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The Essentially Commutative Dilations of Dynamical Semigroups on M_n

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Abstract. For identity and trace preserving one-parameter semigroups $\{T_t\}_{t \geqq 0}$ on the $n \times n$-matrices M_n we obtain a complete description of their "essentially commutative" dilations, i.e., dilations, which can be constructed on a tensor product of M_n by a commutative W^*-algebra.

We show that the existence of an essentially commutative dilation for T_t is equivalent to the existence of a convolution semigroup of probability measures ρ_t on the group $\mathrm{Aut}(M_n)$ of automorphisms on M_n such that $T_t = \int_{\mathrm{Aut}(M_n)} \alpha d\rho_t(\alpha)$, and this condition is then characterised in terms of the generator of T_t. There is a one-to-one correspondence between essentially commutative Markov dilations, weak*-continuous convolution semigroups of probability measures and certain forms of the generator of T_t. In particular, certain dynamical semigroups which do not satisfy the detailed balance condition are shown to admit a dilation. This provides the first example of a dilation for such a semigroup.

Introduction

Dilations of semigroups of completely positive operators on W^*-algebras can be studied under two different points of view: If the W^*-algebras are commutative then the semigroup of (completely) positive operators can be interpreted as a semigroup of transition operators, and its Markov dilation turns out to be the corresponding Markov process. Therefore, from a probability theoretic point of view, a Markov dilation is a non-commutative Markov process or a quantum Markov process.

On the other hand a semigroup of completely positive operators on a W^*-algebra can be interpreted as an operator algebraic description of an irreversibly behaving physical system. In this frame a dilation is a larger reversibly evolving system from which the irreversible system is recovered by coarse graining.

A fundamental problem in non-commutative probability theory is to find all

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stationary quantum Markov processes for a given semigroup of transition probabilities. Physically speaking the question is what are the possible "heat baths" which induce on a given quantum system a given irreversible behaviour.

In classical probability theory this "dilation problem" was solved already by the Kolmogorov–Daniell construction: every semigroup of transition probabilities on a classical probability space admits precisely one minimal Markov dilation.

In the non-commutative case the situation is not so straightforward. Dynamical semigroups may admit many, a few or only one dilation, possibly even none at all.

Dilation theory on non-commutative C^*-algebras was initiated by the interest in constructing heat baths (cf., e.g., [Lew 1]) and started with [Eva 1, Eva 3, Eva 4, Dav 1]. Here C^*-algebraic versions of dilations have been constructed for any semigroup of completely positive operators on a C^*-algebra. However, no invariant states have been taken into consideration. The first dilations in the full sense of our definition appeared in [Emc 1, Eva 2] where the quasifree calculus on the CCR algebra and CAR algebra has been used for the construction. A general theory of dilations has been developed in [Küm 1] to which we refer for their basic properties. The first example of a dilation which does not make use of quasifree techniques appeared in [Küm 2], followed by [Fri 1, Maa 1, Fri 2], where a dilation has been constructed for any semigroup on the $n \times n$-matrices satisfying a so-called detailed balance condition (cf. [Kos 1]). It has been suggested in [Fri 1, Fri 2] that the detailed balance condition is a necessary condition for the existence of a dilation. In the present paper it will be shown that this is not so.

This condition of detailed balance originated from the quantum theory of atoms and molecules, and says that every pair of energy levels, with the probabilities of transitions between them, constitutes a balanced subsystem. Mathematically this amounts to the self-adjointness of the generator of the semigroup on the GNS–Hilbert space, apart from a purely Hamiltonian part. In this paper we shall go one step towards a general solution of the dilation problem. We consider a dynamical semigroup $\{T_t\}_{t \geqq 0}$ on the algebra M_n of all $n \times n$-matrices, the simplest non-commutative system, and then ask for all dilations which do not add any further non-commutativity. We shall call such a dilation "essentially commutative". In the language of probability theory this means that the Markov process involved is essentially a classical stochastic process. A related problem is investigated in [Ali 1] where non-commutative stochastic differential equations are solved by using classical Brownian motion only.

In the physical interpretation we are speaking of a quantum system which evolves irreversibly under the influence of a classical heat bath. A necessary condition for a dynamical semigroup to admit such a dilation is that it preserves the trace on M_n. This is physically reasonable since a classical heat bath can be viewed as a quantum heat bath at infinite temperature, which is reflected by the fact that the trace on M_n describes the thermal equilibrium state at infinite temperature.

A common feature of most dilations constructed so far consists of the fact that these dilations are obtained by coupling to a shift which has the characteristic properties of white noise, i.e., different times are stochastically independent. Moreover, in [Küm 2, Küm 3] it is shown that in various situations any dilation is necessarily of this type. Again, in the present paper a crucial step consists in showing that an essentially commutative dilation can be described as a "coupling to white noise" (cf. 1.2). This description is then used in 1.3 to show that essentially commutative dilations of trace preserving dynamical semigroups $\{T_t\}_{t\geqq 0}$ on M_n correspond in a one-to-one way to stochastic processes $\{\alpha_t\}_{t\geqq 0}$ with independent increments on the group $\mathrm{Aut}(M_n)$ of automorphisms of M_n starting at the identity, via the relationship

$$T_t = \mathbb{E}(\alpha_t).$$

This step is entirely in the spirit of [Alb 1] where semigroups on M_n are related to diffusions on $\mathrm{Aut}(M_n)$. These authors, however, restrict to the case of "symmetric semigroups", i.e., semigroups satisfying the detailed balance condition.

While on commutative W^*-algebras a minimal Markov process is already completely determined by its semigroup of transition probabilities, essentially commutative minimal Markov dilations are no longer unique. However, the processes $\{\alpha_t\}_{t\geqq 0}$ are completely characterised by their probability distributions, forming convolution semigroups $\{\rho_t\}_{t\geqq 0}$ of measures on $\mathrm{Aut}(M_n)$. The problem of classification of essentially commutative minimal Markov dilations can now be completely solved by a theorem of Hunt (Theorem 1.4.1), which characterises all convolution semigroups of probability measures on a compact Lie group by a description of their generators $f \mapsto (d/dt)\rho_t(f)|_{t=0}$. His formula for these generators falls into three parts: a drift term, a Brownian motion term, and a Poisson term. The decomposition into these terms, however, is not determined by the given semigroup $\{T_t\}_{t\geqq 0}$. But if we write the generators in a fixed basis then the particular expression obtained for such a generator forms a complete invariant for essentially commutative minimal Markov dilations (cf. 1.5).

As an illustration let us consider the following example: Let us take $n = 2$, and for simplicity consider maps $\gamma(z)\colon M_2 \to M_2$ given by

$$\gamma(z)\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} & zx_{12} \\ \bar{z}x_{21} & x_{22} \end{pmatrix}, \qquad (z \in \mathbb{C}).$$

For $|z| = 1, \gamma(z)$ is an automorphism, and for $|z| \leqq 1$ it is still a completely positive identity and trace preserving map on M_2. The image $\gamma(\Gamma)$ of the unit circle $\Gamma = \{z : |z| = 1\}$ is a subgroup of $\mathrm{Aut}(M_2)$.

A drift on this subgroup is given by $\alpha_t = \gamma(e^{i\lambda t})$, $(t \geqq 0,\ \lambda \in \mathbb{R})$.

A Brownian motion on $\gamma(\Gamma)$ is for instance $\alpha_t = \gamma(e^{ib_t})$, where b_t is a real-valued realisation of Brownian motion, constituting an essentially commutative dilation of the semigroup $\{T_t\}_{t\geqq 0}$ given by

$$T_t = \mathbb{E}(\gamma(e^{ib_t})) = \gamma(\mathbb{E}(e^{ib_t})) = \gamma(e^{-t/2}).$$

An example of a Poisson process on $\gamma(\Gamma)$ is given by $\gamma((-1)^{N_t})$, where N_t is a realisation of the Poisson process with density $\lambda > 0$, taking values in $\mathbb{N}$. The associated semigroup is given by

$$\mathbb{E}(\gamma((-1)^{N_t})) = \gamma(\mathbb{E}(-1)^{N_t}) = \gamma(e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}(-1)^n) = \gamma(e^{-2\lambda t}).$$

Choosing $\lambda = \frac{1}{4}$ one obtains an example of two processes with independent increments on $\gamma(\Gamma)$ inducing the same dynamical semigroup on M_2.

For clarity we prefer to stick to the simple case of semigroups on M_n, although the result could be generalised into several directions. In particular, replacing $\mathrm{Aut}(M_n)$ by a compact group of automorphisms of some von Neumann algebra would yield similar results.

1. Theory of Essentially Commutative Markov Dilations

1.1. *Notation and Main Result.* As the objects of a category we consider pairs $(\mathscr{A}, \phi)$, where $\mathscr{A}$ is a W^*-algebra and ϕ a faithful normal state on $\mathscr{A}$.

A morphism $M:(\mathscr{A}_1, \phi_1) \rightarrow (\mathscr{A}_2, \phi_2)$ is a completely positive operator $M:\mathscr{A}_1 \rightarrow \mathscr{A}_2$ satisfying: $M(\mathbb{1}) = \mathbb{1}$ and $\phi_2 \circ M = \phi_1$. In particular, a morphism is a normal operator.

If $\{T_t\}_{t \geqq 0}$ is a pointwise weak*-continuous one-parameter semigroup of morphisms of $(\mathscr{A}, \phi)$ into itself with $T_0 = \mathrm{id}_{\mathscr{A}}$, we call $(\mathscr{A}, \phi, T_t)$ a *dynamical system.*

If, moreover, the operators T_t are *-automorphisms, we define $T_{-t} := (T_t)^{-1}$, so that $\{T_t\}_{t \in \mathbb{R}}$ becomes a one-parameter group, and we call $(\mathscr{A}, \phi, T_t)$ a *reversible dynamical system.*

Definition. Let $(\mathscr{A}, \phi, T_t)$ be a dynamical system. If there exist a reversible dynamical system $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t)$ and morphisms $i:(\mathscr{A}, \phi) \rightarrow (\hat{\mathscr{A}}, \hat{\phi})$ and $P:(\hat{\mathscr{A}}, \hat{\phi}) \rightarrow (\mathscr{A}, \phi)$ such that the diagram

$$\begin{array}{ccc} (\mathscr{A}, \phi) & \xrightarrow{T_t} & (\mathscr{A}, \phi) \\ i \downarrow & & \uparrow P \\ (\hat{\mathscr{A}}, \hat{\phi}) & \xrightarrow{\hat{T}_t} & (\hat{\mathscr{A}}, \hat{\phi}) \end{array}$$

commutes for all $t \geqq 0$, then we call $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ a *dilation* of $(\mathscr{A}, \phi, T_t)$. If the diagram only commutes for $t = 0$, we call $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ a *process* over $(\mathscr{A}, \phi)$.

