Supertori are Algebraic Curves*

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Abstract. Super Riemann surfaces of genus 1, with arbitrary spin structures, are shown to be the sets of zeroes of certain polynomial equations in projective superspace. We conjecture that the same is true for arbitrary genus. Properties of superelliptic functions and super theta functions are discussed. The boundary of the genus 1 super moduli space is determined.

1. Introduction

The application of methods from the theory of Riemann surfaces has lead to great progress in string theory [1–3], as physicists have benefited serendipitously from a century of development of this classical branch of mathematics. The theory of super Riemann surfaces (SRS's) should play a similar foundational role in superstring theory. Here, however, physicists have not found the necessary mathematics already developed, but have had to create the theory themselves along with its applications [4–8]. During the past two years, supersymmetric generalizations have been found for many aspects of Riemann surface theory. Such deep results as the representation of surfaces by Fuchsian groups and the structure of the Teichmüller space have been generalized, while some relatively trivial concepts such as the period matrix have resisted generalization.

A basic property of Riemann surfaces is that they are algebraic curves: any compact Riemann surface can be analytically embedded in a complex projective space as the locus of points whose coordinates satisfy some polynomial equations. This allows the study of Riemann surfaces by the techniques of algebraic geometry and is the key to deep connections between Riemann surfaces and number theory. The algebraic aspect of Riemann surfaces has appeared in string theory in the study of orbifolds [9], and is central to the description of fermions on a Riemann surface via the KP hierarchy of soliton equations [10–13]. Friedan and Shenker

have stressed that algebraic or number-theoretic methods may offer the only hope for performing exact nonperturbative calculations in string theory [14]. An understanding of the algebraic nature of super Riemann surfaces should have equal importance for superstring theory.

The polynomial equation representing a Riemann surface of genus 1 (torus) can be explicitly constructed using elliptic functions, leading to the designation "elliptic curves" for such surfaces, whereas the algebraic nature of surfaces of higher genus is established by a more abstract argument [15, 16]. In this paper we will define and construct "superelliptic functions," and use them to derive the equations embedding supertori in projective superspace. This will be done for each of the four possible spin structures. The equations obtained completely characterize the supertori as complex supermanifolds. The superconformal structure of the supertorus (the $2d$ supergravity geometry it represents) is not encoded in the embedding equations, but can be specified by additional algebraic data.

The present work is based on the classification of supertori obtained in [6, 7], and the study of superspace algebraic geometry in [17]. However, we will suppress the rigorous supermanifold theory employed there and use the intuitive superspace language of the physics literature. In particular, bosonic coordinates will be treated as simple complex variables rather than even Grassmann variables containing nilpotent terms. Proofs which cannot be carried out in the intuitive language will be deferred to the Appendix.

The algebraic description of a SRS facilitates the study of the limit of singular surfaces. As an application of our results we can completely describe the boundary points of the genus 1 super moduli space SM_1 which represent such singular surfaces. It is well known that the ordinary moduli space of tori can be compactified by adding a single point at infinity representing a singular torus pinched along one homology cycle. This point is the "compactification divisor" for genus 1. Understanding the structure of the compactification divisor for higher genera is directly relevant for proving that superstring amplitudes are finite and have the correct factorization properties [14]. Indeed, Cohn and Friedan have also determined the boundary of SM_1 by considering the factorization properties of the partition function in superconformal field theory at the divisor [18]. We will show that SM_1 is "compactified" by adding precisely three points at infinity. (When we speak of compactifying a superspace we mean compactification in the bosonic directions only. The superspace is still noncompact in the fermionic directions.) This may give an indication of the nature of the compactification divisor for higher genera, although genus 1 is an exceptional case in many respects, particularly in having an intrinsically distinguished spin structure.

In Sect. 2 we construct superelliptic functions on a supertorus having one of the three nontrivial (even) spin structures. This case is simpler than that of the trivial (odd) spin structure because of the absence of any fermionic supermoduli supplementing the usual modular parameter τ. We prove that the superelliptic functions constructed are complete in the sense that any meromorphic superfunction on the supertorus can be rationally expressed in terms of them. We obtain the polynomial equations connecting our superelliptic functions and show that they give a representation of the supertorus as an affine curve in $C^{2,2}$, complex superspace with two even and two odd dimensions. The modifications necessary to

describe the superconformal structure completely, and to obtain a projective embedding are discussed. Section 3 carries out the same analysis for the odd spin structure, which is characterized by a supermodulus δ in addition to τ. In Sect. 4 these results are applied to determine the global structure of SM_1. Two of the three even spin structures become degenerate at infinity, while the supermodulus describing the odd spin structure becomes irrelevant, so that the compactification divisor consists of precisely three points at infinity. Section 5 contains our conclusions and speculations about the extension of this work to higher genera.

