Commun. Math. Phys. 109, 109–156 (1987)

Communications in
Mathematical
Physics

Scaling Relations for $2D$-Percolation

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Abstract. We prove that the relations

$$\beta=\frac{2\nu}{\delta+1},\quad \gamma=2\nu\frac{\delta-1}{\delta+1},\quad \Delta=2\nu\frac{\delta}{\delta+1},\quad \text{and}\quad \eta=\frac{4}{\delta+1},$$

hold for the usual critical exponents for $2D$-percolation, provided the exponents δ and ν exist. Even without the last assumption various relations (inequalities) are obtained for the singular behavior near the critical point of the correlation length, the percolation probability, and the average cluster size. We show that in our models the above critical exponents have the same value for approach of p to the critical probability from above and from below.

1. Introduction

It is widely believed (see for instance [6, 10, 26]; also [25] for critical exponents in general) that various quantities in percolation behave like powers of $|p-p_c|$ as p approaches the critical probability p_c. To express these conjectures we use the following notation for site percolation on a periodic graph $\mathscr{G}$ in $\mathbb{R}^d$ (see [13, Chaps. 1–3] for a precise description of the terminology). P_p denotes the probability measure according to which all sites of $\mathscr{G}$ are, independently of each other, occupied (vacant) with probability p (respectively $q := 1-p$). E_p denotes expectation with respect to the probability measure P_p. W is the occupied cluster of a certain preselected site w_0, which will be taken to be the origin $\mathbf{O}$ whenever possible.

$$\# W = \text{number of sites in } W,$$

$A \rightsquigarrow B$ means that there exists an occupied path on $\mathscr{G}$ from some site in A to some site in B,

* Research supported by the NSF through grants to Cornell University and to the Institute for Mathematics and its Applications, Minneapolis

$$\theta(p) = \text{percolation probability} = P_p(\# W = -\infty\},$$

$$p_c = \text{critical probability} = \sup\{p: \theta(p) = 0\}.$$

$$\Delta(p) = \text{average number of clusters per site}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} P_p\{\# W = n\} = E_p\{[\# W]^{-1}\}.$$

$$\tau(p, x, y) = P_p\{x \rightsquigarrow y\},$$

$$\chi(p) = E_p\{\# W: \# W < \infty\} = \sum_{n<\infty} n P_p\{\# W = n\},$$

$$\xi(p) = \text{correlation length}$$

$$= \left[\frac{1}{\chi(p)} \sum_y |y|^2 P_p\{w_0 \rightsquigarrow y\} \text{ and } \# W < \infty\right]^{1/2},$$

where

$$|y| = \max\{|y(i)|: 1 \leqq i \leqq d\} \quad \text{when} \quad y = (y(1), \ldots, y(d)).$$

(This choice of distance is the most convenient one for our purposes, but the results would be the same with the Euclidean distance instead.) The correlation length is thought to be a "typical radius" of a finite cluster. It is roughly the root mean square average distance to w_0 of points in the finite cluster of w_0, where the averaging refers to averaging over the finite cluster (see also Corollary 2 below).

We shall abbreviate P_{p_c} to P_{cr}.

Stated in this notation, the principal conjectures concerning power laws are as follows:

$$\xi(p) \approx |p - p_c|^{-\nu} \quad \text{for some} \quad \nu > 0, \tag{1.1}$$

$$|\Delta'''(p)| \approx |p - p_c|^{-1-\alpha} \quad \text{for some} \quad -1 < \alpha < 0, \tag{1.2}$$

$$\theta(p) \approx (p - p_c)^\beta, \quad p > p_H, \text{ for some } 0 < \beta < 1, \tag{1.3}$$

$$\chi(p) \approx |p - p_c|^{-\gamma} \quad \text{for some} \quad \gamma > 0, \tag{1.4}$$

$$\frac{E_p\{[\# W]^k;\ \# W < \infty\}}{E_p\{[\# W]^{k-1};\ \# W < \infty\}} \approx |p - p_c|^{-\Delta_k}, \quad k \geqq 2, \quad \text{for some} \quad \Delta_k > 0, \tag{1.5}$$

$$P_{cr}\{n \leqq \# W < \infty\} \approx n^{-1/\delta}, \quad n \to \infty, \quad \text{for some} \quad \delta > 0, \tag{1.6}$$

and finally, if $\mathscr{G}$ is a d-dimensional graph,

$$\tau(p_c, w_0, x) = P_{cr}\{w_0 \rightsquigarrow x\} \approx |x|^{2-d-\eta}, \quad |x| \to \infty, \quad \text{for some} \quad \eta > 0. \tag{1.7}$$

The meaning of $A(p) \approx |p - p_c|^\zeta$ is that

$$\lim_{p \to p_c} \frac{\log A(p)}{\log |p - p_c|} = \zeta. \tag{1.8}$$

Often one distinguishes approach to p_c from the right and left and associates exponents ζ_+ and ζ_- to the limit as $p \downarrow p_c$, or $p \uparrow p_c$, respectively. In the results here the exponents always come out to be the same for the two sides, so we shall not distinguish the plus and minus versions of the exponents. Equations (1.6) and (1.7) are meant similarly as limit relations for $(\log n)^{-1}$, or $(\log|x|)^{-1}$, respectively, times the logarithm of the left-hand side.

In addition to the power laws (1.1)–(1.7), it is believed that the exponents satisfy the following so-called scaling laws for low enough d:

$$\alpha = 2 - d\nu, \tag{1.9}$$

$$\beta = \frac{d\nu}{\delta+1}, \tag{1.10}$$

$$\gamma = d\nu\frac{\delta-1}{\delta+1}, \tag{1.11}$$

$$\Delta_k = d\nu\frac{\delta}{\delta+1}, \qquad k \geqq 1, \tag{1.12}$$

$$\eta = 2 - d\frac{\delta-1}{\delta+1}. \tag{1.13}$$

The values of these exponents are even supposed to be universal, i.e., dependent on d only, but not on the specific graph.

As far as we know none of the power laws has been proven for percolation, and the only scaling law which has been proven so far is (1.13) for $d=2$ [14] under the assumption that η and δ make sense, i.e., that (1.6) and (1.7) hold. [Actually it suffices to assume (1.7).] A number of inequalities between the exponents have been proven (again under the assumption that the appropriate power laws hold; cf. [8, 9, 17, 19]) but these do not imply any of the *equalities* (1.9)–(1.13). The above conjectures are based on a more fundamental scaling ansatz (see [26, Sect. 3.1.1] and [10, Sect. 4.6]) of the form

$$P_p\{\# W = n\} \sim n^{1-\tau} f((p-p_c)n^\sigma) \tag{1.14}$$

or

$$P_p\{w_0 \rightsquigarrow x,\ \# W < \infty\} \sim |x|^{2-d-\eta} g\left(\frac{|x|}{\xi(p)}\right) \tag{1.15}$$

as $n \to \infty$, respectively, $|x| \to \infty$, $p \to p_c$ for some constants τ, σ, η and some decent functions f, g. g is assumed to satisfy

$$\lim_{s\downarrow 0} g(s) = g(0) > 0, \qquad \lim_{s\to\infty} g(s) = 0. \tag{1.16}$$

Equations (1.14) and (1.15) are supposedly standard asymptotic relations – that is the ratio of the right- and left-hand side tends to 1 – when $p \to p_c$ and $n \to \infty$ or $|x| \to \infty$ such that $(p-p_c)n^\sigma$ or $|x|(\xi(p))^{-1}$, respectively, converge to a constant. In particular, (1.15) and (1.16) show that

$$P_p\{w_0 \rightsquigarrow x,\ \# W < \infty\} \quad \text{and} \quad P_{cr}\{w_0 \rightsquigarrow x,\ \# W < \infty\} \tag{1.17}$$

should be approximately equal when $|x|$ is small with respect to the correlation length. This may be the basis for the folklore that in some vague sense the whole percolation picture on a scale small with respect to the correlation length looks like percolation at criticality.

