcal{N}(\tilde{S}_{12}^{-1}(k))$, i.e.,

$$\sigma-\tilde{T}(k)\,Z(k)\sigma=0.$$

Applying $Z(k)$, setting $\Omega=Z(k)\sigma$ and using (2.7), we get

$$\begin{aligned}\Omega-Z(k)\,\tilde{T}(k)\Omega&=\Omega+(1-WR_2(-k))\,W[\tilde{R}_1(k)-R_1(-k)]\,\Omega\\&=(1+W\tilde{R}_1(k))\Omega=0.\end{aligned}$$

The lemma follows from 1) and 2).

We finally investigate the analyticity properties of the scattering matrix $S_2(k)$ of the pair (H_0, H_2).

Theorem 2.7. *Under the assumptions of Theorem 2.5 the scattering matrix $S_2(k)$ has a meromorphic extension $\tilde{S}_2(k)$ from I to $\mathcal{O}_b \setminus \bar{\Sigma}_i$ with poles at most at the poles of $\tilde{R}_2(k)$.*

Proof. This follows from Theorem 2.5 and the following identity, valid for $k>0$,

$$S_{12}(k)\, S_1(k) = S_2(k), \tag{2.11}$$

which we shall now establish.

Using for $H_{1,\mathrm{ac}}$ the representation as k^2 on $L_2(\mathbb{R}^+, \mathfrak{H}; k^2\, dk)$ defined by $T(k)$, the generalized Fourier transforms $F_{12\pm}$ of the pair $(H_{1,\mathrm{ac}}, H_{2,\mathrm{ac}})$ are given (cf. [8]) for $f \in L_{2,s}$ and $k>0$, $k^2 \notin \sigma_p(H_1 \cup \sigma_p(H_2))$, by

$$\begin{aligned}(F_{12+} f)(k) &= T(k)(1 - WR_{2+}(k)) f = T_0(k)(1 - VR_{1+}(k))(1 - WR_{2+}(k)) f \\ &= T_0(k)(1 - (V+W) R_{2+}(k)) f = (F_{2+} f)(k),\end{aligned} \tag{2.12}$$

where

$$T_0(k) f = (\mathscr{F} f)(k, \cdot),$$

and

$$\begin{aligned}(F_{12-} f)(k) &= T(k)(1 - WR_{2+}(-k)) f = (F_{1+}(1 - WR_{2+}(-k)) f)(k) \\ &= S_1(k)(F_{1-}(1 - WR_{2+}(-k)) f)(k) \\ &= S_1(k)\, T_0(k)(1 - VR_{1+}(-k))(1 - WR_{2+}(-k)) f \\ &= S_1(k)\, T_0(k)(1 - (V+W) R_{2+}(-k)) f = S_1(k)(F_{2-} f)(k).\end{aligned} \tag{2.13}$$

By (2.12) and (2.13), for $f \in L_{2,s}$, $k>0$, $k^2 \notin \sigma_p(H_1) \cup \sigma_p(H_2)$,

$$\begin{aligned}S_2(k)(F_{2-} f)(k) &= (F_{2+} f)(k) = (F_{12+} f)(k) \\ &= S_{12}(k)(F_{12-} f)(k) = S_{12}(k)\, S_1(k)(F_{2-} f)(k).\end{aligned} \tag{2.14}$$

From (2.14) follows (2.11) for $k>0$, $k^2 \notin \sigma_p(H_1) \cup \sigma_p(H_2)$. Since $S_{12}(k)$, $S_1(k)$, and $S_2(k)$ are continuous on $\mathbb{R}^+$ (cf. [7]) and $\sigma_p(H_1)$ and $\sigma_p(H_2)$ are discrete sets in $\mathbb{R}^+$, this implies (2.11) for all $k>0$, and the proof is complete.