2. Even Spin Structures

We begin by reviewing the results on uniformization of genus 1 SRS's obtained in [6, 7]. A supertorus is obtained as the quotient of the complex superplane $C^{1,1}$, with coordinates (z, θ), by a supergroup G of superconformal transformations of the form,

$$\tilde{z} = \frac{az+b}{cz+d} + \theta \frac{\gamma z + \delta}{(cz+d)^2},$$

$$\tilde{\theta} = \frac{\gamma z + \delta}{cz+d} + \frac{\theta}{cz+d}(1 + \tfrac{1}{2}\delta\gamma), \quad ad - bc = 1.$$

(2.1)

Because G is isomorphic to the fundamental group of a torus, it must be Abelian and have precisely two generators. Furthermore, it can be chosen to preserve the flat supergeometry on $C^{1,1}$ characterized by the zweibein

$$E^\theta = d\theta, \quad E^z = dz + \theta d\theta.$$

(2.2)

(We use the convention $\theta d\theta = -d\theta\theta$.) Without loss of generality, the generators can be chosen to be

$$\tilde{z} = z + 1, \quad \tilde{\theta} = \theta,$$

(2.3a)

and

$$\tilde{z} = z + \tau + \theta\delta, \quad \tilde{\theta} = \theta + \delta,$$

(2.3b)

for the odd spin structure, and

$$\tilde{z} = z + 1, \quad \tilde{\theta} = +\theta,$$

(2.4a)

and

$$\tilde{z} = z + \tau, \quad \tilde{\theta} = -\theta,$$

(2.4b)

for one of the even spin structures. The other two even spin structures are obtained by changing the signs in the transformations of θ from $+-$ to $-+$ or $--$. The modular parameters τ (bosonic) and δ (fermionic) are coordinates on ST_1, the genus 1 super Teichmüller space, although the moduli (τ, δ) and $(\tau, -\delta)$ describe the same point and are identified. The super moduli space SM_1 is the quotient of ST_1 by the modular supergroup whose action and fundamental domain will be described later.

We will now construct meromorphic functions on a supertorus with the even spin structure described by Eqs. (2.4). Such functions correspond to "superelliptic" functions on the covering space $C^{1,1}$, that is, to functions invariant under G. Clearly, $F(z, \theta) = A(z) + \theta B(z)$ is invariant under G if

$$A(z+1) = A(z) = A(z+\tau),$$
$$B(z+1) = B(z) = -B(z+\tau). \tag{2.5}$$

These conditions mean that $A(z)$ is an ordinary elliptic function [19] and $B(z)$ is a section of the appropriate spin bundle over the torus. The Weierstrass elliptic function

$$\wp(z) = z^{-2} + \sum_{(m,n)\neq(0,0)} [(z-m-n\tau)^{-2} - (m+n\tau)^{-2}], \tag{2.6}$$

and its derivative $\wp'(z)$ are examples of superelliptic functions with $B(z)=0$. A superelliptic function with $A(z)=0$ can be constructed as follows. The function $\wp(z) - e_r$, with $e_r = \wp(\omega_r)$, where $\omega_1 = 1/2$, $\omega_2 = \tau/2$, and $\omega_3 = (1+\tau)/2$, has a double pole at the origin and a double zero at $z=\omega_r$. Therefore it has a meromorphic square root $\wp_r(z)$, which can be expressed in terms of theta functions as [19],

$$\wp_r(z) = \sqrt{\wp(z) - e_r} = \frac{\Theta'(0;\tau)}{\Theta_r(0;\tau)} \frac{\Theta_r(z;\tau)}{\Theta(z;\tau)}. \tag{2.7}$$

(We use the capital Θ for theta functions throughout this paper to avoid confusion with the superspace coordinate θ.) The periodicity properties of the theta functions show that $\theta\wp_1(z)$ is superelliptic for the $+-$ spin structure under consideration. $\wp_2$ and $\wp_3$ are associated with the $-+$ and $--$ spin structures, respectively [20].

It is well known that any elliptic function is a rational function of $\wp(z)$ and $\wp'(z)$ [19]. Furthermore, if $S(z)$ is another section of the spin bundle, then the ratio $S(z)/\wp_1(z)$ is an elliptic function, so that $\theta S(z)$ is rationally expressed in terms of $\theta\wp_1(z)$, $\wp(z)$, and $\wp'(z)$. Therefore any superelliptic function is rationally expressed in terms of these three functions.