Remark. Considering the above diagram for $t = 0$, one derives that i is an injective *-homomorphism and $i \circ P$ a faithful normal conditional expectation of $\hat{\mathscr{A}}$ onto $i(\mathscr{A})$ leaving $\hat{\phi}$ invariant.

Let $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ be a process over $(\mathscr{A}, \phi)$. For $I \subset \mathbb{R}$ we denote by $\mathscr{A}_I$ the W^*-subalgebra of $\hat{\mathscr{A}}$ generated by $\bigcup_{t \in I} \hat{T}_t \circ i(\mathscr{A})$. In ([Küm 1] 2.1.3) it is shown that there exists a unique conditional expectation P_I of $\hat{\mathscr{A}}$ onto $\mathscr{A}_I$ leaving $\hat{\phi}$ invariant.

Definition. A process $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ over $(\mathscr{A}, \phi)$ is called *minimal* if $\hat{\mathscr{A}} = \mathscr{A}_{\mathbb{R}}$. It is called a *Markov process* if for all $a \in \mathscr{A}_{[0, \infty)}$ we have

$$P_{(-\infty, 0]}(a) = P_{\{0\}}(a).$$

Obviously every process includes a minimal process by restriction to $\mathscr{A}_{\mathbb{R}}$. We note that a Markov process $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ over $(\mathscr{A}, \phi)$ is automatically a Markov dilation of the dynamical semigroup $P \circ \hat{T}_t \circ i$ [Küm 1].

In the present paper we consider the following type of dilation.

Definition. A dilation $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ of $(\mathscr{A}, \phi, T_t)$ is called *essentially commutative* if the relative commutant of $i(\mathscr{A})$ in $\hat{\mathscr{A}}$ is commutative.

Throughout the following we shall take $\mathscr{A}$ to be the algebra M_n of all complex $n \times n$-matrices. In this case every process is a "tensor process"; in the present situation, since $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ is an essentially commutative dilation of $(\mathscr{A}, \phi, T_t)$, this means there exists a commutative W^*-algebra $\mathscr{C}$ with faithful normal state ψ such that $\hat{\mathscr{A}} = \mathscr{A} \otimes \mathscr{C}$, $\hat{\phi} = \phi \otimes \psi$, $i(x) = x \otimes \mathbb{1}$ for $x \in \mathscr{A}$, and the associated conditional expectation P is given by $P(x \otimes f) = \psi(f)x$.

Since every automorphism of M_n leaves the trace invariant, it is easy to see that $\hat{T}_t$ leaves $\mathrm{tr} \otimes \psi$ invariant, where tr denotes the normalised trace on M_n. It follows that tr is invariant under T_t, and we can assume henceforth that $\phi = \mathrm{tr}$.

We are now in a position to state our main result.

Theorem 1.1.1. *Let (M_n, tr, T_t) be a dynamical system. Then the following conditions are equivalent.*

(a) *The dynamical system (M_n, tr, T_t) admits an essentially commutative Markov dilation.*

(b) *There exists a weak*-continuous convolution semigroup $\{\rho_t\}_{t \geqq 0}$ of probability measures on the group $\mathrm{Aut}(M_n)$ of automorphisms on M_n such that*

$$T_t(x) = \int_{\mathrm{Aut}(M_n)} \alpha(x) d\rho_t(\alpha), \qquad (x \in M_n, t \geqq 0).$$

(c) *The generator L of the semigroup $T_t = e^{tL}$ is of the form*

$$L(x) = i[h, x] + \sum_{j=1}^{k} (a_j x a_j - \tfrac{1}{2}(a_j^2 x + x a_j^2)) + \sum_{i=1}^{l} \kappa_i (u_i^* x u_i - x),$$

where h and a_j $(j = 1, \ldots, k)$ are self-adjoint elements of M_n, u_i $(i = 1, \ldots, l)$ are unitaries in M_n and κ_i $(i = 1, \ldots, l)$ are positive numbers.

(d) *The generator L of the semigroup $T_t = e^{tL}$ is in the closure of the cone generated by $\{\alpha - \mathrm{id} \mid \alpha \in \mathrm{Aut}(M_n)\}$.*

(e) *For all $t \geqq 0$, T_t lies in the convex hull $\mathrm{co}(\mathrm{Aut}(M_n))$ of the automorphisms on M_n.*

We shall prove this theorem in Sect. 1.4, making use of the intermediary notions of coupling to white noise and processes with independent increments. These will be introduced in the Sects. 1.2 and 1.3.

1.2. *Coupling to White Noise.* In dilation theory it often occurs that a Markov dilation can be decomposed into a white noise and a system coupled to it [Küm 2, Küm 3, Küm 4]. We shall now show that also in the present situation such a decomposition can be made.

Definition 1.2.1. A reversible dynamical system $(\mathscr{C}, \psi, \sigma_t)$ with $\mathscr{C}$ commutative is called a *white noise* if for t_1, t_2 with $-\infty \leqq t_1 \leqq t_2 \leqq \infty$ there exists a W^*-subalgebra $\mathscr{C}_{[t_1, t_2]}$ of $\mathscr{C}$ such that

(i) $\mathscr{C}$ is generated by $\bigcup_{n \in \mathbb{N}} \mathscr{C}_{[-n, n]}$,

(ii) $\sigma_t(\mathscr{C}_{[t_1, t_2]}) = \mathscr{C}_{[t_1 + t, t_2 + t]}$ for $t \in \mathbb{R}$,

(iii) if $f \in \mathscr{C}_{[t_1, t_2]}$, $g \in \mathscr{C}_{[t_3, t_4]}$, then $\psi(fg) = \psi(f)\psi(g)$ whenever $t_2 \leqq t_3$.

Moreover, if $(\mathscr{C}, \psi, \sigma_t)$ is a white noise, a function α from $\mathbb{R}_+$ into the inner automorphisms of $M_n \otimes \mathscr{C}$ which is continuous for the pointwise weak*-topology on $\operatorname{Aut}(M_n \otimes \mathscr{C})$, is called a *coupling* of M_n to $(\mathscr{C}, \psi, \sigma_t)$ if

(iv) $\alpha_{t_1+t_2} = \alpha_{t_1} \circ (\mathrm{id} \otimes \sigma_{t_1}) \circ \alpha_{t_2} \circ (\mathrm{id} \otimes \sigma_{-t_1})$ for $t_1, t_2 \geqq 0$;

(v) for $0 \leqq t \leqq \infty$ the W^*-algebra $M_n \otimes \mathscr{C}_{[0,t]}$ is generated by

$$\{\alpha_s(x \otimes \mathbb{1}) \mid 0 \leqq s \leqq t, x \in M_n\}.$$

Theorem 1.2.2. *There is a one-to-one correspondence between*

(A) *essentially commutative minimal Markov processes* $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ *over* (M_n, tr), *and*

(B) *white noises* $(\mathscr{C}, \psi, \sigma_t)$ *with a coupling* α_t *to* M_n,

in such a way that $\hat{\mathscr{A}} = M_n \otimes \mathscr{C}$, $\hat{\phi} = \mathrm{tr} \otimes \psi$, $i = \mathrm{id} \otimes \mathbb{1}$, *and*

$$\hat{T}_t = \alpha_t \circ (\mathrm{id} \otimes \sigma_t), \quad (t \geqq 0).$$

Proof. From (A) to (B): As $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ is essentially commutative, it must be of the form $(M_n \otimes \mathscr{C}, \mathrm{tr} \otimes \psi, \hat{T}_t; \mathrm{id} \otimes \mathbb{1})$.

The center $\mathbb{1} \otimes \mathscr{C}$ of $\mathscr{A}$ is globally invariant under $\hat{T}_t, (t \in \mathbb{R})$, and we define $\sigma = \{\sigma_t\}_{t \in \mathbb{R}}$ as the automorphism group of $(\mathscr{C}, \psi)$ induced by the restriction of $\hat{T}_t$ to $\mathbb{1} \otimes \mathscr{C}$.

Then $\alpha_t := \hat{T}_t \circ (\mathrm{id} \otimes \sigma_{-t})$ is an automorphism of the type I algebra $M_n \otimes \mathscr{C}$ leaving its center pointwise fixed, hence it is an inner automorphism. The pointwise weak*-continuity of $t \mapsto \sigma_t$ implies the pointwise weak*-continuity of $t \mapsto \alpha_t$ for $t \in \mathbb{R}$. From the group property of $\hat{T}_t$ and σ_t one easily derives the cocycle property (iv) of α_t with respect to σ_t.

Now we define $\mathscr{C}_{[0,t]} \subset \mathscr{C}$ by

$$M_n \otimes \mathscr{C}_{[0,t]} = \mathscr{A}_{[0,t]}, \quad (0 \leqq t \leqq \infty);$$

$$\mathscr{C}_{[t_1,t_2]} := \sigma_{t_1} \mathscr{C}_{[0,t_2-t_1]}, \quad (-\infty < t_1 \leqq t_2 \leqq \infty);$$

$$M_n \otimes \mathscr{C}_{(-\infty,t]} = \mathscr{A}_{(-\infty,t]} = \bigcup_{m \in \mathbb{N}} M_n \otimes \mathscr{C}_{[-m,t]}.$$

Then $(\mathscr{C}, \psi, \sigma_t)$ is a white noise: condition (i) is an immediate consequence of the minimality property of the dilation, while condition (ii) is satisfied by definition. To prove (iii), let $f \in \mathscr{C}_{[t_1,t_2]}$ and $g \in \mathscr{C}_{[t_3,t_4]}$. Then by the Markov property and the module property of conditional expectations, we obtain

$$\begin{aligned}
\psi(fg) &= \hat{\phi}(\mathbb{1} \otimes fg) = \hat{\phi}(P_{(-\infty,t_3]}(\mathbb{1} \otimes f)(\mathbb{1} \otimes g)) \\
&= \hat{\phi}(\mathbb{1} \otimes f \cdot P_{(-\infty,t_3]}(\mathbb{1} \otimes g)) = \hat{\phi}(\mathbb{1} \otimes f \cdot \hat{T}_{t_3} \circ P_{(-\infty,0]} \circ \hat{T}_{-t_3}(\mathbb{1} \otimes g)) \\
&= \hat{\phi}(\mathbb{1} \otimes f \cdot \hat{T}_{t_3} \circ P_{(-\infty,0]}(\mathbb{1} \otimes \sigma_{-t_3} g)) = \hat{\phi}(\mathbb{1} \otimes f \cdot \hat{T}_{t_3} \circ P_{\{0\}}(\mathbb{1} \otimes \sigma_{-t_3} g)) \\
&= \hat{\phi}(\mathbb{1} \otimes f \cdot \hat{T}_{t_3}(\psi(g) \cdot \mathbb{1} \otimes \mathbb{1})) = \psi(f)\psi(g).
\end{aligned}$$

From (B) to (A): Conversely, let a white noise $(\mathscr{C}, \psi, \sigma_t)$ coupled to M_n by α_t be given. We must show that $(M_n \otimes \mathscr{C}, \mathrm{tr} \otimes \psi, \alpha_t \circ (\mathrm{id} \otimes \sigma_t); \mathrm{id} \otimes \mathbb{1})$ is a minimal Markov process over (M_n, tr).