Throughout this paper we take $d=2$. We shall prove (1.10)–(1.12) for a number of two-dimensional percolation models under the assumption that (1.1) and (1.6) hold. Actually, we shall replace (1.6) by

$$\pi(n)=\pi(p_c,n)\approx n^{-1/\delta_r} \tag{1.18}$$

for some $\delta_r>0$, where

$$\pi(p,n):=P_p\{w_0 \rightsquigarrow \partial S(n)\} \tag{1.19}$$

and

$$S(n)=[-n,n]\times[-n,n]. \tag{1.20}$$

It was shown in [14] that (1.18) is equivalent to (1.7) in the two-dimensional case. Either one of these power laws implies (1.6) and (1.13) with $2\delta_r=(\delta+1)$. At this moment we are unable to prove any scaling relation involving α, but this paper shows that for two-dimensional percolation all the scaling relations (1.10)–(1.13) are valid, once the appropriate power laws hold. Actually, the basis of our results is a number of useful inequalities which do not rely on the power laws (see Theorems 1–3). The most fundamental one is Theorem 1 which gives a weak form of the above mentioned folklore belief on the behavior on a scale small with respect to the correlation length.

For simplicity *we restrict ourselves here to bond- or site percolation on* $\mathbb{Z}^2$ *or percolation on their matching graphs* (see [13, 2.2] for matching pairs of graphs). As done in [13, Sects. 2.5 and 3.1], we treat bond percolation on $\mathbb{Z}^2$ as site percolation on the covering graph of $\mathbb{Z}^2$. For the remainder we take $w_0=\mathbf{O}$, the origin, so that W stands for the occupied cluster of the origin. Site percolation on $\mathbb{Z}^2$ is in a sense more interesting than bond percolation on $\mathbb{Z}^2$, because the former is not self dual. As we shall see in Sect. 4, this will call for special arguments to prove equality of the critical exponents for $p\downarrow p_c$ and $p\uparrow p_c$.

Our arguments seem to go through for site percolation on any matching pair of two-dimensional graphs which are invariant under reflection in one of the coordinate axes and under rotation around the origin by an angle $\phi\in(0,\pi)$. This includes all standard two-dimensional lattices, and in particular, the triangular and honeycomb lattices. The main thing to check is that (2.15) below and the Russo-Seymour-Welsh lemma hold for such graphs. This can be done by the methods of [13, Appl. (v) on p. 66 and Chap. 6] and [21–23].

Despite the preceding definition of correlation length we shall work with another length, $L(p)$, defined in terms of the crossing probabilities of squares. Only later will we show that[1] $L(p)\asymp\xi(p)$. We note that $L(p)$ was already introduced in [5, Sect. 3], where it was shown to be equivalent (up to logarithmic factors) to the

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[1] $A(p)\asymp B(p)$ for two functions A and B means that

$$0<\liminf_{p\to p_c}\frac{A(p)}{B(p)}\leqq\limsup_{p\to p_c}\frac{A(p)}{B(p)}<\infty,$$

so that $A(p)$ and $B(p)$ are of the same order of magnitude correlation length defined in yet another way. When we are dealing with site percolation on the graph $\mathscr{G}$, let $(\mathscr{G}, \mathscr{G}^*)$ be the corresponding matching pair of graphs (see [13, Sect. 2.2]). We then define for $\bar{n}=(n_1, n_2)$ the "sponge crossing probabilities"

$$\begin{aligned}\sigma(\bar{n}; i, p) &= \sigma(\bar{n}; i, p, \mathscr{G}) \\ &= P_p\{\exists \text{ occupied crossing in the } i\text{-direction on } \mathscr{G} \text{ of } [0, n_1]\times[0, n_2]\},\\ \sigma^*(\bar{n}; i, p) &= \sigma^*(\bar{n}; i, p, \mathscr{G}) = \sigma(\bar{n}; i, 1-p, \mathscr{G}^*) \\ &= P_p\{\exists \text{ vacant crossing in the } i\text{-direction on } \mathscr{G}^* \text{ of } [0, n_1]\times[0, n_2]\}.\end{aligned}$$

Here $i=1$ or 2 and a crossing in the 1 (2) direction will usually be called a *horizontal (vertical) crossing*. We remind the reader that a horizontal (vertical) crossing of $[0, n_1]\times[0, n_2]$ is a path with all its vertices, except for its endpoints, in the *interior* $(0, n_1)\times(0, n_2)$. We define

$$L(p) = L(p, \varepsilon) = L(p, \varepsilon, \mathscr{G}) = \begin{cases}\min\{n\colon \sigma((n, n); 1, p)\} \geqq 1-\varepsilon & \text{if } \quad p > p_c, \\ \min\{n\colon \sigma((n, n); 1, p)\} \leqq \varepsilon & \text{if } \quad p < p_c.\end{cases} \tag{1.21}$$

For the graphs which we are considering there is symmetry between the horizontal and vertical direction so that $L(p)$ would not change if we used vertical crossing probabilities instead of horizontal crossing probabilities in (1.21). Furthermore, it holds for our graphs that

$$\begin{aligned}\sigma((n, n); 1, p) + \sigma^*((n, n)); 2, p) &\geqq 1, \\ \sigma((n, n); 1, p) + \sigma^*((n-1, n); 2, p) &\leqq 1.\end{aligned} \tag{1.22}$$

Therefore, $L(p, \varepsilon, \mathscr{G})$ for $p < p_c = p_c(\mathscr{G})$ is essentially the same as $L(1-p, \varepsilon, \mathscr{G}^*)$ for $1-p > 1-p_c(\mathscr{G}) = p_c(\mathscr{G}^*)$. For the self dual case of bond percolation on $\mathbb{Z}^2$ (for which $p_c = \frac{1}{2}$) we find that $L(\frac{1}{2}+\delta, \varepsilon) \sim L(\frac{1}{2}-\delta, \varepsilon)$, $\delta > 0$. However, it is far less obvious that even for site-percolation on $\mathbb{Z}^2$

$$L(p_c+\delta, \varepsilon) \asymp L(p_c-\delta, \varepsilon), \qquad \delta \downarrow 0, \tag{1.23}$$

for fixed $\varepsilon \leqq$ some small ε_0, defined below. This implies $\nu_+ = \nu_-$, where ν_+ and ν_- are the exponents in (1.1) corresponding to $p \downarrow p_c$ and $p \uparrow p_c$, respectively. This is also the reason why the other critical exponents (assuming they exist) are the same for $p \downarrow p_c$ and $p \uparrow p_c$. Equation (1.23) will be proven in Sect. 4, (see Theorem 4), where it will also be shown that

$$L(p, \varepsilon_1) \asymp L(p, \varepsilon_2) \quad \text{for fixed} \quad 0 < \varepsilon_1, \varepsilon_2 \leqq \varepsilon_0. \tag{1.24}$$

It should be noted that *no symmetry between $p < p_c$ and $p > p_c$ is used in Sects. 2 and 3*. These sections only have to deal with p on one side of p_c at a time.

Here are our other principal results.