The Weierstrass $\wp$ function and its derivative satisfy the equation,

$$\wp'^2(z) = 4\wp^3(z) - g_2\wp(z) - g_3,$$
$$= 4(\wp - e_1)(\wp - e_2)(\wp - e_3), \tag{2.8}$$

where $g_2(\tau)$ and $g_3(\tau)$ are the standard modular forms of weights 4 and 6 respectively. One might suppose that the map

$$(z, \theta) \rightarrow [\wp(z), \wp'(z), \theta\wp_1(z)] = (x, y, \phi), \tag{2.9}$$

from the supertorus into the superspace $C^{2,1}$ with global coordinates (x, y, ϕ) would embed the supertorus as the set of points satisfying

$$y^2 = 4x^3 - g_2 x - g_3, \tag{2.10}$$

but this is wrong for two reasons. First, the map is undefined at points with $z=0$, where all the superelliptic functions have poles. This problem is not serious and will be solved below by passing to a projective superspace which contains points at infinity. The fatal objection is that $\wp_1$ vanishes at $z=1/2$, so the map is not an

embedding at such points. To solve this problem we must add more fermionic dimensions to the target space and construct the embedding from several sections of the spin bundle which do not vanish simultaneously.

Consider the map into $C^{2,2}$,

$$(z, \theta) \rightarrow [\wp(z), \wp'(z), \theta \wp_1(z), \theta \wp'_1(z)] = (x, y, \phi, \psi). \tag{2.11}$$

From $\wp_1^2 = \wp - e_1$ we have $2\wp_1 \wp'_1 = \wp'$ or, multiplying by $\wp_1$,

$$2(\wp - e_1)\wp'_1 = \wp' \wp_1, \tag{2.12}$$

or

$$2(x - e_1)\psi = y\phi. \tag{2.13}$$

Because $\wp_1$ has a *simple* zero at $z = 1/2$, $\wp'_1$ does not vanish there, so $\wp_1$ and $\wp'_1$ are two sections which do not vanish simultaneously. The map (2.11) embeds the supertorus (minus the points with $z = 0$) in $C^{2,2}$. The image of the embedding is almost characterized by the two equations (2.10, 2.13), except at the points $z = 1/2$, where (2.13) fails to put a constraint on ϕ and ψ because their coefficients both vanish. We can add another equation which does constrain ϕ and ψ at these points by multiplying (2.12) by $\wp'$ to get

$$2(\wp - e_1)\wp'\wp'_1 = (\wp')^2 \wp_1 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)\wp_1, \tag{2.14}$$

and cancelling the common factor $(\wp - e_1)$. The image is then described by the three

$$y^2 = 4x^3 - g_2 x - g_3, \quad 2(x - e_1)\psi = y\phi, \quad y\psi = 2(x - e_2)(x - e_3)\phi. \tag{2.15}$$

To include the points $z = 0$ which were mapped to infinity by (2.11), we must enlarge $C^{2,2}$ to a projective superspace $SP^{2,3}$. The definition of projective superspace $SP^{m,n}$ follows that of ordinary projective space [21, 22]. Begin with $C^{m+1,n}$, with coordinates $(z^\mu; \theta^\alpha)$, and delete the points where all of the z^μ vanish. Then identify $(z^\mu; \theta^\alpha)$ with $(kz^\mu; k\theta^\alpha)$ for every nonzero bosonic k. The resulting space is $SP^{m,n}$, and $(z^\mu; \theta^\alpha)$ are homogeneous coordinates on this space.

Now consider the map into $SP^{2,3}$ given in terms of homogeneous coordinates by

$$(z, \theta) \rightarrow [\wp(z), \wp'(z), 1; \theta \wp_1(z), \theta \wp'_1(z), \theta \wp_1(z)\wp(z)] = (x, y; 1; \phi, \psi, \zeta). \tag{2.16}$$

This embedding is well defined even at $z = 0$ since we can divide the homogeneous coordinates by the function $\wp'$ with the worst pole to obtain

$$(0, \theta) \rightarrow (0, 1, 0; 0, 0, -\theta/2). \tag{2.17}$$

In order to obtain this embedding it was necessary to add a third odd coordinate ζ obeying

$$\zeta = x\phi \tag{2.18}$$

in order to accommodate a section $\wp_1 \wp$ having a triple pole at $z = 0$ as $\wp'$ does. The equations (2.15, 2.18) give the representation of the supertorus as an algebraic curve. They are written as equations for the affine coordinates but can be converted into projective equations as usual by homogenization: write each

coordinate $x, y, \ldots$ as a ratio x/k, $y/k, \ldots$ and clear denominators in all equations [23].