The group property of $\hat{T}_t := \alpha_t \circ (\mathrm{id} \otimes \sigma_t)$ follows from the cocycle property (iv) of α_t. By (v), $\bigvee_{t \geqq 0} \hat{T}_t \circ i(M_n) = M_n \otimes \mathscr{C}_{[0,\infty)}$, and since $\hat{T}_{-t}(\mathbb{1} \otimes \mathscr{C}_{[0,\infty)}) = \mathbb{1} \otimes \mathscr{C}_{[-t,\infty)}$, the minimality property follows from (i). As $\mathscr{A}_{(-\infty,0]} = M_n \otimes \mathscr{C}_{(-\infty,0]}$ it follows from (iii) that for $x \otimes f \in \mathscr{A}_{[0,\infty)} = M_n \otimes \mathscr{C}_{[0,\infty)}$ we obtain

$$P_{(-\infty,0]}(x \otimes f) = x \otimes \mathbb{1} \cdot P_{(-\infty,0]}(\mathbb{1} \otimes f) = \psi(f) \cdot x \otimes \mathbb{1} = P_{\{0\}}(x \otimes f),$$

which proves the Markov property. It is now clear that the stated correspondence is one-to-one.

1.3. *Processes on* $\mathrm{Aut}(M_n)$ *with (Stationary) Independent Increments.* We shall now show that a white noise with coupling to M_n can be considered as a stochastic process with values in $\mathrm{Aut}(M_n)$, having independent increments.

Indeed, because $\mathscr{C}$ is a commutative von Neumann algebra, it can be written as $L^\infty(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ), such that μ induces the state ψ. In this setting the inner automorphisms $\{\alpha_t\}_{t \geqq 0}$ of $M_n \otimes \mathscr{C}$, as they leave its center $\mathbb{1} \otimes L^\infty(\Omega, \Sigma, \mu)$ pointwise fixed, are to be interpreted as random variables on (Ω, Σ, μ) with values in $\mathrm{Aut}(M_n)$. We shall see that this defines a process with stationary independent increments $\alpha_s^{-1} \cdot \alpha_t$, $(s \leqq t)$. Its transition probabilities are described by the measures $\rho_t = \rho_t^\alpha$, $(t \geqq 0)$, given by

$$\rho_t^\alpha(B) = \int_\Omega \chi_B(\alpha_t(\omega))\, d\mu(\omega), \tag{3.1}$$

where χ_B denotes the characteristic function of a Borel subset B of $\mathrm{Aut}(M_n)$.

Theorem 1.3.1. *The relation* $\alpha \mapsto \rho^\alpha$ *defines a one-to-one correspondence between*

(B) *white noises* $(\mathscr{C}, \psi, \sigma_t)$ *with coupling* α_t *to* M_n, *and*
(C) *weak*-continuous convolution semigroups of probability measures* $\{\rho_t\}_{t \geqq 0}$ *on* $\mathrm{Aut}(M_n)$.

Proof. From (B) to (C): Define the sub-σ-algebras $\Sigma_{[s,t]}$ of Σ by

$$\mathscr{C}_{[s,t]} = L^\infty(\Omega, \Sigma_{[s,t]}, \mu).$$

Then for all $x \in M_n$ we have $\alpha_t(x \otimes \mathbb{1}) \in M_n \otimes \mathscr{C}_{[0,t]}$. Therefore the function $\omega \mapsto (\alpha_t(x \otimes \mathbb{1}))(\omega)$, and hence also $\omega \mapsto \xi(\alpha_t(x \otimes \mathbb{1})(\omega)) = f_{x,\xi} \circ \alpha_t(\omega)$ with $f_{x,\xi}(\alpha) := \xi(\alpha(x))$ is $\Sigma_{[0,t]}$-measurable for all $\xi \in M_n^*$. As the functions $\{f_{x,\xi} | x \in M_n, \xi \in M_n^*\}$ algebraically generate $L^\infty(\mathrm{Aut}(M_n), \eta)$, where η denotes the Haar measure on $\mathrm{Aut}(M_n)$, we have $f \circ \alpha_t \in \mathscr{C}_{[0,t]}$ for all $f \in L^\infty(\mathrm{Aut}(M_n), \eta)$. Similarly, as

$$(\mathrm{id} \otimes \sigma_t) \circ \alpha_s \circ (\mathrm{id} \otimes \sigma_{-t})(x \otimes \mathbb{1}) = (\mathrm{id} \otimes \sigma_t) \circ \alpha_s(x \otimes \mathbb{1}) \in M_n \otimes \mathscr{C}_{[t,t+s]},$$

we have for all $g \in L^\infty(\mathrm{Aut}(M_n), \eta)$:

$$g \circ ((\mathrm{id} \otimes \sigma_t) \circ \alpha_s \circ (\mathrm{id} \otimes \sigma_{-t})) \in \mathscr{C}_{[t,t+s]}.$$

Therefore

$$\psi((f \circ \alpha_t) \cdot (g \circ ((\mathrm{id} \otimes \sigma_t) \circ \alpha_s \circ (\mathrm{id} \otimes \sigma_{-t})))) = \psi(f \circ \alpha_t) \cdot \psi(g \circ ((\mathrm{id} \otimes \sigma_t) \circ \alpha_s \circ (\mathrm{id} \otimes \sigma_{-t}))),$$

hence α_t and $(\mathrm{id} \otimes \sigma_t) \circ \alpha_s \circ (\mathrm{id} \otimes \sigma_{-t})$ are independent random variables on (Ω, Σ, μ). By definition α_t induces ρ_t and from the invariance of ψ under σ_t it follows that $(\mathrm{id} \otimes \sigma_t) \circ \alpha_s \circ (\mathrm{id} \otimes \sigma_{-t})$ induces ρ_s on $\mathrm{Aut}(M_n)$. Now from the cocycle property (iv) of α_t the semigroup property $\rho_t * \rho_s = \rho_{t+s}$ follows.

It remains to show the required continuity. Since $\{\rho_t\}_{t\geqq 0}$ is a semigroup, it suffices to prove that for all closed subsets E of $\mathrm{Aut}(M_n)$ not containing id, we have

$$\lim_{t\downarrow 0} \rho_t(E) = 0.$$

We may confine ourselves to E of the form

$$E_{x,\delta} = \{\alpha\in\mathrm{Aut}(M_n) \,|\, \mathrm{tr}((\alpha(x)-x)^*(\alpha(x)-x)) \geqq \delta\}, \quad (x\in M_n, \delta>0),$$

and we shall use a version of Chebyshev's inequality.

We have for all $\alpha\in\mathrm{Aut}(M_n)$,

$$\chi_{E_{x,\delta}}(\alpha) \leqq \frac{1}{\delta}\mathrm{tr}((\alpha(x)-x)^*(\alpha(x)-x)).$$

Therefore we obtain for $t>0$,

$$\rho_t(E_{x,\delta}) = \int_\Omega \chi_{E_{x,\delta}}(\alpha_t(\omega))d\mu(\omega) \leqq \frac{1}{\delta}\int_\Omega \mathrm{tr}((\alpha_t(\omega)x-x)^*(\alpha_t(\omega)x-x))d\mu(\omega)$$

$$= \frac{1}{\delta}\psi((\alpha_t(x\otimes\mathbb{1}) - x\otimes\mathbb{1})^*(\alpha_t(x\otimes\mathbb{1}) - x\otimes\mathbb{1})),$$

which tends to zero as $t\downarrow 0$.

From (C) to (B): Conversely, let a weak*-continuous semigroup of measures $\{\rho_t\}_{t\geqq 0}$ on $\mathrm{Aut}(M_n)$ be given. Then a left-invariant transition probability Π_t on $\mathrm{Aut}(M_n)$ is defined by $\Pi_t(\beta,B)=\rho_t(\beta^{-1}B)$, where $\beta\in\mathrm{Aut}(M_n)$ and B is a Borel subset of $\mathrm{Aut}(M_n)$. By the Kolmogorov–Daniell construction there exists a minimal process $\{\alpha_t\}_{t\geqq 0}$ with values in $\mathrm{Aut}(M_n)$ on a probability space $(\Omega_+,\Sigma_+,\mu_+)$ with transition probabilities $\{\Pi_t\}_{t\geqq 0}$ and initial value $\alpha_0 = \mathrm{id}$. This process is unique up to equivalence. We may realise Ω_+ as

$$\Omega_+ = \{\omega\colon \mathbb{R}_+ \to \mathrm{Aut}(M_n) \,|\, \omega(0)=\mathrm{id}\},$$

and $\alpha_t\colon\Omega_+\to\mathrm{Aut}(M_n)$ by $\alpha_t(\omega)=\omega(t)$. Then time translation $\tilde{\sigma}_t^+\colon(\Omega_+,\mu_+)\to(\Omega_+,\mu_+)\colon(\tilde{\sigma}_t^+\omega)(s)=\omega(t)^{-1}\cdot\omega(s+t)$, $(t\geqq 0)$, induces a semigroup of injective *-homomorphisms σ_t^+ from $L^\infty(\Omega_+,\Sigma_+,\mu_+)$ into itself leaving μ_+ invariant. This shift σ_t^+ is uniquely determined by the requirement (iv): $\sigma_t^+(f\circ\alpha_s) = f\circ(\alpha_t^{-1}\circ\alpha_{t+s})$. It is well known that there exists a unique extension (Ω,Σ,μ) to negative times of $(\Omega_+,\Sigma_+,\mu_+)$ on which σ_t^+ extends to a group σ_t of automorphisms of $(\mathscr{C},\psi)$, where $\mathscr{C}=L^\infty(\Omega,\Sigma,\mu)$ and ψ is induced by μ. Finally, define the sub-algebras $\mathscr{C}_{[s,t]}$ by

$$\mathscr{C}_{[s,t]} = \bigvee_{0\leqq r\leqq t-s} \{\sigma_s(f\circ\alpha_r) \,|\, f\in L^\infty(\mathrm{Aut}(M_n),\eta)\}, \quad (-\infty<s\leqq t\leqq\infty);$$

$$\mathscr{C}_{(-\infty,t]} = \bigvee_{m\in\mathbb{N}} \mathscr{C}_{[t-m,t]}.$$

Let us now check the validity of the conditions (i)–(v) in Definition 1.2.1. Condition (i) follows from the minimality of the process α_t, while conditions (ii) and (iv) are satisfied by construction. Condition (iii) is a consequence of the independence of the increments α_t and $\alpha_t^{-1}\cdot\alpha_{t+s} = (\mathbb{1}\otimes\sigma_t)\circ\alpha_s\circ(\mathbb{1}\otimes\sigma_{-t})$ together with the invariance of ψ under the time translation σ_t.