Theorem 1. *There exist constants*[2] $0 < C_1, C_2 < \infty$ *such that*

$$C_1 \leqq \frac{\pi(p, n)}{\pi(p_c, n)} \leqq C_2 \quad \textit{for all} \quad n \leqq L(p, \varepsilon_0).$$

[2] In the sequel C_i will always stand for a strictly positive finite constant whose precise value is of little importance. The value of C_i may be different at different appearances. Practically all these constants depend on ε_0, to be fixed in Sect. 2

Theorem 2. *For* $p > p_c$

$$\pi(p_c, L(p, \varepsilon_0)) \leqq \frac{1}{C_1}\pi(p, L(p, \varepsilon_0)) \leqq C_3\theta(p) \leqq C_3\pi(p, L(p, \varepsilon_0))$$
$$\leqq C_4\pi(p_c, L(p, \varepsilon_0)).$$

Corollary 1. *If* (1.18) *holds, then*

$$\theta(p) \approx [L(p, \varepsilon_0)]^{-1/\delta_r} = [L(p, \varepsilon_0)]^{-2/(\delta+1)}.$$

If in addition (1.1) *holds, then* $\beta = 2\nu/(\delta+1)$.

Theorem 3. *For* $t > \frac{1}{3}$

$$E_p\{[\# W]^t;\ \# W < \infty\} \asymp [L(p, \varepsilon_0)]^{2t}[\pi(p_c, L(p, \varepsilon_0))]^{t+1}, \tag{1.25}$$

and for all $t > 0$

$$\sum_y |y|^t P_p\{\mathbf{O} \rightsquigarrow y \text{ and } \# W < \infty\} \asymp [L(p, \varepsilon_0)]^{t+2}[\pi(p_c, L(p, \varepsilon_0))]^2. \tag{1.26}$$

Corollary 2. $\xi(p) \asymp L(p, \varepsilon_0)$,

$$\frac{E_p\{[\# W]^t;\ \# W < \infty\}}{E_p\{[\# W]^{t-1};\ \# W < \infty\}} \asymp [\xi(p)]^2\pi(p_c, \xi(p)), \quad t > \frac{4}{3},$$

and

$$\left[\frac{1}{\chi(p)}\sum_y |y|^t P_p\{\mathbf{O} \rightsquigarrow y \text{ and } \# W < \infty\}\right]^{1/t} \asymp \xi(p), \quad t > 0.$$

If (1.1) *and* (1.18) *hold then*

$$\gamma = 2\nu\frac{\delta-1}{\delta+1}, \quad \Delta_k = 2\nu\frac{\delta}{\delta+1}, \quad k \geqq 2, \quad \nu \geqq \frac{\delta+1}{\delta},$$

and

$$\nu_k := \lim_{p \to p_c} \frac{-1}{\log|p-p_c|}\log\left[\frac{1}{\chi(p)}\sum_y |y|^k P_p\{\mathbf{O} \rightsquigarrow y \text{ and } \# W < \infty\}\right]^{1/k} = \nu, \quad k \geqq 1.$$

The proof of Theorem 1 is in Sect. 2. Theorems 2 and 3 and Corollary 1.2 are proven in Sect. 3. The proofs in Sect. 3 reinforce the idea that $L(p)$ is the fundamental length scale; contributions to (1.25) and (1.26) from points y with $|y|$ much larger than $L(p)$ are negligible.

To close this introduction we tabulate some rigorous bounds which were already known or follow from the results here for the critical exponents. We also include in the table the exact values predicted by the theories of den Nijs [7], Nienhuis et al. [18], and Pearson [20], as well as the exact values on the Bethe tree.

Rigorous bounds for 2D-percolation	"Exact" values of [7, 18, 20]	Bethe tree
$\alpha<0$	$\alpha=-2/3$	$\alpha=-1$
$\beta<1$	$\beta=5/36$	$\beta=1$
$\gamma\geqq 8/5$	$\gamma=43/18$	$\gamma=1$
$\delta\geqq 5$	$\delta=91/5$	$\delta=2$
$\nu>1$	$\nu=4/3$	$\nu=1/2$

In particular, none of the exponents $\beta, \gamma, \delta, \nu$ have in dimension 2 the same values as on a Bethe tree. Section 5 contains some comments to this table.

2. The Behavior of $\pi(p, n)$

This section contains the proof of Theorem 1. The proof is based on a differential inequality. Aizenman and Newman [3] seem to have been the first to apply such a method in percolation theory; they used it to obtain the inequality $\gamma\geqq 1$. Differential inequalities have also been used recently with great success by Chayes and Chayes [4] (to prove $\beta\leqq 1$) and by Aizenman and Barsky [1] (to prove $p_T=p_H$). To obtain an estimate for a derivative one starts with Russo's formula [22; 13, Chap. 4.2]. We shall need later a simple extension of this formula which we state here as Lemma 1 without proof. No essential change in the original proof is needed for this extension.

We remind the reader of some standard notion which we shall use and a couple of definitions. For a rectangle R

$$\partial R = \text{(topological) boundary of } R,$$

$$\mathring{R} = \text{interior of } R,$$

$$I_A = \text{indicator function of the event } A.$$

A site v is *pivotal* for an event A if $I_A(\omega)$ changes its value when v is changed from occupied to vacant or vice versa. Here ω stands for the configuration of occupied and vacant sites. The event $\{v \text{ is pivotal for } A\}$ depends only on the sites other than v. A is an *increasing (decreasing)* event if changing any site from vacant to occupied can only increase (decrease) the value of I_A. We shall consider a family $\hat{P}_t$ of product measures (i.e., measures under which all vertices are independent) and with probability $p(v,t)$ that v is occupied. $p'(v,t)$ denotes $\frac{d}{dt}p(v,t)$.

Lemma 1. *Let A and B be an increasing and decreasing event, respectively, each of which depends on finitely many sites only. Then*

$$\frac{d}{dt}\hat{P}_t\{A\cap B\} = \sum_v p'(v,t)[\hat{P}_t\{v \text{ is pivotal for } A\text{, but not for } B\text{, and } B \text{ occurs}\}$$
$$-\hat{P}_t\{v \text{ is pivotal for } B\text{, but not for } A\text{, and } A \text{ occurs}\}]. \quad (2.1)$$

We apply this lemma here with $B=$ the certain event (which is the original Russo formula) and $A=A(n)=\{\mathbf{O} \rightsquigarrow \partial S(n)\}$. It follows from Whitney's theorem (cf. [24, Theorem 2.1]; also [13, Proposition 2.1]) that a site v is pivotal for $A(n)$ if and only if all of the following hold (see Fig. 1):

$$v \in S(n);$$

there exists a path r on $\mathscr{G}$ and inside $S(n)$ from $\mathbf{O}$ to some point on $\partial S(n)$ which passes through v, and has all its vertices other than v occupied;
there exists a circuit C on $\mathscr{G}^*$ and inside $S(n)$ which passes through v, contains $\mathbf{O}$ in its interior, and has all its vertices other than v vacant.