Equations (2.15, 2.18) realize the supertorus as an algebraic subset of $SP^{2,3}$. The complex supermanifold structure of the supertorus is determined by that of $SP^{2,3}$, since meromorphic superfunctions on $SP^{2,3}$ restrict to meromorphic superfunctions on the supertorus. However, a super Riemann surface has more structure than just that of a complex supermanifold. It admits a superconformal structure, namely a set of charts in which the 1-forms $dz + \theta d\theta$ in each chart are proportional when charts overlap. (In other words, $dz + \theta d\theta$ spans a line subbundle of the cotangent bundle.) This genus 1 case is special in that $dz + \theta d\theta$ is actually a global 1-form. We know that the supertorus admits a superconformal structure, but this structure is not induced by the algebraic embedding in any obvious way. $SP^{2,3}$ itself does not have a superconformal structure; in fact the notion of a superconformal structure is undefined for supermanifolds of dimension other than $(1,1)$ [or $(1,N)$ for extended supersymmetry]. We can, however, supplement the embedding equations by additional information which will specify the superconformal structure. The additional information will be a rational, meromorphic 1-form on $SP^{2,3}$ which agrees with $dz + \theta d\theta$ on the supertorus itself. From

$$d\wp(z)/\wp'(z) = dz \qquad (2.19)$$

and

$$\theta\wp_1 d(\theta\wp_1) = \theta d\theta \wp_1^2 = \theta\delta\theta(\wp - e_1), \qquad (2.20)$$

we see that

$$dz + \theta d\theta = \frac{dx}{y} + \frac{\phi d\phi}{x - e_1} \qquad (2.21)$$

is one of infinitely many 1-forms which works.

3. The Odd Spin Structure

In this section we will construct superelliptic functions on the supertorus with the odd spin structure, and use them to obtain an algebraic embedding. This case is more interesting than that of the even spin structures because of the presence of the supermodulus δ.

A superelliptic function for the odd spin structure obeys

$$R(z + \tau + \theta\delta, \theta + \delta) = R(z, \theta) = R(z + 1, \theta). \qquad (3.1)$$

Such a function can be constructed from the Weierstrass function $\wp(z; \tau)$, where the dependence on the modular parameter has been shown explicitly, namely,

$$R(z, \theta) = \wp(z; \tau + \theta\delta) \qquad (3.2)$$

is superelliptic. Further, the covariant derivatives $D^n R$ are all superelliptic, where

$$D = \partial_\theta + \theta\partial_z. \qquad (3.3)$$

In terms of the Weierstrass $\wp$ function, the first few of these superelliptic functions are:

$$R(z, \theta) = \wp(z) + \theta \delta \dot{\wp}(z),$$

$$DR(z, \theta) = \delta \dot{\wp}(z) + \theta \wp'(z),$$

$$D^2 R(z, \theta) = \wp'(z) + \theta \delta \dot{\wp}'(z), \qquad (3.4)$$

$$D^3 R(z, \theta) = \delta \dot{\wp}'(z) + \theta \wp''(z),$$

where a dot indicates differentiation with respect to τ. Once again any superelliptic function can be rationally expressed in terms of R, DR, and $D^2 R$. The proof of this fact requires some rigorous superspace theory and is given in the Appendix.

We now define a map from the supertorus to $C^{2,2}$ by

$$(z, \theta) \rightarrow (R, D^2 R, DR, D^3 R) = (x, y, \phi, \psi). \qquad (3.5)$$

Points in the image of this map satisfy two independent polynomial equations, one even and one odd:

$$y^2 - 4x^3 + g_2 x + g_3 - 2\phi\psi = 0,$$

$$2y\psi - 12x^2\phi + g_2\phi + \delta\dot{g}_2 x + \delta\dot{g}_3 = 0. \qquad (3.6)$$

These equations can be verified by writing out their components using Eqs. (3.4); these components are the Weierstrass equation (2.8) and its derivatives with respect to z and τ. The equations may also be derived as the Weierstrass equation is normally derived, by writing out the Laurent expansions of the functions $D^n R$ and forming holomorphic combinations. It is an easy consequence of the completeness of the superelliptic functions that any holomorphic superelliptic function is of the form $a + \theta b$ with a and b constants such that $b\delta = 0$. This means that a combination of superelliptic functions whose Laurent series vanishes up through zeroth order in z is identically zero. As shown in the Appendix, Eqs. (3.6) give an embedding of the supertorus, minus the points $z = 0$, in $C^{2,2}$.

To obtain a projective embedding of the entire supertorus, it is necessary to add extra coordinates so that the orders of the highest poles involved in the even and odd embedding functions agree. Here this can be achieved by adding another even coordinate u to obtain the embedding in $SP^{3,2}$,

$$(z, \theta) \rightarrow (R, D^2 R, D^4 R, 1; DR, D^3 R) = (x, y, u, 1; \phi, \psi). \qquad (3.7)$$

Since $D^4 R$ and $D^3 R$ both have fourth order poles at $z = 0$, we can divide the homogeneous coordinates by $D^4 R$ and obtain

$$(0, \theta) \rightarrow (0, 0, 1, 0; 0, \theta), \qquad (3.8)$$

which is nonsingular. The equation satisfied by u is

$$2yu = 12yx^2 - g_2 y + \delta\dot{g}_2 \phi. \qquad (3.9)$$

This equation determines u except at points where $y = 0$. As in the discussion surrounding Eq. (2.13), it is possible to eliminate y between Eqs. (3.9) and (3.6), obtaining another equation which does determine u at the points $y = 0$.