In order to check (v), put $\mathscr{A}_{[0,t]} = \bigvee \{\alpha_s(M_n \otimes \mathbb{1}) | s \in [0, t]\}$. Choose $x \in M_n$ and a pure state ξ on M_n, and define as before $f_{x,\xi}$: $\mathrm{Aut}(M_n) \to \mathbb{C}: \alpha \mapsto \xi(\alpha(x))$. Denote by p the support projection of ξ, and let u be a unitary in M_n with the property

$$p + u^* p u + (u^*)^2 p u^2 + \cdots + (u^*)^{n-1} p u^{n-1} = \mathbb{1}.$$

Since $p \otimes \mathbb{1}, u \otimes \mathbb{1}$ and $\alpha_s(x)$, $(s \in [0, t])$, are elements of $\mathscr{A}_{[0,t]}$, the element

$$\mathbb{1} \otimes (f_{x,\xi} \circ \alpha_s) = \mathbb{1} \otimes \xi(\alpha_s(x)) = \sum_{k=0}^{n-1} (u^* \otimes \mathbb{1})^k (p \otimes \mathbb{1}) \alpha_s(x) (p \otimes \mathbb{1})(u \otimes \mathbb{1})^k$$

again is an element of $\mathscr{A}_{[0,t]}$. As the functions $f_{x,\xi}$ generate $L^\infty(\mathrm{Aut}(M_n), \eta)$, the elements $\mathbb{1} \otimes f \circ \alpha_s$ are in $\mathscr{A}_{[0,t]}$ for all $f \in L^\infty(\mathrm{Aut}(M_n), \eta)$. Since also $M_n \otimes \mathbb{1} \subset \mathscr{A}_{[0,t]}$, condition (v) follows.

Finally, the bijectivity of the correspondence between ρ_t and α_t follows from the observation that in the above construction from (C) to (B), ρ_t is the image measure of α_t, so that it is regained in the construction from (B) to (C).

1.4. *Proof of the Main Theorem.* (a)$\Leftrightarrow$(b): Let $(\hat{\mathscr{A}}, \hat{\phi}, \hat{T}_t; i)$ be an essentially commutative minimal Markov dilation of (M_n, tr, T_t). We have constructed in Theorems 1.2.2 and 1.3.1 a weak*-continuous convolution semigroup of measures $\{\rho_t\}_{t \geqq 0}$ on $\mathrm{Aut}(M_n)$. Using the notation introduced in Sects. 1.2 and 1.3 we may now write, for all $x \in M_n$ and $t \geqq 0$,

$$T_t(x) = P \circ \hat{T}_t \circ i(x) = P(\alpha_t(x \otimes \mathbb{1})) = \int_\Omega \alpha_t(\omega)(x) d\mu(\omega) = \int_{\mathrm{Aut}(M_n)} \alpha(x) d\rho_t(\alpha).$$

Conversely, suppose that T_t is of the above form. Then an essentially commutative minimal Markov dilation of (M_n, tr, T_t) is obtained by the reconstructions in Theorems 1.2.2 and 1.3.1.

(b)$\Rightarrow$(c): This part of the proof mainly consists of an application of a theorem by Hunt [Hun 1]. In order to formulate it we need some notation.

Let $\mathscr{L}_n$ be the Lie algebra of $\mathrm{Aut}(M_n)$, to be represented by the skew-adjoint elements in M_n with vanishing trace. We choose a basis $\{d_1, \ldots, d_{n^2-1}\}$ of $\mathscr{L}_n$ with $\mathrm{tr}(d_j^* d_k) = \delta_{jk}$.

Let $C_k(\mathrm{Aut}(M_n))$ be the space of all k times continuously differentiable complex functions on $\mathrm{Aut}(M_n)$. The basis element d_i induces a derivation on $C_2(\mathrm{Aut}(M_n))$ given by

$$(D_i f)(\alpha) = \lim_{t \downarrow 0} \frac{1}{t}(f(\mathrm{Ad}\, e^{t d_i} \cdot \alpha) - f(\alpha)), \quad (\alpha \in \mathrm{Aut}(M_n)), \quad f \in C_2(\mathrm{Aut}(M_n)).$$

In $C_2(\mathrm{Aut}(M_n))$ there exists a function ϕ with the properties

$$\phi(\mathrm{id}) = 0; \qquad (D_i \phi)(\mathrm{id}) = 0;$$
$$(D_i D_j \phi)(\mathrm{id}) = \delta_{ij}; \qquad \phi > 0 \quad \text{on } \mathrm{Aut}(M_n) \backslash \{\mathrm{id}\}. \tag{1.4.1}$$

Note that for a measure ν on $\mathrm{Aut}(M_n) \backslash \{\mathrm{id}\}$ the condition

$$\int_{\mathrm{Aut}(M_n) \backslash \{\mathrm{id}\}} \phi(\alpha) d\nu(\alpha) < \infty$$

does not depend on the choice of ϕ. If a measure ν satisfies the above condition, it is called a *Lévy measure*. Finally, we choose functions $g_i \in C_2(\mathrm{Aut}(M_n))$ with the properties

$$g_i(\mathrm{id}) = 0; \quad D_i g_j(\mathrm{id}) = \delta_{ij}.$$

In all the above formulae the indices i and j range between 1 and n^2-1. For short let us denote the integral $\int_{\mathrm{Aut}(M_n)} f(\alpha)d\rho_t(\alpha)$ by $\rho_t(f)$, $f \in C_0(\mathrm{Aut}(M_n))$.

Theorem 1.4.1 (Hunt). *Let $\{\rho_t\}_{t\geqq 0}$ be a weak*-continuous semigroup of probability measures on* $\mathrm{Aut}(M_n)$.

Then there exist real numbers c_i $(i=1,\dots,n^2-1)$, a positive semidefinite symmetric $(n^2-1)\times(n^2-1)$-matrix $b=\{b_{ij}\}$ and a Lévy measure ν such that for all $f\in C_2(\mathrm{Aut}(M_n))$ the derivative $(d/dt)\rho_t(f)|_{t=0}$ exists and is given by

$$\frac{d}{dt}\rho_t(f)|_{t=0} = \sum_{i=1}^{n^2-1} c_i D_i f(\mathrm{id}) + \sum_{i,j=1}^{n^2-1} b_{ij} D_i D_j f(\mathrm{id})$$
$$+ \int_{\mathrm{Aut}(M_n)\setminus\{\mathrm{id}\}} \Big(f(\alpha) - f(\mathrm{id}) - \sum_{i=1}^{n^2-1} g_i(\alpha) D_i f(\mathrm{id})\Big) d\nu(\alpha). \tag{1.4.2}$$

Conversely, if $\{c_i\}$, $\{b_{ij}\}$ and ν satisfy the above conditions, then (1.4.2) *determines exactly one convolution semigroup $\{\rho_t\}_{t\geqq 0}$ of measures on* $\mathrm{Aut}(M_n)$.

We now apply this theorem to the function $f_{x,\xi}: \mathrm{Aut}(M_n)\to\mathbb{C}: \alpha\mapsto\xi(\alpha(x))$, $x\in M_n$, $\xi\in M_n^*$. Note that $f_{x,\xi}\in C_2$, and therefore we easily find that

$$\rho_t(f_{x,\xi}) = \int_{\mathrm{Aut}(M_n)} \xi(\alpha(x))d\rho_t(\alpha) = \xi(T_t(x)),$$
$$(D_i f_{x,\xi})(\mathrm{id}) = \xi([-d_i, x]), \quad \text{and} \quad (D_i D_j f_{x,\xi})(\mathrm{id}) = \xi([d_i,[d_j,x]]),$$

where $[a,b]$ denotes as usual the commutator $ab-ba$ of a and b in M_n.

Since this is true for all linear forms ξ on M_n we find

$$\frac{d}{dt}T_t(x)|_{t=0} = \sum_{i=1}^{n^2-1} c_i[-d_i,x] + \sum_{i,j=1}^{n^2-1} b_{ij}[d_i,[d_j,x]]$$
$$+ \int_{\mathrm{Aut}(M_n)\setminus\{\mathrm{id}\}} \Big(\alpha(x) - x + \sum_{i=1}^{n^2-1} g_i(\alpha)[d_i,x]\Big)d\nu(\alpha). \tag{1.4.3}$$

We first cast the second term on the right-hand side in its final form by diagonalisation. Since $\{b_{ij}\}$ is a real positive semidefinite matrix there exists an orthogonal matrix $\{U_{ij}\}$ such that

$$b_{ij} = \sum_{k=1}^{n^2-1} \beta_k U_{ki} U_{kj}$$

for some $\beta_k \geqq 0$, $(k=1,\dots,n^2-1)$. Putting $a_k := \sqrt{\beta_k}\sum_j U_{kj} d_j$ and writing out the double commutator, one obtains

$$\sum_{i,j} b_{ij}[d_i,[d_j,x]] = \sum_j \big(a_j x a_j - \tfrac{1}{2}(a_j^2 x + x a_j^2)\big).$$

Next we shall discuss the third term in (1.4.3), which we shall call $G(x)$, G being an element of the space $\mathscr{B}(M_n)$ of linear maps $M_n\to M_n$. Let D denote the linear map from $\mathscr{B}(M_n)$ to the space $\mathscr{B}(M_n, M_n; M_n)$ of bilinear maps $M_n\times M_n\to M_n$ given by

$$D(L)(x,y) = L(xy) - xL(y) - L(x)y, \quad (L\in\mathscr{B}(M_n); x, y\in M_n).$$

The bilinear form $D(L)$ was introduced by G. Lindblad [Lin 1] and is called the *dissipator* of L. Obviously, $D(L)=0$ if and only if L is a derivation, i.e. if L is real then $L(x)=i[h,x]$ for some self-adjoint element h of M_n.