(If $v=\mathbf{O}$, then the third condition – requiring the existence of C – should be dropped.) Therefore, if $\mathbf{O} \neq v \in S(n)$ is pivotal for $A(n)$, then for any rectangle R for which $\mathbf{O}$ lies outside R, v lies inside R, and $R \subset S(n)$, the following two events must occur (see Fig. 2):

a) $\Gamma(v, R)$: = there exist two paths r_1 and r_3 on $\mathscr{G}$ from v to ∂R and two paths r_2^* and r_4^* on $\mathscr{G}^*$ from v to ∂R such that any two of these four paths only have the vertex v in common; all vertices of $r_1 \cup r_3$ other than v are occupied and all vertices of $r_2^* \cup r_4^*$ other than v are vacant; the paths r_1, r_3, r_2^*, and r_4^* all are contained in $\mathring{R}$

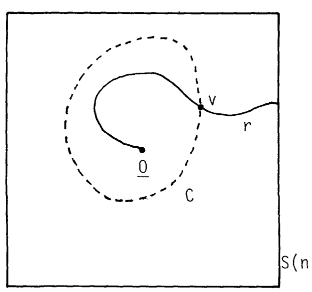

Fig. 1. v is pivotal for $A(n)$

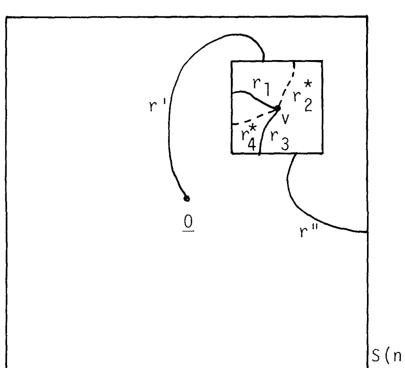

Fig. 2. Illustration of $\Gamma(v, R)$ and $\Xi(R)$

except for their endpoint on ∂R; r_1 and r_3 separate r_2^* and r_4^* (i.e., $r_1 \cup r_3$ forms a crosscut of R such that $r_2^* \backslash v$ and $r_4^* \backslash v$ lie in different components of $R \backslash r_1 \cup r_3$).

b) $\Xi(R) :=$ there exist two disjoint paths r' and r'' from $\mathbf{O}$ to ∂R and from ∂R to $\partial S(n)$, respectively, and except for their endpoints on ∂R, r' and r'' have only occupied vertices, all of which lie in $S(n) \backslash R$.

To see this one merely takes for r' and r'' the pieces of r from $\mathbf{O}$ to the first intersection with ∂R and from the last intersection with ∂R to $\partial S(n)$, respectively. r_1 and r_3 are pieces of r inside R connecting v to ∂R. Finally, r_2^* and r_4^* are two pieces of C, which must exist because C contains $\mathbf{O}$ in its interior, so that C must leave R.

Since $\Gamma(v, R)$ and $\Xi(R)$ depend only on vertices inside R and outside R, respectively, they are independent. Thus for any fixed choice for $R = R(v)$,

$$\hat{P}_t\{v \text{ is pivotal for } A(n)\} \leqq \hat{P}_t\{\Gamma(v, R)\} \hat{P}_t\{\Xi(R)\}.$$

Thus, if we set

$$\hat{\pi}(t, n) = \hat{P}_t\{\mathbf{O} \rightsquigarrow \partial S(n)\} = \hat{P}_t\{A(n)\},$$

then we obtain from (2.1)

$$\left|\frac{d}{dt} \hat{\pi}(t, n)\right| \leqq \sum_{v \in S(n)} |p'(t, v)| \hat{P}_t\{\Gamma(v, R)\} \hat{P}_t\{\Xi(R)\}. \tag{2.2}$$

We now must make a specific choice of R for each v. For simplicity we take $n = 2^k$ for some large integer k (we later show that this is permissible). It will also turn out that we can restrict ourselves to families $\hat{P}_t$ with $p'(t, v) = \frac{d}{dt} p(t, v) = 0$ for all v with $|v| > 2^{k-3}$. Thus in (2.2) we only have to consider terms with $|v| \leqq 2^{k-3}$. If $v = (v_1, v_2)$ and $|v_1| \leqq |v_2| \leqq 2^{k-3}$ and $16 \leqq 2^{j+1} < v_2 \leqq 2^{j+2}$, then let l_1, l_2 be such that

$$l_1 2^{j-2} < v_1 \leqq (l_1 + 1) 2^{j-2}, \quad l_2 2^{j-2} < v_2 \leqq (l_2 + 1) 2^{j-2}, \tag{2.3}$$

and take

$$R = R(v) = [(l_1 - 2) 2^{j-2}, (l_1 + 2) 2^{j-2}] \times [l_2 2^{j-2} - 2^j, l_2 2^{j-2} + 2^j]. \tag{2.4}$$

If $v_2 < 0$, then we choose $R(v)$ as the mirror image of $R((v_1, -v_2))$ with respect to the x-axis. If $|v_2| < |v_1|$, then we interchange the roles of the first and second coordinates in (2.3) and (2.4). As will become apparent, the precise choice of $R(v)$ is unimportant. What matters is that the ratio of the long and short side of R is bounded (in fact, it is $\leqq 2$ for our choice), that $R \subset S(2^{k-1})$, and that the distances from v to ∂R as well as from $\mathbf{O}$ to ∂R are of the same order as the sides of R in (2.4). Note that if $|v_1| \leqq v_2 \leqq 2^{j+2} \leqq 2^{k-3}$ and $2^{j+1} < v_2$, then under (2.3) we must have

$$j \leqq k - 5 \quad \text{and} \quad |l_1| \leqq 17, \quad 8 \leqq |l_2| \leqq 17. \tag{2.5}$$

Therefore, our choice of R has the listed properties when $16 < |v| \leqq 2^{k-3}$. For $|v| \leqq 16$ take $R(v) = R_0 := [-16, 16] \times [-16, 16]$. This contains $\mathbf{O}$, but if v is pivotal for $A(2^k)$ there must still exist an occupied path from ∂R_0 to $\partial S(2^k)$, so that we still have (2.2) (even though $\mathbf{O} \in R_0$) if we define $\Xi(R_0)$ as $\{\partial R_0 \rightsquigarrow \partial S(\tau)\}$ and take $\hat{P}\{\Gamma(v, R_0)\} = 1$ for $|v| \leqq 16$.

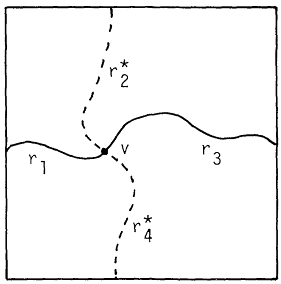

Fig. 3

We can now outline the main steps of the proof of Theorem 1. We first show that the probability of $A(2^k)$ is not changed by more than a bounded factor if $P\{v$ is occupied$\}$ is changed from p_c to p for every $v \in S(2^k) \backslash S(2^{k-3})$ [as long as $2^k \leqq L(p, \varepsilon_0)$].[3] We then define $\hat{P}_t$ as the product measure corresponding to

$$p(t, v) = \begin{cases} p & \text{for} \quad v \notin S(2^{k-3}), \\ tp + (1-t)p_c & \text{for} \quad v \in S(2^{k-3}). \end{cases} \tag{2.6}$$

for $0 \leqq t \leqq 1$. Thus $p'(t, v) = p - p_c$ for $v \in S(2^{k-3})$ and $=0$ otherwise. We next show that

$$\hat{P}_t\{\Xi(R(v))\} \leqq C_3 \hat{\pi}(t, 2^k) \tag{2.7}$$

for some constant C_3 independent of p, v, k, t. These first two steps are easy and were essentially proved in [15]. Combining these results with (2.2) yields, after dividing by $\hat{\pi}(t, 2^k)$,

$$\left| \frac{d}{dt} \log \hat{\pi}(t, 2^k) \right| \leqq C_3 |p - p_c| \left[1 + \sum_{j=3}^{k-5} \sum_{2^{j+1} < |v| \leqq 2^{j+2}} \hat{P}_t\{\Gamma(v, R(v))\} \right]. \tag{2.8}$$