Finally, the superconformal structure is defined by a rational 1-form on $C^{2,2}$ which agrees with $dz + \theta d\theta$ on the supertorus. This can be found by computing the differentials of the functions R and DR using

$$d = (dz + \theta d\theta)\partial_z + d\theta D \tag{3.10}$$

and solving the resulting equations for $dz + \theta d\theta$. The result is

$$dz + \theta d\theta = \frac{y\,dx + \phi\,d\phi}{y^2 + \phi\psi}. \tag{3.11}$$

Because this 1-form has no explicit dependence on τ or δ, all the dependence of the SRS structure on these parameters is captured in Eqs. (3.6, 3.9).

As a check on these results, note that replacing δ by $-\delta$ in Eqs. (3.6, 3.9) is equivalent to a change of coordinates $\phi \to -\phi$, $\psi \to -\psi$ which preserves Eq. (3.11). This confirms that both signs of δ describe the same supertorus. In fact, δ and $i\delta$ also describe the same supertorus, because of the invariance of Eqs. (3.6, 3.9) under

$$\delta \to i\delta, \quad \phi \to i\phi, \quad \psi \to -i\psi, \quad y \to -y. \tag{3.12}$$

This symmetry generalizes the hyperelliptic involution $y \to -y$ of the ordinary torus. The fact that the form (3.11) changes sign under the symmetry does not matter, because the superconformal structure is defined by the bundle it generates rather than the form itself.

4. The Super Moduli Space of Supertori

Recall the relationship between Teichmüller space and moduli space for ordinary tori. The Teichmüller space T_1 is the upper half plane with coordinate τ. The moduli space M_1 is the quotient of T_1 by the modular group. The action of this group is generated by the two transformations,

$$S: \tau \to \tau + 1;$$
$$T: \tau \to -1/\tau, \tag{4.1}$$

and a fundamental domain is shown in Fig. 1. The transformation S identifies the vertical sides of the fundamental domain, while T makes identifications on the
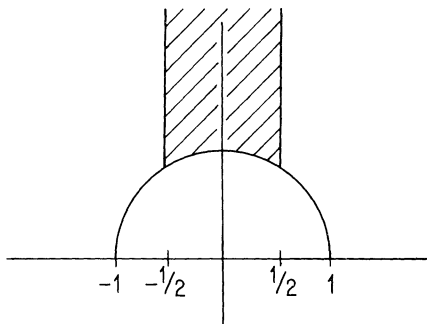

Fig. 1. The standard fundamental domain for the action of the modular group on the upper half plane is the region $|\tau| \geq 1$, $-1/2 \leq \mathrm{Re}\,\tau \leq 1/2$
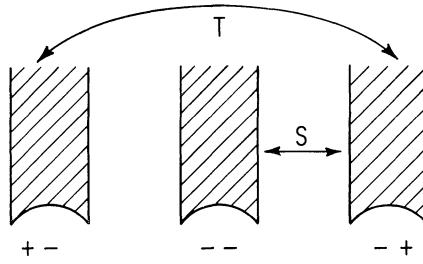
Fig. 2. The fundamental domain for the action of the modular supergroup on the three even spin structures consists of three copies of the standard domain of Fig. 1, labeled by spin structure. The generators S and T interchange the indicated spin structures

circular edge. Two points, $\tau = i$ and $\tau = e^{i\pi/3}$, are fixed by some modular transformations and become singular points of the quotient space. M_1 is not quite compact, because τ can go to infinity along the imaginary axis. M_1 can be compactified by adding a single point at infinity, representing a torus pinched along one homology cycle.

The super Teichmüller space ST_1 consists of four disconnected pieces, corresponding to the four spin structures. The three pieces representing the even spin structures are copies of the upper half plane. The sheet representing the odd spin structure is obtained from $C^{1,1}$, with coordinates (τ, δ), by restricting τ to lie in the upper half plane and identifying (τ, δ) with $(\tau, -\delta)$. The modular supergroup is isomorphic to the ordinary modular group and does not connect the odd sheet with the even ones, so they can be considered separately and lead to two disconnected pieces of super moduli space SM_1 [7].

A fundamental domain for the modular supergroup acting on the even spin structures consists of three copies of the usual fundamental domain labeled by spin structure, as shown in Fig. 2. The group acts on the τ coordinate as in Eqs. (4.1), but also changes the spin structure: S interchanges $--$ with $-+$, leaving $+-$ fixed, while T interchanges $+-$ with $-+$, leaving $--$ fixed. The quotient space is a three-sheeted cover of M_1. Some sheets now cross at the points $\tau = i, e^{i\pi/3}$.