Let $\mathscr{K}$ denote the (non-closed) convex cone in $\mathscr{B}(M_n)$ generated by $\{\alpha-\mathrm{id}\,|\,\alpha\in\mathrm{Aut}(M_n)\}$. Since D is linear and continuous we have

$$D(G)=\int_{\mathrm{Aut}(M_n)\setminus\{\mathrm{id}\}} D(\alpha-\mathrm{id})d\nu(\alpha).$$

Now define the norm $|||\cdot|||$ on $\mathscr{B}(M_n)$ by

$$|||L|||^2:=\frac{1}{n}\sum_{i,j=1}^{n}\mathrm{tr}(L(e_{ij})^*L(e_{ij})),$$

where $\{e_{ij}\}_{i,j=1}^n$ are the matrix units spanning M_n. Calculations show that the function $\phi\in C_2(\mathrm{Aut}(M_n))$ defined by

$$\phi(\alpha):=\tfrac{1}{4}|||\alpha-\mathrm{id}|||^2$$

satisfies the conditions (1.4.1), so that $\phi d\nu$ is a finite measure on $\mathrm{Aut}(M_n)$. On the other hand, for all $\alpha\in\mathrm{Aut}(M_n)$ and $x,y\in M_n$ we have

$$D(\alpha-\mathrm{id})(x,y)=(\alpha(x)-x)(\alpha(y)-y).$$

As all norms on the finite dimensional vector space $\mathscr{B}(M_n)$ are equivalent,

$$\left\|\frac{D(\alpha-\mathrm{id})(x,y)}{\phi(\alpha)}\right\|\leqq\frac{\|\alpha-\mathrm{id}\|^2\cdot\|x\|\cdot\|y\|}{\frac{1}{4}|||\alpha-\mathrm{id}|||^2}$$

is uniformly bounded on $\mathrm{Aut}(M_n)\setminus\{\mathrm{id}\}$. Therefore the integral

$$D(G)=\int_{\mathrm{Aut}(M_n)\setminus\{\mathrm{id}\}}\frac{D(\alpha-\mathrm{id})}{\phi(\alpha)}\phi(\alpha)d\nu(\alpha)$$

is well-defined and lies in $D(\mathscr{K})$. It follows that $G=G_1+i[h_1,\cdot]$, where $G_1\in\mathscr{K}$ and $h_1=h_1^*\in M_n$. As $\mathscr{K}$ is finite-dimensional there exist $l\in\mathbb{N}$, $\kappa_i>0$, $(i=1,\ldots,l)$ and unitaries u_i $(i=1,\ldots,l)$ such that

$$G_1(x)=\sum_{i=1}^{l}\kappa_i(u_i^*xu_i-x).$$

Finally, putting $h=\sum c_id_i+h_1$ we obtain the desired conclusion.

(c)$\Rightarrow$(b): Let $T_t=e^{tL}$ with L as in (c). There exists a real positive semidefinite matrix $\{b_{ij}\}_{1\leqq i,j\leqq n^2-1}$, such that

$$\sum_{j=1}^{k}(a_jxa_j-\tfrac{1}{2}(a_j^2x+xa_j^2))=\sum_{i,j=1}^{n^2-1}b_{ij}[d_i,[d_j,x]].$$

Let us define, for a unitary $u\in M_n$, an element $\mathrm{Ad}\,u\in\mathrm{Aut}(M_n)$ by $(\mathrm{Ad}\,u)(x)=u^*xu$. By the converse part of Hunt's theorem (cf. Theorem 1.4.1), there exists a weak*-continuous convolution semigroup of probability measures $\{\rho_t\}_{t\geqq0}$ with the property that for all $f\in C_2(\mathrm{Aut}(M_n))$,

$$\frac{d}{dt}\rho_t(f)|_{t=0}=\frac{d}{dt}f(\mathrm{Ad}\,e^{-ith})|_{t=0}$$

$$+\frac{d}{dt}\frac{d}{ds}\sum_{i,j=1}^{n^2-1}b_{ij}f((\mathrm{Ad}\,e^{-itd_i})(\mathrm{Ad}\,e^{-isd_j}))|_{s=t=0}$$

$$+\sum_{j=1}^{l}\kappa_j(f(\mathrm{Ad}\,u_j)-f(\mathrm{id})).$$

Now let $T'_t(x):=\int_{\mathrm{Aut}(M_n)}\alpha(x)d\rho_t(\alpha)$, $(x\in M_n)$. In the proof of (b)⇒(c) it was shown that

$$\frac{d}{dt}T'_t(x)|_{t=0}=i[h,x]+\sum_{i,j=1}^{n^2-1}b_{ij}[d_i,[d_j,x]]+\sum_{j=1}^{l}\kappa_j(u_j^*xu_j-x)=L(x).$$

Hence $T'_t=T_t$.

(c)⇔(d): Denote by $\mathscr{K}$ the cone generated by $\{\alpha-\mathrm{id}\,|\,\alpha\in\mathrm{Aut}(M_n)\}$ and let $\bar{\mathscr{K}}$ be its closure.

In order to prove the implication (c)⇒(d) it suffices to show that $x\mapsto i[h,x]$ and $x\mapsto axa-\frac{1}{2}(a^2x+xa^2)$ are in $\bar{\mathscr{K}}$ for self-adjoint h and a in $M_n, x\in M_n$. Indeed, we have

$$i[h,x]=\frac{d}{dt}(e^{iht}\cdot x\cdot e^{-iht})|_{t=0}=\lim_{t\downarrow 0}\frac{1}{t}(e^{iht}\cdot x\cdot e^{-iht}-x),$$

and

$$axa-\tfrac{1}{2}(a^2x+xa^2)=\tfrac{1}{2}[ia,[ia,x]]=\frac{1}{2}\frac{d^2}{dt^2}(e^{iat}\cdot x\cdot e^{-iat})|_{t=0}$$

$$=\lim_{t\downarrow 0}\frac{1}{2t^2}((e^{iat}\cdot x\cdot e^{-iat}-x)+(e^{-iat}\cdot x\cdot e^{iat}-x)).$$

To prove that (d) implies (c), let the linear functional τ on $\mathscr{B}(M_n,M_n;M_n)$ be defined by

$$\tau(A):=\frac{1}{4n}\sum_{i,j=1}^{n}\mathrm{tr}(A(e_{ij},e_{ji})),\quad A\in\mathscr{B}(M_n,M_n;M_n),$$

where $\{e_{ij}\}_{1\leq i,j\leq n}$ is a system of matrix units spanning M_n. Note that $\tau(D(\alpha-\mathrm{id}))=\phi(\alpha)$ for $\alpha\in\mathrm{Aut}(M_n)$. Let $\mathscr{H}$ denote the hyperplane in $\mathscr{B}(M_n,M_n;M_n)$ given by $\{A\in\mathscr{B}(M_n,M_n;M_n)\,|\,\tau(A)=1\}$.

Then for all $\alpha\in\mathrm{Aut}(M_n)\backslash\{\mathrm{id}\}$ the ray $\{\lambda D(\alpha-\mathrm{id})\,|\,\lambda\geq 0\}$ intersects $\mathscr{H}$ in the point $D(\alpha-\mathrm{id})/\phi(\alpha)$. As was noted in the proof (b)⇒(c), the set $\mathscr{E}:=\{D(\alpha-\mathrm{id})/\phi(\alpha)\,|\,\alpha\in\mathrm{Aut}(M_n)\backslash\{\mathrm{id}\}\}$ and hence $\mathscr{C}:=D(\mathscr{K})\cap\mathscr{H}=\mathrm{co}(\mathscr{E})$ is a bounded set in $\mathscr{B}(M_n,M_n;M_n)$ forming a base for the cone $D(\mathscr{K})$. The closure $\bar{\mathscr{C}}=\overline{D(\mathscr{K})}\cap\mathscr{H}=D(\bar{\mathscr{K}})\cap\mathscr{H}$ is a compact convex set now forming a base for $D(\bar{\mathscr{K}})$. (Note that in finite-dimensional vector spaces the image of closed sets under linear maps is closed).

Obviously it is enough to prove the assertion (c) for an element L on an extreme ray of $\bar{\mathscr{K}}$. Now either $D(L)=0$, in which case $L(x)=i[h,x]$ for some self-adjoint

$h\in M_n$ and all $x\in M_n$ so that (c) holds, or $D(L)\neq 0$. Then, by multiplication by a suitable positive number we may assume that $D(L)\in\overline{\mathscr{C}}$, and hence that $D(L)=:A$ is an extreme point of $\overline{\mathscr{C}}$. As $\mathscr{C}$ is a subset of a finite-dimensional vector space, it is well known that $\overline{\mathscr{C}}=\overline{\mathrm{co}}(\mathscr{E})=\overline{\mathrm{co}}(\overline{\mathscr{E}})=\mathrm{co}(\overline{\mathscr{E}})$. Therefore we have $A\in\overline{\mathscr{E}}$.