Now the occurrence of $\Gamma(v, R(v))$ means that v is connected to $\partial R(v)$ by four paths, which have only the vertex v in common. Two of these paths, r_1 and r_3, are occupied while r_2^* and r_4^* are vacant (outside v). If the endpoints of r_1 and r_3 would lie on the left and right edge of $R(v)$, respectively, and r_2^* and r_4^* would have endpoints on the top and bottom edge of $R(v)$, respectively (see Fig. 3), then v would be pivotal for the event

$$C(R) := \{\exists \text{ an occupied horizontal crossing on } \mathcal{G} \text{ of } R\}.$$

The principal step in our proof is to show that the additional restriction on the endpoints of r_1, r_3, r_2^*, r_4^* does not lower the probability of $\Gamma(v, R)$ too much, i.e., that

$$\hat{P}_t\{\Gamma(v, R(v))\} \leqq C_4 \hat{P}_t\{v \text{ is pivotal for } C(R(v))\} \tag{2.9}$$

[3] This first step is a technicality. Unfortunately, the form of our main estimate (Lemma 4) forces us to treat vertices "near the boundary" of $S(2^k)$ differently from vertices in the "central part" of $S(2^k)$. The latter fact will continue to plague us throughout the paper

when $v \in S(2^{k-3})$, $2^k \leqq L(p, \varepsilon_0)$. Once we have this we can continue (2.8) with

$$\begin{aligned}\left|\frac{d}{dt}\log\hat{\pi}(t,2^k)\right| &\leqq C_5|p-p_c|\left[1+\sum_j\sum_v \hat{P}_t\{v \text{ is pivotal for } C(R(v))\}\right]\\ &\leqq C_5|p-p_c|\left[1+\sum_R\sum_{R(v)=R}\hat{P}_t\{v \text{ is pivotal for } C(R)\}\right]\\ &= C_5\left[1+\sum_R\left|\frac{d}{dt}\hat{\sigma}(R;1,t)\right|\right],\end{aligned} \tag{2.10}$$

with the slightly abusive notation,

$$\hat{\sigma}(R;1,t)=\hat{P}_t\{C(R)\}.$$

Note that the equality in the last step of (2.10) is just (2.1) applied to $A=C(R)$ and $B=$certain event. For the sake of argument take $p>p_c$. Then each $\hat{\sigma}(R;1,t)$ is increasing in t. Moreover, $\hat{P}_1=P_p$. Integration of (2.10) from $t=0$ to $t=1$ then gives

$$\left|\log\frac{\pi(p,2^k)}{\pi(p_c,2^k)}\right| \leqq C_6+\left|\log\frac{\pi(p,2^k)}{\hat{\pi}(0,2^k)}\right| \leqq C_7\left[1+\sum_R|\hat{\sigma}(R;1,1)-\hat{\sigma}(R;1,0)|\right]. \tag{2.11}$$

Finally, we note that the rectangles R which can occur as an $R(v)$ are of the form (2.4) (or rotations or reflections of these) and that $|l_1|$, and $|l_2|$ in (2.4) are at most 17. It follows that the right-hand side of (2.11) is at most

$$C_8\left[1+\sum_{j=3}^{k-5}|\hat{\sigma}((3.2^j,2^{j+1});1,1)-\hat{\sigma}((3.2^j,2^{j+1});1,0)|\right]. \tag{2.12}$$

Since $L(p)$ is chosen such that

$$0\leqq\hat{\sigma}(S(L(p));1,1)-\hat{\sigma}(S(L(p));1,0)\leqq\sigma((2L(p),2L(p));1,p)-\sigma((2L(p),2L(p));1,p_c)$$

lies somewhere strictly between 0 and 1, it stands to reason that for 2^j small with respect to $2^k \leqq L(p)$ the summand in (2.12) is also small. In fact, we believe that there exists a constant $\zeta>0$ such that

$$|\sigma((3.2^j,2^{j+1});1,p)-\sigma((3.2^j,2^{j+1};1,p_c)|\leqq C_9\left[\frac{2^j}{L(p)}\right]^\zeta. \tag{2.13}$$

If (2.13) would hold, then we would obtain from (2.11), (2.12) that

$$\left|\log\frac{\pi(p,2^k)}{\pi(p_c,2^k)}\right|\leqq C_{10} \quad\text{for}\quad 2^k\leqq L(p), \tag{2.14}$$

which is essentially Theorem 1. We do not quite prove (2.13), but only the weaker inequality (2.58). Nevertheless, this is good enough to yield (2.14), essentially by the indicated route.

Before turning to the nitty-gritty of the proof we assemble some basic tools. First, there exists some C_3 such that

$$\sigma((n,n);i,p_c,\mathscr{G})\geqq C_3 \quad\text{and}\quad \sigma^*((n,n);i,p_c,\mathscr{G})\geqq C_3, \quad i=1,2, \quad n\geqq 1 \tag{2.15}$$

(see [22] or [13, Theorem 5.1]). Now let $p>p_c$ again for the sake of argument. Then, since $\sigma((n,n);i,s,\mathscr{G})$ is increasing in s, we have from the definition of

$L(p)=L(p,\varepsilon_0)$ that

$$C_3 \leqq \sigma((n,n);i,s,\mathcal{G}) \leqq 1-\varepsilon_0 \quad \text{for} \quad p_c \leqq s \leqq p \quad \text{and} \quad 1 \leqq n < L(p,\varepsilon).$$

By virtue of (1.22) we then also have

$$\sigma^*((n,n);i,s,\mathcal{G}) \geqq \varepsilon_0, \qquad p_c \leqq s \leqq p, \qquad 1 \leqq n < L(p,\varepsilon_0).$$

These inequalities plus the Russo-Seymore-Welsh theorem [21–23; 13, Theorem 6.1] imply that for each $k \geqq 1$ there exists a $\delta_k = \delta_k(\varepsilon_0, C_3) > 0$ such that

$$\sigma((kn,n);1,s,\mathcal{G}) \geqq \delta_k \quad \text{and} \quad \sigma^*((kn,n);1,s,\mathcal{G}) \geqq \delta_k \tag{2.16}$$

for $p_c \leqq s \leqq p$, $n \leqq L(p,\varepsilon_0)$, and the same holds when the roles of the first and second coordinate are interchanged. Also, the same argument works for $p \leqq s \leqq p_c$. Even more, simple monotonicity arguments show that if $\hat{P}$ is any probability measure according to which the site v is occupied with probability $p(v)$ and all sites v are independent, then, if $p(v)$ lies between p_c and p for all v and $n \leqq L(p,\varepsilon_0)$, one has

$$\hat{P}\{\exists \text{ occupied horizontal crossing of } [0,kn]\times[0,n] \text{ on } \mathcal{G}\} \geqq \delta_k, \tag{2.17}$$

$$\hat{P}\{\exists \text{ vacant horizontal crossing of } [0,kn]\times[0,n] \text{ on } \mathcal{G}^*\} \geqq \delta_k, \tag{2.18}$$

and the same holds for vertical crossings of $[0,n]\times[0,kn]$. It also follows from this that

$$P\{\exists \text{ occupied circuit on } \mathcal{G} \text{ surrounding } (n,(k-1)n)^2 \text{ in } [0,kn]^2 \backslash (n,(k-1)n)^2\} \geqq \delta_k^4, \tag{2.19}$$

and similarly for vacant circuits on $\mathcal{G}^*$. By the same argument as used to prove (6) in [15] we now have

$$\hat{P}\{\mathbf{O} \rightsquigarrow \partial S(n)\} \quad \text{is decreasing in } n, \text{ but} \quad \hat{P}\{\mathbf{O} \rightsquigarrow \partial S(2n)\} \geqq C_4 \hat{P}\{\mathbf{O} \rightsquigarrow \partial S(n)\} \tag{2.20}$$

for a suitable $0 < C_4 < \infty$ independent of p, $\hat{P}$, and n, as long as $n \leqq L(p,\varepsilon_0)$ and

$$p(v) \text{ lies between } p_c \text{ and } p \text{ for all } v. \tag{2.21}$$

(We stress that $p < p_c$ is allowed here.) It is also easy to obtain from this the first two steps of our proof, namely, if $\hat{P}_t$ is defined by (2.6), then

$$\left|\log \frac{\hat{\pi}(0,2^k)}{\pi(p_c,2^k)}\right| \leqq C_5, \tag{2.22}$$

and (2.7) holds. Equation (2.22) follows for $p > p_c$ from

$$\pi(p_c,2^k) \leqq \hat{\pi}(0,2^k) \leqq \hat{\pi}(0,2^{k-3}) = \pi(p_c,2^{k-3}) \leqq C_4^{-3}\pi(p_c,2^k) \quad [\text{see } (2.20)].$$

[The equality here holds because $p(0,v)=p_c$ for $v \in S(2^{k-3})$.]