Appendix 1

Lemma A 1. *Let f and g be functions in L_2 with compact support, and let $\alpha_0 \in \mathbb{R} \setminus \{0\}$. Then there exists $0 < \delta_0 < |\alpha_0|$ such that the following limits exist, uniformly in $\{k = \alpha + i\beta \mid |\alpha - \alpha_0| \leqq \delta_0,\ 0 \leqq \beta \leqq \delta_0\}$,*

$$\lim_{\varepsilon \downarrow 0} (f, R_0(\beta, k + i\varepsilon) g).$$

Proof. Taking Fourier transforms, we have

$$I = (f, R_0(\beta, k + i\varepsilon) g) = \int_{\mathbb{R}^3} \hat{f}(\xi)\, \bar{\hat{g}}(\xi)\, (\xi^2 - \alpha^2 + 2\varepsilon\beta + \varepsilon^2 + 2i(\beta\xi_1 - \beta\alpha - \varepsilon\alpha))^{-1}\, d\xi.$$

Since $\hat{f}(\xi)$ and $\bar{\hat{g}}(\bar{\xi})$ are entire analytic functions of ξ_1, ξ_2, ξ_3, we can deform the manifold of integration in $\mathbb{C}^3$ as follows.

Let $\alpha_0>0$ ($\alpha_0<0$ is analogous). It suffices to consider $\beta=0$.

We write the integral in spherical coordinates as follows.

$$I=I(\varepsilon)=\int_{S^2} \sin\theta\, d\theta\, d\varphi \int_{\mathbb{R}^+} dr \hat{f}(r,\theta,\varphi)\,\bar{\hat{g}}(r,\theta,\varphi)\,(r^2-\alpha^2+\varepsilon^2-2i\varepsilon\alpha)^{-1}.$$

Deforming the radial integration path $\mathbb{R}^+$ into the curve Γ indicated on Fig. 1, where $0<\delta<\alpha$, we get

$$I(\varepsilon)=\int_{S^2} \sin\theta\, d\theta\, d\varphi \int_{\Gamma} dz \hat{f}(z,\theta,\varphi)\,\bar{\hat{g}}(\bar{z},\theta,\varphi)$$
$$\times (z^2-\alpha^2+2\varepsilon\beta+\varepsilon^2+2i(\beta z\sin\theta\cos\varphi-\beta\alpha-\varepsilon\alpha)^{-1},$$

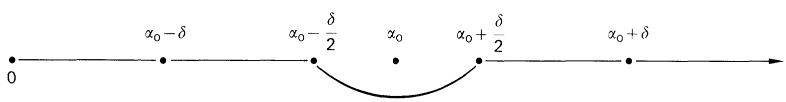

Fig. 1. This yields $I(\varepsilon)\xrightarrow[\varepsilon\downarrow 0]{} \int_{S^2} \sin\theta\, d\theta\, d\varphi \int_{\Gamma} dz\hat{f}(z,\theta,\varphi)\,\bar{\hat{g}}(\bar{z},\theta,\varphi)(z^2-\alpha^2)^{-1}$, uniformly for $|\alpha-\alpha_0| \leq \delta/4$

Appendix 2

Lemma A2.1. *Let $b_0>0$ be fixed. For $|\beta|<\beta_0$ and $\mathbb{C}^{b_0}=\{a+ib\,|\,b>b_0\}$, we have*

$$\sigma_d(H(i\beta))\cap\{z^2\,|\,z\in\mathbb{C}^{b_0}\}=\{-b^2\in\sigma_d(H)\,|\,b>b_0\}.$$

Proof. We set

$$H(z)=e^{-iz}He^{iz}=-\Delta-2iz\frac{\partial}{\partial x_1}+z^2+V.$$

For β fixed

$$H(\alpha+i\beta)=e^{-i\alpha x_1}H(i\beta)e^{i\alpha x_1}.$$

The operators $H(z)$ form an entire self-adjoint, analytic family of operators of type A. For fixed β the operators $H(\alpha+i\beta)$ are unitarily equivalent. The essential spectrum $\sigma_e(z)$ of $H(z)$ is the parabolic region $\{\zeta^2|\mathrm{Im}\,\zeta|\leq|\beta|\}$ (for $\beta=0$ coinciding with $\overline{\mathbb{R}^+}$). Thus, $\mathbb{C}^{b_0}\cap\sigma_e(z)=\emptyset$ for $|\beta|<b_0$. A discrete eigenvalue λ of $H(\alpha'+i\beta')$, $|\beta'|<b_0$, remains a discrete eigenvalue of $H(\alpha+i\beta')$ for all $\alpha\in\mathbb{R}$ and hence, by analyticity of $H(z)$, for all $\alpha\in\mathbb{R}$, $|\beta|<b_0$.

The lemma follows.