Let $\{\alpha_i\}_{i\in\mathbb{N}}$ be a sequence of automorphisms in $\mathrm{Aut}(M_n)$ such that $\{D(\alpha_i-\mathrm{id})/\phi(\alpha_i)\}_{i\in\mathbb{N}}\subset\mathscr{E}$ tends to A as $i\to\infty$. By restriction to a subsequence we may assume that $\{\alpha_i\}_{i\in\mathbb{N}}$ itself converges to some automorphism α_0. Either $\alpha_0\neq\mathrm{id}$, then $A\in\mathscr{E}$ and hence $L(x)=i[h,x]+(1/\phi(\alpha_0))(\alpha_0-\mathrm{id})(x)$ for $x\in M_n$, proving (c). Or $\alpha_0=\lim_{i\to\infty}\alpha_i=\mathrm{id}$. Since

$$\begin{aligned}\phi(\alpha)=\tau(D(\alpha-\mathrm{id}))&=\frac{1}{4n}\sum_{i,j=1}^{n}\mathrm{tr}((\alpha-\mathrm{id})(e_{ij})^*(\alpha-\mathrm{id})(e_{ij}))\\&=\frac{1}{4n}\sum_{i,j=1}^{n}\mathrm{tr}(\alpha((\mathrm{id}-\alpha^{-1})(e_{ij})^*\cdot(\mathrm{id}-\alpha^{-1})(e_{ij})))\\&=\frac{1}{4n}\sum_{i,j=1}^{n}\mathrm{tr}((\alpha^{-1}-\mathrm{id})(e_{ij})^*(\alpha^{-1}-\mathrm{id})(e_{ij}))=\phi(\alpha^{-1}),\end{aligned}$$

for $\alpha\in\mathrm{Aut}(M_n)$, we obtain

$$\begin{aligned}A(x,y)&=\lim_{i\to\infty}(\alpha_i-\mathrm{id})(x)\cdot(\alpha_i-\mathrm{id})(y)/\phi(\alpha_i)\\&=\lim_{i\to\infty}\alpha_i((\alpha_i^{-1}-\mathrm{id})(x)\cdot(\alpha_i^{-1}-\mathrm{id})(y))/\phi(\alpha_i^{-1})\\&=\lim_{i\to\infty}(\alpha_i^{-1}-\mathrm{id})(x)\cdot(\alpha_i^{-1}-\mathrm{id})(y)/\phi(\alpha_i^{-1}).\end{aligned}$$

Therefore

$$\begin{aligned}A&=\lim_{i\to\infty}D(\alpha_i-\mathrm{id})/\phi(\alpha_i)=\lim_{i\to\infty}D(\alpha_i^{-1}-\mathrm{id})/\phi(\alpha_i)\\&=\lim_{i\to\infty}\tfrac{1}{2}(D(\alpha_i-\mathrm{id})/\phi(\alpha_i)+D(\alpha_i^{-1}-\mathrm{id})/\phi(\alpha_i^{-1}))=\lim_{i\to\infty}D\left(\frac{\alpha_i+\alpha_i^{-1}}{2}-\mathrm{id}\right)\Big/\phi(\alpha_i)\end{aligned}$$

is the limit of dissipators of self-adjoint generators, i.e. generators L' satisfying $\mathrm{tr}(xL'(y))=\mathrm{tr}(L'(x)y)$ for $x,y\in M_n$. As the set of self-adjoint generators is closed in $\mathscr{B}(M_n)$ it follows that $A=D(L_0)$ for some self-adjoint L_0. It is well-known (see, e.g., [Kos 1]) that such an L_0 can be written as $L_0(x)=\sum_{j=1}^{k}a_jxa_j-\frac{1}{2}(a_j^2x+xa_j^2)$ for elements $a_j=a_j^*\in M_n$, $(1\leqq j\leqq k, x\in M_n)$. Finally, $L(x)=i[h,x]+L_0(x)$ with $h\in M_n$ self-adjoint and $x\in M_n$, which proves our assertion for this case.

Note that L, being mapped into an extreme point of $\overline{\mathscr{C}}$, even has to be of the form $L(x)=i[h,x]+axa-\frac{1}{2}(a^2x+xa^2)$ for a single self-adjoint $a\in M_n$.

As the implication (b)$\Rightarrow$(e) is trivial, we end the proof by showing (e)$\Rightarrow$(d).

If $T_t=e^{tL}$, then $L=\lim_{t\downarrow 0}(T_t-\mathrm{id})/t$. Assuming that $T_t\in\mathrm{co}(\mathrm{Aut}(M_n))$, we obtain for each fixed $t>0$: $T_t=\sum_{i=1}^{m}\lambda_i\alpha_i$ for some positive numbers λ_i with $\sum\lambda_i=1$ and

$\alpha_i \in \text{Aut}(M_n)$. Hence $(T_t - \text{id})/t = (1/t) \sum_{i=1}^{m} \lambda_i(\alpha_i - \text{id})$, which shows that L can be approximated by elements in the cone $\mathscr{K}$.

1.5. *Classification of Essentially Commutative Dilations.* As was demonstrated in the introduction there may be several convolution semigroups of probability measures on $\text{Aut}(M_n)$ leading to the same dynamical semigroup $\{T_t\}$ on M_n. However, there is a one-to-one correspondence between essentially commutative dilations, convolution semigroups on $\text{Aut}(M_n)$ and certain forms of generators of semigroups on M_n.

Theorem 1.5.1. *Let $d_1, \ldots, d_{n^2-1}$ be a fixed basis of the Lie algebra $\mathscr{L}_n$ of $\text{Aut}(M_n)$ such that $\text{tr}(d_j^* d_k) = \delta_{jk}$. Choose functions $g_1, \ldots, g_{n^2-1} \in C_2(\text{Aut}(M_n))$ with the properties $g_i(\text{id}) = 0$ and $(D_i g_j)(\text{id}) = \delta_{ij}$, where D_i is the derivation on $C_2(\text{Aut}(M_n))$ induced by d_i (compare Theorem 1.4.1). Let (M_n, tr, T_t) with $T_t = e^{tL}$ be a dynamical system. Then there exists a natural one-to-one correspondence between*

(a) *essentially commutative minimal Markov dilations of (M_n, tr, T_t),*
(b) *convolution semigroups $\{\rho_t\}_{t \geq 0}$ of measures on $\text{Aut}(M_n)$, such that*

$$T_t(x) = \int_{\text{Aut}(M_n)} \alpha(x) d\rho_t(\alpha),$$

(c) *triples $(h, \{b_{ij}\}_{1 \leq i,j \leq n^2-1}, \nu)$, where h is a self-adjoint element of M_n, $\{b_{ij}\}$ a real positive semidefinite matrix and ν a Lévy measure (cf. 1.4), such that for all $x \in M_n$:*

$$L(x) = i[h, x] + \sum_{i,j=1}^{n^2-1} b_{ij}[d_i, [d_j, x]]$$

$$+ \int_{\text{Aut}(M_n) \setminus \{\text{id}\}} \Big(\alpha(x) - x + \sum_{i=1}^{n^2-1} g_i(\alpha)[d_i, x]\Big) d\nu(\alpha).$$

Proof. The one-to-one correspondence proved in Theorems 1.2.2 and 1.3.1 leads to the biunique correspondence between (a) and (b). From the converse of Hunt's Theorem 1.4.1 any triple $(h, \{b_{ij}\}, \nu)$ uniquely determines a semigroup $\{\rho_t\}$ such that $T_t(x) = \int_{\text{Aut}(M_n)} \alpha(x) d\rho_t(\alpha)$. On the other hand we have shown in Sect. 1.4, (b) $\Rightarrow$ (c), that every ρ_t comes from some triple as in (c).

2. Discussion

In this chapter we discuss various aspects of the main result.

2.1. *The Constituents of an Essentially Commutative Dilation.* The generator in Theorem 1.1.1(c) is built out of three different types of elementary generators of the following forms:

$$L_1(x) = axa - \tfrac{1}{2}(a^2 x + x a^2), \quad (a \text{ self-adjoint});$$

$$L_2(x) = u^* x u - x, \quad (u \text{ unitary});$$

$$L_3(x) = i(xh - hx), \quad (h \text{ self-adjoint}).$$

The corresponding semigroups can easily be brought into the form $x \mapsto \int_{\mathrm{Aut}(M_n)} \alpha(x) d\rho_t(\alpha)$ as

$$e^{L_1 t}(x) = \int_{\mathbb{R}} e^{-ias} \cdot x \cdot e^{ias} \frac{1}{\sqrt{2\pi t}} e^{-s^2/2t} ds,$$

$$e^{L_2 t}(x) = \sum_{n=0}^{\infty} (u^*)^n \cdot x \cdot u^n \frac{t^n}{n!} e^{-t},$$

$$e^{L_3 t}(x) = e^{-iht} \cdot x \cdot e^{iht}.$$

For these elementary cases the construction in Sects. 1.2 and 1.3 reads as follows:

Since $e^{L_3 t}$ already is a group of automorphisms it is its own dilation.
For $i = 1, 2$ we define $(\Omega_i, \Sigma_i, \mu_i; \tilde{\sigma}_t^i; \{X_t^i\}_{t \geqq 0})$:
$(\Omega_i, \Sigma_i, \mu_i)$ is a probability space $(i = 1, 2)$:

$\{X_t^1\}_{t \geqq 0}$ is a Brownian motion inducing the measures $\frac{1}{\sqrt{2\pi t}} e^{-s^2/2t} ds$ on $\mathbb{R}$;

$\{X_t^2\}_{t \geqq 0}$ is a Poisson process inducing the measures $\sum_{n=0}^{\infty} \frac{t^n}{n!} e^{-t} \delta_n$ on $\mathbb{N} \subset \mathbb{R}$;

$\{\tilde{\sigma}_t^i\}_{t \in \mathbb{R}}$ is a group of measure preserving transformations of $(\Omega_i, \Sigma_i, \mu_i)$ such that $X_{t+s}^i = X_t^i + X_s^i \circ \tilde{\sigma}_t^i$ $(t, s \geqq 0)$ and Σ_i is generated by $\{X_t^i \circ \tilde{\sigma}_s^i \mid t \geqq 0, s \in \mathbb{R}\}$, $(i = 1, 2)$.

Now we put $\mathscr{C}_i = L^\infty(\Omega_i, \Sigma_i, \mu_i)$, define ψ_i as the state on $\mathscr{C}_i$ induced by μ_i, and define $\sigma_t^i(f) := f \circ \tilde{\sigma}_t^i$ for $f \in \mathscr{C}_i$.

Then $(\mathscr{C}_i, \psi_i, \sigma_t^i)$ becomes in a canonical way a white noise in the sense of 1.2. Moreover, define $\alpha_t^1(x) := \exp(-iaX_t^1) \cdot x \cdot \exp(iaX_t^1)$ and $\alpha_t^2(x) := (u^*)^{X_t^2} \cdot x \cdot u^{X_t^2}$, then α_t^i is a coupling of M_n to $(\mathscr{C}_i, \psi_i, \sigma_t^i)$ which obviously yields an essentially commutative minimal Markov dilation of $\exp(L_i t)$ as in 1.2.2. Finally, we remark that the above forms of the generators, the convolution semigroups of measures and the constructed dilations correspond to each other in the sense of Theorem 1.5.1.