How many points at infinity must be added to the quotient space to compactify it? One might expect that one would be needed for each sheet. The correct answer is that one point is needed for the $+-$ sheet, while the $-+$ and $--$ sheets cross at a single additional point at infinity: the points $\tau = i\infty$ on these sheets are identified by the transformation S. The algebraic equations of Sect. 2 are not needed to obtain this result, but they confirm it. The equations describing the three even spin structures differ only in that a different one of the e_i is distinguished in each case. The spin structures $-+$ and $--$ which are identified at infinity correspond respectively to $e_2 = \wp(\tau/2)$ and $e_3 = \wp\left(\dfrac{1+\tau}{2}\right)$, which indeed become identical as $\tau \to i\infty$.

This identification of certain pairs of spin structures at the boundary of super moduli space occurs quite generally for higher genera as well. Indeed, it occurs already in spin moduli space, the bosonic part of super moduli space. When a Riemann surface X is pinched along a particular cycle, one can distinguish
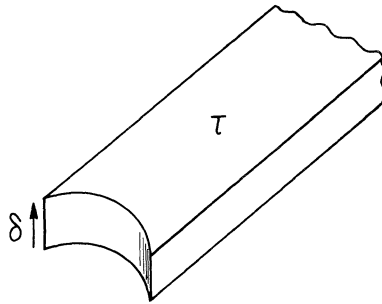
Fig. 3. The fundamental domain for the action of the modular supergroup on the odd spin structure. It is a fiber bundle over the standard domain of Fig. 1, with the fiber coordinates δ and $-\delta$ identified. The fibers have been drawn as one-dimensional although δ is a complex fermionic parameter

between spin structures whose holonomy around that cycle is trivial or not. Dehn twists around the pinching cycle will fix all trivial spin structures but may interchange pairs of nontrivial ones. If the pinching cycle is homologically nontrivial, and if another nontrivial cycle c passes through the neck, the spin structures related by the element c in $H^1(X, Z_2)$ are interchanged, hence identified in the pinched limit. These conditions obtain at the boundary component D_0 of moduli space, where surfaces remain connected after pinching, but not at the other components D_i, where the pinching cycle bounds a surface of genus i. In the present example the identified spin structures $-+$ and $--$ have antiperiodic boundary conditions (nontrivial holonomy) around the pinching cycle $\tilde{z} = z + 1$ and opposite boundary conditions around the other cycle, both cycles being homologically nontrivial.[1]

The modular supergroup acts on the odd spin structure by [7]

$$S: \tau \to \tau + 1, \quad \delta \to \delta;$$

$$T: \tau \to -1/\tau, \quad \delta \to \delta\tau^{-3/2}.$$

(4.2)

The transformation T^2 identifies δ with $i\delta$ and accounts for the symmetry (3.12) pointed out in Sect. 3. The fundamental domain is shown in Fig. 3. Except for the identification of δ and $-\delta$, it is a fiber bundle with fiber coordinate δ over the standard bosonic domain. It is possible to extend this bundle by adding a fiber over the point $\tau = i\infty$. This is most easily seen by applying the transformation T to bring $\tau = i\infty$ to the origin and adding a fiber over the origin. A point with finite coordinate along the fiber over the origin corresponds to a point on the fiber at infinity such that δ blows up like $\tau^{3/2}$ as $\tau \to i\infty$.

Although it is possible to add a fiber at infinity as described above, this is *not* the correct way to compactify this piece of SM_1. This can be demonstrated using the algebraic equations (3.6, 3.9) for the supertorus with odd spin structure. The equations depend on τ through the modular forms $g_i(\tau)$ and on δ only through the

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[1] We thank the referee for suggesting this discussion

combination $\delta \dot{g}_i(\tau)$. The modular forms have expansions

$$g_2(\tau) = (2\pi)^4 \left[\tfrac{1}{12} + 20 \sum_{n=1}^{\infty} \sigma_3(n) e^{2\pi i n \tau} \right],$$

$$g_3(\tau) = (2\pi)^6 \left[\tfrac{1}{216} - \tfrac{7}{3} \sum_{n=1}^{\infty} \sigma_5(n) e^{2\pi i n \tau} \right], \tag{4.3}$$

where the σ_i are certain arithmetic functions [19]. In the limit $\tau \to i\infty$, g_i remains finite, while $\dot{g}_i \to 0$ exponentially. Therefore, $\delta \dot{g}_i(\tau) \to 0$ exponentially even if δ goes as $\tau^{3/2}$ when $\tau \to i\infty$. All dependence on δ disappears in this limit, so that there is only one singular supertorus at $\tau = i\infty$ rather than a family labeled by δ. Consequently the correct compactification of this piece of SM_1 is obtained from the fiber bundle described earlier, with all points on the fiber at infinity identified – a "stratified" space with worse than orbifold singularities.