Lemma A2.2. *Let $\beta>0$ be fixed and let $k=a+ib$, $a\in\mathbb{R}$, $b>\beta$. Then the equation*

$$\phi+R_0(\beta,k)V\phi=0 \qquad (1)$$

has a solution $\phi\in H_{2,-s}$, $\phi\neq 0$, if and only if $a=0$ and $k^2=-b^2\in\sigma_d(H(\beta))$ with

$$(H(i\beta)+b^2)\phi=0. \qquad (2)$$

Proof. 1) If $\phi \in H_2$ satisfies (2), then applying $R_0(\beta, ib) \in \mathscr{B}(L_2, H_2)$ to (2), we get, since $V\phi \in L_2$, (1) where $\phi \in H_{2,-s}$.

2) Let $\phi \in H_{2,-s}$ and assume (1). Then $V\phi \in L_{2,s} \subset L_2$ and hence $\phi = -R_0(\beta, k) V\phi \in H_2$. Applying $H_0(\beta) - k^2$ to (1), we get (2) and hence by Lemma A2.1, $a=0$ and $k^2 = -b^2 \in \sigma_d(H(i\beta)$.

Lemma A2.3. *For* $\beta > 0$, $k = \alpha + i\beta$,

$$R_{0+}(\beta, k) = \lim_{\beta' \uparrow \beta} \left(-\Delta - 2\beta' \frac{\partial}{\partial x_1} - \beta'^2 - k^2 \right)^{-1}.$$

Proof. This follows from the fact, proved in Theorem 1.3, that

$$\| R_0(\beta, k + i\varepsilon) - R_{0+}(\beta, k) \| \mathbb{B}(L^1_{2,s}, H_{2,-s}) \xrightarrow[\varepsilon \to 0]{} 0,$$

uniformly on compact sets, together with the norm-continuity of $R_{0+}(\beta, k)$.

Lemma A2.4.

$$\{k = \alpha + i\beta \in \mathbb{C}^+ \mid \mathcal{N}(1 + R_{0+}(\beta, z^2) V) \neq \{0\}\} = \{i\beta \mid -\beta^2 \in \sigma_d(H)\}.$$

Proof. Fix $k = \alpha + i\beta$, $\alpha \in \mathbb{R}$, $\beta > 0$. By Lemma A2.1, for $0 \leqq \beta' < \beta$,

$$\sigma_d(H(i\beta')) \cap \{k^2 \mid \operatorname{Im} k \geqq \beta\} = \{-b^2 \in \sigma_d(H) \mid b \geqq \beta\}.$$

Hence, by Lemma A2.2, there exists a circle C with center -1, separating -1 from the rest of the spectrum of the operator $R_0(\beta', k) V \in \mathscr{C}(H_{2,-s})$ for $\alpha \in \mathbb{R}$, $0 \leqq \beta' < \beta$.

Let

$$P(\beta', k) = -\frac{1}{2\pi i} \int_C (-\lambda + R_0(\beta', k) V)^{-1} d\lambda.$$

By Lemma A2.3

$$\lim_{\beta' \uparrow \beta} R_0(\beta', k) V = R_{0+}(\beta, k) V$$

in the uniform operator topology of $\mathscr{B}(H_{2,-s})$.

It follows that

$$(-\lambda + R_0(\beta', k) V)^{-1} \xrightarrow[\beta' \uparrow \beta]{} (-\lambda + R_0(\beta, k) V)^{-1}$$

in the uniform operator topology of $\mathscr{B}(H_{2,-s})$, uniformly for $\lambda \in C$. Hence, in the same topology

$$P(\beta', k) \xrightarrow[\beta' \uparrow \beta]{} -\frac{i}{2\pi i} \int_C (-\lambda + R_{0+}(\beta, k) V)^{-1} d\lambda = P_+(\beta, k),$$

where $P_+(\beta, k)$ is a projection on the algebraic null space of $1 + R_{0+}(\beta, k) V$.

It follows that $P_+(\beta, k) \neq 0$ if and only if $P(\beta', k) \neq 0$ for all $\beta' < \beta$. By Lemma A2.1 this holds if and only if $\alpha = 0$ and $-\beta^2 \in \sigma_d(H)$, and the lemma is proved.

Acknowledgement. I want to thank Ira Herbst for a valuable discussion, suggesting the possibility of proving the estimates of Sect. 1 for fixed $\omega \in S^2$. I also thank Erik Skibsted for pointing out an error in the original proof of Lemma A 2.3 and indicating the proof given.

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Communicated by B. Simon

Received June 16, 1987