As a further step let L be a finite sum of generators of the above-mentioned three elementary types. Then on the tensor product of the dilations of these components we can construct a dilation of e^{Lt} by a well known technique from perturbation theory ([Fri 2]).

Therefore, even in the general case where the generator is of the form described in 1.5.1(c), containing infinite sums or integrals, the dilation may still be understood as being composed of the constituents described above. In particular, the dilation consists of a Brownian motion, a Poisson process, and a drift.

2.2. *Dynamical Semigroups on* (M_n, tr) *Admitting an Essentially Commutative Dilation.* A natural question in view of the main result in Sect. 1.1 is whether or not all dynamical systems (M_n, tr, T_t) admit an essentially commutative dilation, or equivalently, whether all generators L of dynamical semigroups on (M_n, tr) are of the form

$$L(x) = i[h, x] + \sum_{j=1}^{k} (a_j x a_j - \tfrac{1}{2}(a_j^2 x + x a_j^2)) + \sum_{l=1}^{m} \kappa_l (u_l^* x u_l - x),$$

with $h, a_1, \ldots, a_k \in M_n$ self-adjoint, $u_1, \ldots, u_m \in M_n$ unitary and $\kappa_1, \ldots, \kappa_m$ positive real numbers.

In the present section we show that this is not the case, and add a few remarks discussing this result.

For $c \in M_n$ we denote by L_c the generator given by

$$L_c(x) := c^*xc - \tfrac{1}{2}(c^*cx + xc^*c), \quad (x \in M_n).$$

If c is normal and given by $\sum_{i=1}^{n} \gamma_i e_{ii}$ for some system of matrix units $\{e_{ij}\}_{i,j=1}^{n}$ of M_n and complex numbers γ_i, then one easily calculates that the dissipator D satisfies:

$$D(L_c)(e_{ij}, e_{jk}) = \overline{(\gamma_i - \gamma_j)}(\gamma_k - \gamma_j)e_{ik}, \quad (1 \leqq i,j,k \leqq n). \tag{2.2.1}$$

Proposition 2.2.1. *The following conditions are equivalent:*

(a) *The semigroup e^{tL_c} admits an essentially commutative dilation.*
(b) *The element c is normal and its spectrum lies either on a circle or on a straight line in $\mathbb{C}$.*

Proof. (b)$\Rightarrow$(a): Condition (b) implies the existence of complex numbers λ and μ such that $c = \lambda b + \mu \cdot \mathbb{1}$ with b either unitary or self-adjoint. Since L_c can be rewritten as $L_c(x) = i[h, x] + |\lambda|^2 \cdot L_b$, where $h = (1/2i)(\mu\bar{\lambda}b^* - \bar{\mu}\lambda b)$, the assertion follows from the main theorem.

(a)$\Rightarrow$(b): As $D(L_c)$ is on an extreme ray of the cone of dissipators, it follows from the main theorem that $D(L_c) = D(L_b)$, where b is either a multiple of a unitary or self-adjoint. We may write $b = \sum_{i=1}^{n} \beta_i e_{ii}$ as above. It follows that $D(L_c)(e_{ii}, e_{jj}) = 0$, hence $L_c(e_{ii}) = 0$, since L_c is a bounded derivation on the maximal commutative algebra $\mathscr{F}$ generated by the projections e_{ii}, $1 \leqq i \leqq n$. Therefore $\frac{1}{2}(c^*cx + xc^*c) = c^*xc$ is positive on the positive elements x of $\mathscr{F}$, hence $c^*c \in \mathscr{F}$. It follows that for all i, $1 \leqq i \leqq n$, there exists $\lambda_i \geqq 0$ with $c^*e_{ii}c = \frac{1}{2}(c^*ce_{ii} + e_{ii}c^*c) = \lambda_i e_{ii}$, hence $c \in \mathscr{F}$. Now we apply (2.2.1) to both b and c and conclude that

$$|\beta_i - \beta_j|^2 = |\gamma_i - \gamma_j|^2; \quad (1 \leqq i, \ j \leqq n).$$

Hence there exists an isometric transformation in the complex plane between the spectra of b and c. As the spectrum of b is either on a circle or on a line, the assertion follows.

Remarks. If the spectrum of $c \in M_n$ lies on a line, the semigroup e^{tL_c} admits a dilation involving a Brownian motion (and a drift). If it lies on a circle, e^{tL_c} can be dilated using a Poisson process (and a drift). The approximation of a line in the complex plane by circles thus corresponds to the approximation of a Brownian motion by (compensated) Poisson processes.

If the spectrum of c lies neither on a circle nor on a line, e^{tL_c} does not admit an essentially commutative dilation. From the equivalence of (a) and (e) in the main Theorem 1.1.1 it now follows that for some $t > 0$ (and this must be for t small) the operator e^{tL_c} does not lie in the convex hull of the automorphisms on M_n. It does not seem easy to reach a conclusion like the latter by other means.

The equivalence of (b) and (e) in Theorem 1.1.1 can be looked upon as the solution of a lifting problem: every continuous semigroup in co(Aut(M_n)) can be lifted in a continuous way to a *convolution semigroup* of measures on Aut(M_n).

2.3. *Dilations and the Detailed Balance Condition.* Up to the present day dilations have been constructed of dynamical semigroups (M_n, ϕ, T_t) only for the case where $\{T_t\}_{t \geqq 0}$ satisfies the detailed balance condition with respect to ϕ (cf., for instance, [Küm 2, Fri 1]). This condition may be formulated as follows ([Kos 1]). Let for any operator N on M_n the ϕ-adjoint N^+ be defined by the identity $\phi(x^*N(y)) = \phi(N^+(x^*)y)$. The semigroup e^{tL} is said to satisfy the detailed balance condition with respect to ϕ if $L^+ - L$ is a derivation, or equivalently if $D(L^+) = D(L)$.

Moreover, it was conjectured that only these dynamical systems possess a dilation. The following proposition, in conjunction with Proposition 2.2.1, disproves this conjecture.

Proposition 2.3.1. *Let c be a normal element of M_n. The following statements are equivalent:*

(a) *The semigroup e^{tL_c} satisfies the detailed balance condition with respect to the trace on M_n.*
(b) *The spectrum of c lies on a straight line in the complex plane.*

Proof. Let $\{e_{ij}\}_{i,j=1}^n$ denote a basis of matrix units in M_n on which c has the form $c = \sum_{i=1}^{n} \gamma_i e_{ii}$. Since $D(L_c^+) = D(L_{c^*})$ and $c^* = \sum \bar{\gamma}_i e_{ii}$, we have by (2.2.1):

$$D(L_c^+)(e_{ij}, e_{jk}) = \overline{D(L_c)(e_{ij}, e_{jk})}.$$

The detailed balance condition $D(L_c^+) = D(L_c)$ is therefore equivalent to

$$\overline{(\gamma_i - \gamma_j)}(\gamma_k - \gamma_j) \in \mathbb{R}, \quad (i, j, k = 1, \dots, n).$$

This is the case if and only if the differences $\{\gamma_i - \gamma_j\}_{i,j=1}^n$ all have the same phase, i.e. if the points $\gamma_1, \dots, \gamma_n$ lie on a line.

A consequence of the above proposition is that the semigroup e^{tL_u}, where u is a unitary in M_n, satisfies the detailed balance condition with respect to the trace if and only if the spectrum of u contains only two points. Consequently, for $n \geqq 3$, there exist semigroups on (M_n, tr) which do not satisfy the detailed balance condition and still admit an essentially commutative Markov dilation.

A specific example (cf. also Sect. 2.6) is provided as follows: Put $q := e^{2\pi i/3}$ and define unitaries in M_3 as

$$u_1 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad u_2 := \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q^2 \\ 1 & 0 & 0 \end{pmatrix}, \quad u_3 := \begin{pmatrix} 0 & q^2 & 0 \\ 0 & 0 & q \\ 1 & 0 & 0 \end{pmatrix}.$$

Defining the generator L as $L(x) = \sum_{i=1}^{3} \frac{1}{3}(u_i^* x u_i - x)$, $(x \in M_3)$, then L is explicitly given as

$$L\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{33} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{22} \end{pmatrix} - \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$$

and we obtain

$$\tfrac{1}{2}(L - L^+)(x) = \tfrac{1}{2}\begin{pmatrix} x_{33} - x_{22} & 0 & 0 \\ 0 & x_{11} - x_{33} & 0 \\ 0 & 0 & x_{22} - x_{11} \end{pmatrix}.$$

Assume now that $\frac{1}{2}(L - L^+)(x) = i(xh - hx)$ for some self-adjoint element $h \in M_3$. Then we obtain $he_{ij} = e_{ij}h$ for the matrix unit e_{ij} $(i \neq j)$, hence also $he_{ii} = he_{ij}e_{ji} = e_{ij}he_{ji} = e_{ij}e_{ji}h = e_{ii}h$ which contradicts the requirement $i(e_{11}h - he_{11}) = \frac{1}{2}(e_{22} - e_{33})$. Therefore L violates the detailed balance condition, as one already expects when interpreting the generator L in terms of transition probabilities (cf. Sect. 2.6).

2.4. *Quantum Stochastic Differential Equations.* Many Markov processes in continuous time satisfy stochastic differential equations. Also in the non-commutative setting the Markov dilations known up to now satisfy certain quantum stochastic differential equations in the sense of Hudson and Parthasarathy [Hud 1, Maa 1]. It is not to be expected in general that the solutions of these equations exhaust the class of all stationary quantum Markov processes. However, the results in this paper show that at least in the essentially commutative case for semigroups on M_n this is indeed so. To make this clear it suffices to indicate here how all three types of dilations (Brownian, Poisson and drift) come out as special cases.

Consider the symmetric Fock space $\mathscr{F}$ over $L^2(\mathbb{R})$ with vacuum vector Ω. Let $\{\sigma_t\}_{t\in\mathbb{R}}$ denote the group of automorphisms of $\mathscr{B}(\mathscr{F})$ induced by the right shift on $L^2(\mathbb{R})$. Hudson and Parthasarathy define a stochastic integration on $M_n \otimes \mathscr{B}(\mathscr{F})$ with respect to $dA_t^*, d\Lambda_t$ and dA_t, thereby giving a meaning to equations involving these differentials and dt.