For $p < p_c$ similar estimates apply. Equation (2.7) follows by an argument similar to that for (4) in [14]. Consider the case with $\mathbf{O} \notin R = R(v) \subset S(2^k)$,

$$\begin{aligned}\hat{P}_t\{\mathbf{O} \rightsquigarrow \partial S(2^k)\} \geqq \hat{P}_t\{&\mathbf{O} \rightsquigarrow \partial R \text{ and } \partial R \rightsquigarrow \partial S(2^k) \\ &\text{and there exists an occupied circuit in } \tilde{R}\backslash R \\ &\text{and surrounding } R\},\end{aligned}$$

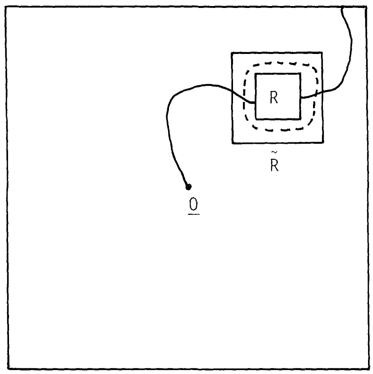

Fig. 4

where $\tilde{R}$ is any rectangle containing R but $\mathbf{O} \notin \tilde{R} \subset S(2^k)$ (see Fig. 4). Thus by the Harris-FKG inequality

$$\hat{P}_t\{\mathbf{O} \rightsquigarrow \partial S(2^k)\} \geqq \hat{P}_t\{\Xi(R)\} \cdot P_t\{\exists \text{ occupied circuit in } \tilde{R}\backslash R \text{ surrounding } R\}.$$

By (2.3)–(2.5) we can choose $\tilde{R}$ such that $\mathbf{O} \notin \tilde{R} \subset S(2^k)$ and such that $\tilde{R}\backslash R$ has width at least 2^j, so that the last factor is uniformly bounded away from 0 by (2.19). Thus (2.7) holds.

Finally, we fix ε_0. It follows from the proofs of Lemma 5.2, 5.3 in [13] or [2, Lemma 4.12 and Eq. (4.60)] (see also [19, Lemma 1]) that there exist $0 < C_6, C_7 < \infty$ and $\varepsilon_1 > 0$ with the following property. As soon as for some L

$$\sigma((L, 3L); 1, p, \mathscr{G}) \leqq \varepsilon_1 \quad \text{and} \quad \sigma((3L, L); 2, p, \mathscr{G}) \leqq \varepsilon_1, \tag{2.23}$$

then

$$P_p\{[-L, L]^2 \rightsquigarrow \partial S(kL)\} \leqq C_6 \exp(-C_7 k). \tag{2.24}$$

Similarly, if (2.23) holds with σ replaced by σ^*, then

$$P_p\{\exists \text{ vacant path on } \mathscr{G}^* \text{ from } [-L, L]^2 \text{ to } \partial S(kL)\} \leqq C_6 \exp(-C_7 k). \tag{2.25}$$

Finally, by the Russo-Seymour-Welsh lemma [21–23] or [13, Chap. 6] we can choose $0 < \varepsilon_0 \leqq \frac{1}{2} C_3$ [C_3 as in (2.15)] so small that $\sigma((L, L-1); 1, p, \mathscr{G}) \leqq \varepsilon_0$ implies (2.23), and similarly with σ replaced by σ^*. *For the remainder of the paper we fix* $\varepsilon_0 > 0$ *with this property*. By the definition of $L(p, \varepsilon_0)$ and (1.22), we then also have (2.24) for $L = L(p, \varepsilon_0)$ if $p < p_c$, and (2.25) for $L = L(p, \varepsilon_0)$ if $p > p_c$.

We now begin our proof of a strengthened version of (2.9), first only for $v = \mathbf{O}$ and $R(v)$ a square centered at $\mathbf{O}$ (see Lemma 4). This requires further definitions. Let r be a path on $\mathscr{G}$ from $\mathbf{O}$ to $\partial S(2^k)$ in $S(2^k)$, for some $k \geqq 2$. Assume that all vertices of r other than $\mathbf{O}$ are occupied. For the sake of argument assume that the endpoint of r lies on the right edge of $S(2^k)$, i.e., on $\{2^k\} \times [-2^k, 2^k]$. Let r' be the piece of r from its last intersection with the line $x = 2^{k-1} + 1$ to the right edge of $S(2^k)$ (see Fig. 5). r' is a crosscut of the strip $\mathscr{S}_R := [2^{k-1} + 1, 2^k] \times [-2^k, 2^k]$. Let $\mathscr{C} = \mathscr{C}(r, k)$ be the occupied component of r' on $\mathscr{G}$ in $\mathscr{S}_R$. Further, denote by $a = a(\mathscr{C})$ the lowest point of $\mathscr{C}$ on the right edge of $S(2^k)$. Note that a is not necessarily the

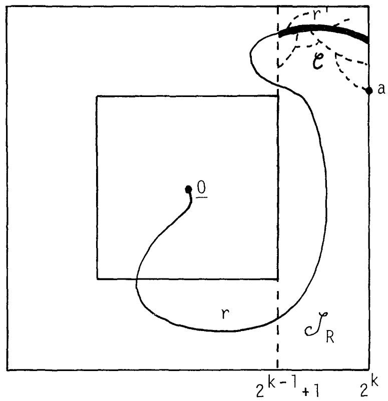

Fig. 5. r' is the boldly drawn piece of r; $\mathscr{C}$ consists of r' and the dashed parts

endpoint of r (cf. Fig. 5). In fact, one easily sees that $a=(a(1), a(2))$ has to be the right endpoint of the lowest occupied horizontal crossing $\mathscr{R}$ of $\mathscr{S}_R$ in $\mathscr{C}$. Here the lowest crossing of $\mathscr{S}_R$ in $\mathscr{C}$ is defined as follows. Let $\mathring{\mathscr{S}}_R$ be the interior of $\mathscr{S}_R$. For any crosscut $\tilde{r}$ or $\mathscr{S}_R$, connecting the left edge and right edge of $\mathscr{S}_R$, $\mathring{\mathscr{S}}_R \backslash \tilde{r}$ consist of an upper and lower component, $\mathscr{S}^+ = \mathscr{S}^+(\tilde{r})$ and $\mathscr{S}^- = \mathscr{S}^-(\tilde{r})$. $\mathscr{S}^+$ and $\mathscr{S}^-$ are those components with $[2^{k-1}+1, 2^k] \times \{2^k\}$ and $[2^{k-1}+1, 2^k] \times \{-2^k\}$ as parts of their boundary, respectively. $\mathscr{R}$ is that crossing of $\mathscr{S}_R$ which lies in $\mathscr{C}$ and for which $\mathscr{S}^-(\mathscr{R})$ is minimal. For the precise definition and existence proof of $\mathscr{R}$ we should go over to the planar modification of $\mathscr{G}$ as in Sect. 2.3 of [13] and apply Proposition 2.3 of [13]. (See also Lemma 1 of [12].) We do not go into further detail since this would only obfuscate the argument at this moment.