5. Discussion

In this paper we have established some results in the algebraic geometry of super Riemann surfaces. Any SRS of genus 1 was shown to be a projective supervariety, that is, the locus of zeroes of a set of polynomials in projective superspace. The polynomials in question were explicitly constructed for all choices of spin structure using the theory of superelliptic functions. Although the polynomials only capture the complex supermanifold structure of the SRS, the superconformal structure could also be described algebraically, by giving a rational 1-form on projective superspace which agrees with the even component of the supergravity frame field $E^z = dz + \theta d\theta$ on the supertorus itself. Given the equations representing an algebraic curve, the limit of singular curves can be investigated. This procedure was used to obtain a complete picture of the compactification divisor representing singular supertori on the boundary of the genus 1 super moduli space. This divisor consists of precisely three points, since two of the four spin structures degenerate at infinity and the supermodulus characterizing the odd spin structure becomes irrelevant.

These tools, and their generalization to higher genera, are relevant to several problems in superstring theory. First, as emphasized by Friedan and Shenker, it is very desirable to formulate string theory completely in algebraic or even number-theoretic terms. Such a formulation might lend itself to exact nonperturbative calculations, in addition to clarifying the fundamental mathematical principles underlying string theory. Second, detailed knowledge of the nature of the compactification divisor for all genera is necessary for a proof that Polyakov superstring amplitudes are finite and have the correct factorization properties. In one approach to these questions, due to Friedan and Shenker, the super moduli spaces for all genera are glued together along their compactification divisors to form a universal super moduli space [14]. Another approach would simply investigate the asymptotic behavior of the relevant functional determinants as the divisor is approached [24, 25].

Many more mathematical questions remain to be addressed before the algebraic geometry of SRS's can compete with its bosonic counterpart as a useful tool in superstring theory. The simplest of these is whether the superconformal structure of a projective SRS has a more profound description than that adopted here. A superconformal structure on a complex $(1,1)$ supermanifold is a certain distinguished $(1,0)$ subbundle of its cotangent bundle, here specified by the explicit generator $dz + \theta d\theta$ which we expressed as the pullback of a meromorphic 1-form via the projective embedding. At higher genus the natural generalization would be a distinguished $(1,0)$ bundle on projective superspace, which restricts to the superconformal subbundle on the embedded SRS. Can this information be encoded in the embedding in some more natural way? One idea would be to pair off the even and odd coordinates on projective superspace and demand that the embedding give a superconformal map from the SRS coordinates (z, θ) to each pair. We have not succeeded in implementing this idea, and further thought shows it to be quite unnatural. The supertorus example shows that the target space $SP^{m,n}$ need not have $m = n$, although of course extra coordinates could be added to achieve this. More telling is the fact that

$$SP^{m,m} \neq SP^{1,1} \times SP^{1,1} \times \ldots \times SP^{1,1}, \tag{5.1}$$

so that no such global pairing of coordinates exists. Is the subbundle description the best possible or can the superconformal structure be encoded in some other way?

The theory of superelliptic functions could be developed further. Notably missing from our discussion was an addition law for superelliptic functions. Certainly a supertorus has a supergroup structure, but is there a geometric realization of this analogous to the addition law on ordinary elliptic curves? The techniques usually employed to derive the addition law involve the properties of line bundles and divisors [15, 16], which have yet to be generalized to SRS's [26]. Also, the superelliptic function $R(z, \theta)$ discussed in Sect. 4 is closely related to the super theta function

$$\Theta(z, \theta; \tau, \delta) = \Theta(z; \tau + \theta\delta), \tag{5.2}$$

whose properties follow from those of the ordinary theta function. For example, its behavior under the modular transformation T is

$$\sqrt{\frac{\tau + \theta\delta}{i}} \, \Theta(z, \theta; \tau, \delta) = ie^{-\pi iz^2/(\tau + \theta\delta)} \Theta\left(\frac{z}{\tau + \theta\delta}, \theta\tau^{-1/2}; -1/\tau, \delta\tau^{-3/2}\right). \tag{5.3}$$

This definition of the super theta function depends on the fact that the numbers of bosonic moduli and fermionic supermoduli are equal, so that the combination $\tau + \theta\delta$ can be formed, and this will not generalize to higher genus. At higher genus the super theta function should be defined on the Jacobian supervariety of a SRS, and the construction of this object in turn is unknown. The Picard supervariety, a closely related notion, makes sense as the group of super line bundles on a SRS, but the important embedding of the SRS in its Picard supervariety depends on properties of divisors and has not been worked out.

We conjecture that our result that supertori are projective algebraic super-varieties will generalize to higher genera [26]. The usual proof that a Riemann

surface is a projective algebraic variety has two steps [16], neither of which has yet been carried out for SRS's. First one proves the existence of an "ample" line bundle over the Riemann surface, whose sections can serve as the functions defining a projective embedding as in Sects. 3 and 4. The second step is Chow's theorem, which guarantees that any projective embedding is in fact algebraic. Chow's theorem depends only on properties of projective space, so it should be the easier of the two steps to generalize. Manin has shown that the projectivity of a complex manifold does not guarantee the projectivity of its super generalization [21], but there is no obvious reason why SRS's should behave like his counterexample, super Grassmannians.