Now let us consider a dynamical semigroup e^{tL} on M_n whose generator L is given by

$$L(x) = v^*xv - \tfrac{1}{2}(v^*vx + xv^*v), \quad (v \in M_n).$$

Then a C^*-version of a Markov dilation

$$(M_n \otimes \mathscr{B}(\mathscr{F}), \hat{T}_t; \mathrm{id} \otimes \mathbb{1})$$

is obtained by putting

$$\hat{T}_t = \mathrm{Ad}\, u_t \circ (\mathbb{1} \otimes \sigma_t), \quad (t \geqq 0),$$
$$\hat{T}_t = \hat{T}_{-t}^{-1}, \quad (t < 0),$$

where $\{u_t\}_{t \geqq 0}$ is the solution with initial condition $u_0 = \mathbb{1}$ of the quantum stochastic differential equation

$$du_t = (v \otimes dA_t^* - v^*w \otimes dA_t + (w - \mathbb{1}) \otimes d\Lambda_t - \tfrac{1}{2}v^*v \otimes \mathbb{1} \cdot dt)u_t \tag{2.4.1}$$

for some unitary element w of M_n.

Now if we substitute $w = \mathbb{1}$ and $v = ia$, we obtain

$$du_t = (ia \otimes dB_t - \tfrac{1}{2}a^2 \otimes \mathbb{1} \cdot dt)u_t, \quad (a = a^* \in M_n), \tag{2.4.2}$$

where $B_t := A_t^* + A_t$ is isomorphic to classical Brownian motion with respect to the state $\mathrm{tr} \otimes \mathrm{vac}$, where vac denotes the Fock vacuum state on $\mathscr{B}(\mathscr{F})$. On the other hand, putting $v = \lambda(w - \mathbb{1})$, $(\lambda > 0)$ we obtain

$$du_t = ((w - \mathbb{1})dN_t)u_t, \tag{2.4.3}$$

where $N_t = \lambda B_t + \Lambda_t + \lambda^2 t \cdot \mathbb{1}$ is isomorphic to the classical Poisson process with intensity λ^2 with respect to $\mathrm{tr} \otimes \mathrm{vac}$([Hud 1]).

On the other hand, it can be shown that the requirement that the von Neumann algebra generated by $M_n \otimes \mathbb{1}$ and $\{u_t\}_{t \geqq 0}$ is of the form $M_n \otimes \mathscr{C}$ with $\mathscr{C}$ commutative imposes the following restrictions:

either $\quad w = \mathbb{1}$ and $v^* = e^{i\theta}v, \quad (\theta \in \mathbb{R})$,

or $\quad v = \lambda(w - \mathbb{1}), \quad (\lambda \in \mathbb{C})$.

Equations (2.4.2) and (2.4.3) yield dilations of e^{tL} with

$$L(x) = axa - \tfrac{1}{2}(a^2 x + xa^2), \quad \text{and} \quad L(x) = \lambda^2(w^* x w - x)$$

respectively. We note that in these cases $\mathrm{tr} \otimes \mathrm{vac}$ is $\hat{T}_t$-invariant only in its restriction $\hat{\phi}$ to $M_n \otimes \mathscr{C}$. In general the invariance of $\hat{\phi}$ is not guaranteed in the Hudson–Parthasarathy construction.

2.5. *Tensor Dilations of Semigroups on* $L^\infty(\mathrm{Aut}(M_n), \eta)$. Let $\mathscr{B} = L^\infty(\mathrm{Aut}(M_n), \eta)$, where η is the Haar measure on $\mathrm{Aut}(M_n)$ inducing a state, χ say, on $\mathscr{B}$. Then to a convolution semigroup $\{\rho_t\}_{t \geqq 0}$ of probability measures on $\mathrm{Aut}(M_n)$ there corresponds in a one-to-one fashion a left-invariant dynamical semigroup $\{S_t\}_{t \geqq 0}$ on $(\mathscr{B}, \chi)$ given by

$$(S_t f)(\alpha) = \int_{\mathrm{Aut}(M_n)} f(\alpha\beta) d\rho_t(\beta), \quad (\alpha \in \mathrm{Aut}(M_n)).$$

The (commutative) Markov dilation of $(\mathscr{B}, \chi, S_t)$ is closely related to the essentially commutative Markov dilation of $(M_n, \mathrm{tr}, \int \alpha d\rho_t(\alpha))$ which corresponds to $\{\rho_t\}$ according to Theorem 1.5.1. In this section we shall indicate how.

In the part from (C) to (B) of the proof of Theorem 1.3.1 a probability space (Ω, Σ, μ) was constructed with a group of measure-preserving transformations $\tilde{\sigma}_t$ (we realise Ω as $\{\omega: \mathbb{R} \to \mathrm{Aut}(M_n) \mid \omega(0) = \mathrm{id}\}$ and extend $\tilde{\sigma}_t^+$ to the whole of Ω). This leads to a white noise $(\mathscr{C}, \psi, \sigma_t)$ with coupling $\alpha_t(\omega) = \omega(t)$ to M_n, where $\mathscr{C} = L^\infty(\Omega, \Sigma, \mu)$ and $\sigma_t f = f \circ \tilde{\sigma}_t$, and ψ is the state on $\mathscr{C}$ induced by μ.

The Markov dilation of $(\mathscr{B}, \chi, S_t)$ is now given by $(\mathscr{B} \otimes \mathscr{C}, \chi \otimes \psi, \hat{S}_t; \mathrm{id} \otimes \mathbb{1})$, where

$$(\hat{S}_t g)(\beta, \omega) = g(\beta\alpha_t(\omega), \tilde{\sigma}_t(\omega)), \quad (g \in \mathscr{B} \otimes \mathscr{C}, \beta \in \mathrm{Aut}(M_n), \omega \in \Omega, t \geqq 0),$$

$$\hat{S}_t = \hat{S}_{-t}^{-1}, \quad (t < 0).$$

Note that this dilation is minimal, because the functions $\hat{S}_t(f \otimes \mathbb{1}): (\beta, \omega) \mapsto f(\beta\alpha_t(\omega))$,

($t \in \mathbb{R}$), generate $\mathscr{B}' \otimes \mathscr{C}$. Hence we have here a minimal tensor dilation in the sense of [Küm 1] of a classical diffusion semigroup on $\mathrm{Aut}(M_n)$.

This dilation is easily recognised as a stationary Markov process on $\mathrm{Aut}(M_n)$ as follows. Let Ω' be the space of all paths $\omega': \mathbb{R} \to \mathrm{Aut}(M_n)$, without the restriction that $\omega'(0) = \mathrm{id}$. Then we may identify a pair $(\beta, \omega) \in \mathrm{Aut}(M_n) \times \Omega$ with the path $\omega' \in \Omega' : s \mapsto \beta\omega(s)$, so that the transformation on $\mathrm{Aut}(M_n) \times \Omega$ given by $(\beta, \omega) \mapsto (\beta\alpha_t(\omega), \tilde{\sigma}_t(\omega)) = (\beta\omega(t), \omega(t)^{-1}\omega(\cdot + t))$ reduces to a straightforward shift $(\tilde{\sigma}'_t\omega')(s) = \omega'(s+t)$ on Ω' and moreover $\hat{S}'_t g = g \circ \tilde{\sigma}'_t$. The measure $\eta \otimes \mu$ on $\mathrm{Aut}(M_n) \times \Omega$ becomes a shift-invariant measure μ' on Ω' and $\mathscr{B} \otimes \mathscr{C} \cong L^\infty(\Omega', \mu')$.

The probability measure μ' is the Kolmogorov–Daniell measure on the space Ω' of all paths in $\mathrm{Aut}(M_n)$ associated to the Markov process with transition probabilities $\Pi_t(\beta, B) = \rho_t(\beta^{-1}B)$, $\beta \in \mathrm{Aut}(M_n)$, B a Borel subset of $\mathrm{Aut}(M_n)$. The fact that it is possible to construct a *tensor* dilation of $(\mathscr{B}, \chi, S_t)$ comes from the property that S_t commutes with left translations on $\mathrm{Aut}(M_n)$. This means that the process looks the same, independently of its starting point.

2.6. *The Violation of Detailed Balance.* One of the physical interpretations of an essentially commutative Markov dilation of a dynamical semigroup on M_n is that of an atom or molecule with n energy levels, coupled to a stochastic classical radiation field ([Küm 2], compare also [Maa 1]). Usually such a system is described by means of the Schrödinger equation

$$i\frac{d}{dt}u_t(\omega) = \left(h + \sum_{j=1}^{3} E_t^j(\omega)p_j\right)u_t(\omega).$$

Here $p_j \in M_n$ ($j = 1, 2, 3$) are the components of the dipole moment of the molecule and $h = h^* \in M_n$ is its free Hamiltonian. The electric field $E_t: \Omega \to \mathbb{R}^3$ and the unitary evolution $u_t: \Omega \to U(n) \subset M_n$ are random variables on a probability space (Ω, μ). We have seen that, for the evolution of the molecule in the Heisenberg picture to be a Markov process, E_t has to be a white noise, i.e. a sum of derivatives of Poisson processes and Brownian motions.

Now, in the physical literature the point of view is widely taken that the transition probabilities of such a molecule have to satisfy the detailed balance condition. In the present context this condition means that for each pair of energy levels i and j the probability per second of a transition from level i to level j equals that from j to i. In Sect. 2.3 we have seen that this condition need not be satisfied if the random field contains a suitable Poisson component. We gave an example of a 3-level system which, far from obeying detailed balance, only performs transitions from level 1 to level 2, from 2 to 3 and from 3 to 1, and none at all in the reverse directions.

It is therefore an interesting question, whether such non-detailed-balanced dynamical semigroups can indeed be physically realised. The following requirement on the random field E_t has to be met: the field has to consist of an exponentially distributed sequence of δ-like spikes, taking place in very short time intervals $\Delta_1, \Delta_2, \Delta_3, \ldots$, the duration $|\Delta_j|$ of a spike being much shorter than ν^{-1}, where ν is a typical transition frequency of the Hamiltonian h. This condition is needed in order that the jumps in Aut(M_n) can have an asymmetric probability distribution. This may be realisable by collisions with other atoms.

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