We use similar definitions if r ends on the top, left or bottom edge of $S(2^k)$. We define in these cases the component $\mathscr{C}(r, k)$ as the occupied component of the final piece of r in the strip $\mathscr{S}_T := [-2^k, 2^k] \times [2^{k-1}+1, 2^k]$, $\mathscr{S}_L := [-2^k, -2^{k-1}-1] \times [-2^k, 2^k]$ or $\mathscr{S}_B := [-2^k, 2^k] \times [-2^k, -2^{k-1}-1]$, respectively. $a(\mathscr{C})$ in these cases is the endpoint of the most "clockwise" crossing in $\mathscr{C}$ (i.e., the right most, highest and left most crossing, respectively). For a vacant crossing r^* we have similar definitions, obtained by replacing "occupied and $\mathscr{G}$" by "vacant and $\mathscr{G}^*$." The corresponding component and endpoint will be denoted by $\mathscr{C}^*(r^*, k)$ and $a^*(\mathscr{C}^*)$, respectively. We further use the notation

$$S(b, n) = [b(1)-n, b(1)+n] \times [b(2)-n, b(2)+n],$$

$$\mathring{S}(b, n) = (b(1)-n, b(1)+n) \times (b(2)-n, b(2)+n)$$

when $b=(b(1), b(2))$. $\mathring{S}(n) = \mathring{S}(\mathbf{O}, n)$.

We now define an (η, k)-*fence* for r, or rather for $\mathscr{C}$. We say that r, or $\mathscr{C}$, has an (η, k)-*fence* if all three of the following conditions hold (see Fig. 6):

> if t is any path on $\mathscr{G}$ from $\mathbf{O}$ to $\partial S(2^k)$ which lies in $\mathring{S}(2^k)$, except for its endpoint, and which has all its vertices other than $\mathbf{O}$ occupied, and corresponding component $\mathscr{C}(t, k)$, and if $\mathscr{C}(t, k) \cap \mathscr{C}(r, k) = \emptyset$, then $|a(\mathscr{C}(t, k)) - a(\mathscr{C}(r, k))| > 2\sqrt{\eta}\, 2^k$, (2.26)

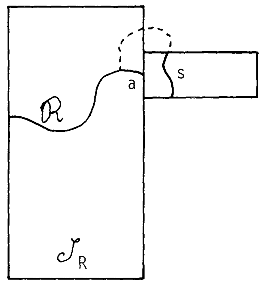

Fig. 6. The small rectangle of the right is $(a(1), a(1)+\sqrt{\eta}2^k] \times [a(2)-\eta 2^k, a(2)+\eta 2^k]$. The crossing s is connected to $\mathscr{C}$ by the dashed path

if r^* is any path on $\mathscr{G}^*$ from $\mathbf{O}$ to $\partial S(2^k)$ which lies in $\mathring{S}(2^k)$, except for its endpoint, and which has all its vertices other than $\mathbf{O}$ vacant and corresponding component $\mathscr{C}^*(r^*, k)$, then $|a^*(\mathscr{C}^*)-a(\mathscr{C}(r,k))|>2\sqrt{\eta}2^k$, (2.27)

there exists an occupied vertical crossing s on $\mathscr{G}$ of the rectangle $(a(1), a(1)+\sqrt{\eta}2^k] \times [a(2)-\eta 2^k, a(2)+\eta 2^k]$, which is connected to $\mathscr{C}(r,k)$ by an occupied path on $\mathscr{G}$ in $S(a, \sqrt{\eta}2^k)$. [Here $a=a(\mathscr{C})$.] (2.28)

[Equations (2.26) and (2.27) say that other components than $\mathscr{C}(r,k)$ cannot come too close to $a(\mathscr{C}(r,k))$.]

As we shall see, the existence of an (η, k)-fence is useful because it allows, with a probability bounded away from zero, a "nice" extension of r to an occupied path from $\mathbf{O}$ to $\partial S(2^{k+1})$. Note that the fence is defined in terms of $\mathscr{C}$ only, and does not explicitly refer to r. We can therefore talk about an occupied component $\mathscr{C}$ of $\mathscr{S}_R$ having a fence. We can also define an (η, k)-fence for a vacant component $\mathscr{C}^*$ of $\mathscr{S}_R$ on $\mathscr{G}^*$ by interchanging occupied and vacant everywhere in the above. Of course, if r^* is a path on $\mathscr{G}^*$ from $\mathbf{O}$ to the right edge of $S(2^k)$, with all its vertices except $\mathbf{O}$ vacant, then we say that r^* has an (η, k)-fence if the component $\mathscr{C}^*$ of the last piece of r^* crossing $\mathscr{S}_R$ has an (η, k)-fence. Finally, we define an (η, k)-fence for a path from $\mathbf{O}$ to $\partial S(2^k)$ which ends on the top edge of $S(2^k)$ by replacing $\mathscr{S}_R$ by the strip $\mathscr{S}_T$, and the lowest crossing in $\mathscr{C}$ by the right most crossing. Also, in (2.28) we now require a horizontal crossing of $[a(1)-\eta 2^k, a(1)+\eta 2^k] \times (a(2), a(2)+\sqrt{\eta}2^k]$ which is connected to $\mathscr{C}$ in $S(a, \sqrt{\eta}2^k)$. It is similar for paths ending on the left or bottom edge of $S(2^k)$.

We shall now show that for small η the probability that every crossing of $\mathscr{S}_R$ has an (η, k) fence is close to 1. More specifically we have the following lemma.

Lemma 2. *For each $\delta>0$ there exists an $\eta=\eta(\delta)>0$ such that, uniformly in $2^k \leqq L(p, \varepsilon_0)$ and $\hat{P}$ satisfying* (2.21), *we have*

$$\hat{P}\{\exists \text{ occupied horizontal crossing on } \mathscr{G} \text{ of } \mathscr{S}_R=[2^{k-1}+1, 2^k]\times[-2^k, 2^k] \text{ whose occupied component } \mathscr{C} \text{ in } \mathscr{S}_R \text{ does not have an } (\eta,k)\text{-fence}\} \leqq \delta. \tag{2.29}$$

Remark. The same statement holds with "occupied on $\mathscr{G}$" replaced by "vacant on $\mathscr{G}^*$."

Proof. Assume that there exists an occupied horizontal crossing on $\mathscr{S}_R$. Let $\mathscr{R}_1$ be the lowest such crossing and $\mathscr{C}_1$ its occupied component in $\mathscr{S}_R$. Let $a_1 = (a_1(1), a_1(2))$ be the endpoint of $\mathscr{R}_1$, and for fixed η consider the events

$E_j = E_j(\eta, k, \mathscr{R}_1) := \{\exists$ vertical crossings on $\mathscr{G}$ of the strips $[a_1(1) - \eta 2^{k+j},\ a_1(1) - \eta 2^{k+j-1}] \times [a_1(2) - \eta 2^{k+j},\ a_1(2) + \eta 2^{k+j}]$ and $[a_1(1) + \eta 2^{k+j-1},\ a_1(1) + \eta 2^{k+j}] \times [a_1(2) - \eta 2^{k+j},\ a_1(2) + \eta 2^{k+j}]$, as well as horizontal crossings of $[a_1(1) - \eta 2^{k+j},\ a_1(1) + \eta 2^{k+j}] \times [a_1(2) + \eta 2^{k+j-1},\ a_1(2) + \eta 2^{k+j}]$ and $[a_1(1) - \eta 2^{k+j},\ a_1(1) + \eta 2^{k+j}] \times [a_1(2) - \eta 2^{k+j},\ a_1(2) - \eta 2^{k+j-1}]$; all sites on these crossings outside the closure of $\mathscr{S}^-(\mathscr{R}_1)$ are occupied$\}$.