Finally there are open questions regarding the structure of the super moduli space for higher genus. Does this space have a compactification? If so, what is the structure of the compactification divisor [18]? Answering this question probably requires a deep understanding of spin structures on singular Riemann surfaces, and the action of the modular supergroup on spin structures as well as on the supermoduli. Looming beyond these issues is the very difficult question of whether the compactified super moduli space itself may be a projective supervariety.

Appendix

The purpose of this Appendix is to give proofs of some results which require a more rigorous approach to superspace than was adopted in the main text. These results are the completeness of the superelliptic functions R, DR, and D^2R, and the fact that the supertorus with odd spin structure is actually embedded in $C^{2,2}$ by the map defined in Sect. 3.

For each of these proofs it is necessary to introduce explicitly the Grassmann algebra within which z, θ, δ, and all functions under consideration take their values. This Grassmann algebra will be generated by L anticommuting elements v_i, $i = 1, 2, ..., L$. The value of L is immaterial and can be taken arbitrarily large [27]. Any element of the algebra can be expanded as

$$A = A_0 + A_i v_i + A_{ij} v_i v_j + \ldots \qquad (A.1)$$

with ordinary complex numbers as coefficients. z takes even values in this algebra while θ takes odd ones. For some purposes z may be regarded as an ordinary complex variable z_0, since a function of z is uniquely determined by analytic continuation once it is known for pure complex values $z = z_0$.

To show the completeness of R, DR, and D^2R, let $F(z, \theta) = A(z) + \theta B(z)$ be any superelliptic function, so that

$$F(z + \tau + \theta\delta, \theta + \delta) = F(z, \theta) = F(z + 1, \theta), \qquad (A.2)$$

or

$$A(z + 1) = A(z) = A(z + \tau) + \delta B(z + \tau),$$
$$B(z + 1) = B(z) = B(z + \tau) + \delta A'(z + \tau). \qquad (A.3)$$

As explained above, we consider only pure complex values of z. If the functions A and B are expanded in the generators v_i as in Eq. (A.1), the components $A_0(z)$ and

$B_0(z)$ will be ordinary elliptic functions, since δ is at least of first order in the v_i. Express these elliptic functions as rational functions of $\wp(z)$ and $\wp'(z)$, and then replace $\wp$ by R and $\wp'$ by D^2R. The resulting expressions $\tilde{A}_0(z, \theta)$ and $\tilde{B}_0(z, \theta)$ are superelliptic functions which agree with $A_0(z)$ and $B_0(z)$ at lowest order in the v_i. Noting that

$$\frac{DR}{D^2R} = \theta + \delta\,\frac{\hat{\wp}(z)}{\hat{\wp}'(z)}, \tag{A.4}$$

write

$$F(z, \theta) = \tilde{A}_0(z, \theta) + \frac{DR}{D^2R}\,\tilde{B}_0(z, \theta) + \hat{F}(z, \theta). \tag{A.5}$$

Then $\hat{F}$ will be superelliptic with components of order v_i or higher, and by applying the same procedure inductively to these components we eventually express F as a rational function of R, DR, and D^2R as desired.

We turn now to the proof that the supertorus with odd spin structure, minus the points $z = 0$, is in fact embedded in $C^{2,2}$ as the set of points obeying Eqs. (3.6). Certainly the supertorus is mapped into this set of points; what must be shown is that each point satisfying (3.6) is in fact the image of a unique point of the supertorus. Specifically, given a point (x, y, ϕ, ψ) satisfying (3.6), we must find a unique point (z, θ) with

$$R(z, \theta) = \wp(z) + \theta\delta\hat{\wp}(z) = x,$$

$$D^2R(z, \theta) = \wp'(z) + \theta\delta\hat{\wp}'(z) = y,$$

$$DR(z, \theta) = \delta\hat{\wp}(z) + \theta\wp'(z) = \phi, \tag{A.6}$$

$$D^3R(z, \theta) = \delta\hat{\wp}'(z) + \theta\wp''(z) = \psi.$$

These equations can always be solved by expanding z, θ, δ, x, y, ϕ, and ψ in the v_i and treating the v_i as "small" perturbation parameters. The zeroth order equations have a unique solution for z_0 because the ordinary Weierstrass equation does give an embedding of a torus. The first order equations can be solved for θ_i because either $\wp'(z_0)$ or $\wp''(z_0)$ must be nonzero, and the proof is completed inductively. The embedding is necessarily nonsingular, because it is nonsingular at zeroth order and essentially linear at higher orders.

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