Figure 7 makes it evident that if E_j occurs for some $j \geqq 1$ with $2^j \leqq \eta^{-1/2}$, then also (2.28) occurs. The events E_j are independent of each other since they depend on annuli with disjoint interiors. Also, conditionally on $\mathscr{R}_1$ (and hence with a_1 also given), the probability of E_j occurring is at least $\delta_4^4 > 0$. This follows from (2.17) and the fact that conditioning on $\mathscr{R}_1 = \tilde{r}$ gives no information about any vertices outside the closure of $\mathscr{S}^-(\tilde{r})$ (see [13, Proposition 2.3]). Since there are[4] $\lfloor -(2\log 2)^{-1} \log \eta \rfloor$ values of j with $2^j \leqq \eta^{-1/2}$, we see that

$$\hat{P}\{\exists \text{ lowest occupied horizontal crossing } \mathscr{R}_1 \text{ of } \mathscr{S}_R, \text{ but (2.28) fails for its component } \mathscr{C}_1\} \leqq (1 - \delta_4^4)^{-C_3 \log \eta}. \tag{2.30}$$

Now assume that $\mathscr{R}_i, \mathscr{C}_i$, $1 \leqq i \leqq \sigma$, are given occupied horizontal crossings of $\mathscr{S}_R$, with their corresponding occupied components in $\mathscr{S}_R$. Assume further that $\mathscr{R}_i$

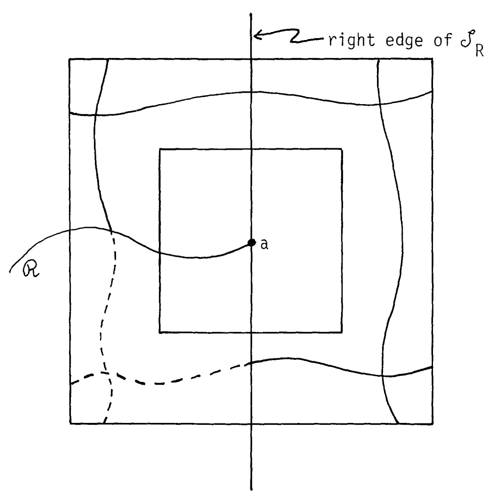

Fig. 7. The inner and outer square centered at a are $S(a, \eta 2^{k+j-1})$ and $S(a, \eta 2^{k+j})$, respectively. The solid parts of the four crossings in $S(a, \eta 2^{k+j}) \backslash S(a, \eta 2^{k+j-1})$ have to be occupied for E_j to occur

[4] $\lfloor a \rfloor$ denotes the largest integer $\leqq a$

is disjoint from $\mathscr{C}_j$ for $i \neq j$. Then, for $i \neq j$, $\mathscr{C}_i$ lies in $\mathscr{S}_R \backslash \mathscr{R}_j$, and hence belongs to exactly one of $\mathscr{S}^-(\mathscr{R}_j)$ or $\mathscr{S}^+(\mathscr{R}_j)$. Assume the $\mathscr{R}_i$ are ordered such that $\mathscr{C}_i \subset \mathscr{S}^-(\mathscr{R}_j)$ for $i<j$, so that $\mathscr{R}_\sigma$ is the highest crossing among the $\mathscr{R}_i$, $1 \leqq i \leqq \sigma$. If there exists still another occupied crossing of $\mathscr{S}_R$ in $\mathscr{S}_R \Big\backslash \bigcup_1^\sigma \mathscr{C}_i$, let $\mathscr{R}_{\sigma+1}$ be the lowest such crossing. Denote its endpoint on the right edge of $\mathscr{S}_R$ by $a_{\sigma+1}$, and its occupied component in $\mathscr{S}_R$ by $\mathscr{C}_{\sigma+1}$. Note that $\mathscr{C}_i \subset$ closure of $\mathscr{S}^-(\mathscr{R}_{\sigma+1})$ for $1 \leqq i \leqq \sigma$ in this case, and hence fixing $\mathscr{R}_i$ and $\mathscr{C}_i$ for $1 \leqq i \leqq \sigma$ and fixing $\mathscr{R}_{\sigma+1} = \tilde{r}$ still gives no information about vertices outside the closure of $\mathscr{S}^-(\tilde{r})$. We can therefore repeat the argument for (2.30) to obtain

$$\begin{aligned}&\hat{P}\{\exists \text{ lowest occupied crossing } \mathscr{R}_{\sigma+1} \text{ of } \mathscr{S}_R \text{ in } \mathscr{S}^+(\mathscr{R}_\sigma),\\ &\quad \text{but (2.28) fails for its component } \mathscr{C}_{\sigma+1} | \mathscr{R}_i, \mathscr{C}_i, 1 \leqq i \leqq \sigma\}\\ &\leqq (1-\delta_4^4)^{-C_3 \log \eta}. \end{aligned} \tag{2.31}$$

Consequently,

$$\begin{aligned}&\hat{P}\{\exists \text{ any occupied crossing } \mathscr{R} \text{ of } \mathscr{S}_R \text{ such that (2.28) fails for its component } \mathscr{C}\}\\ &\quad \leqq \hat{P}\{\exists \text{ more than } \varrho \text{ disjoint occupied horizontal crossings of } \mathscr{S}_R\}\\ &\qquad + \varrho(1-\delta_4^4)^{-C_3 \log \eta}. \end{aligned} \tag{2.32}$$

By virtue of [28, Corollary 3.10] [or by a direct argument similar to (2.31)] the first term in the right-hand side of (2.32) is at most

$$\begin{aligned}&[P\{\exists \text{ at least one occupied horizontal crossing of } \mathscr{S}_R\}]^\varrho\\ &\quad \leqq [1 - P\{\exists \text{ vacant vertical crossing on } \mathscr{G}^* \text{ of } [2^{k-1}+2, 2^k-1] \times [-2^k, 2^k]\}]^\varrho\\ &\qquad [\text{by (1.22)}] \leqq (1-\delta_5)^\varrho \quad [\text{by (2.18)}]. \end{aligned}$$

Combining this with (2.32), and taking first ϱ large, then η small, we see that we can obtain

$$\begin{aligned}&\hat{P}\{\exists \text{ any occupied horizontal crossing of } \mathscr{S}_R\\ &\quad \text{whose occupied component } \mathscr{C} \text{ in } \mathscr{S}_R \text{ does not satisfy (2.28)}\}\\ &\leqq \tfrac{1}{4}\delta. \end{aligned} \tag{2.33}$$

We next show that a minor strengthening of the derivation of (2.33) actually yields (2.29). Let $\mathscr{R}$ be an occupied horizontal crossing of $\mathscr{S}_R$, with occupied component $\mathscr{C}$ in $\mathscr{S}_R$ and endpoint a on the right edge of $\mathscr{S}_R$. Assume that one of the events E_j occurs with $2\eta^{-1/2} \leqq 2^{j-1} \leqq \eta^{-3/4}$. It is easy to see (compare Fig. 7 again) that in this case there can be no vacant path r^* on $\mathscr{G}^*$, nor any occupied path s on $\mathscr{G}$ which does not intersect $\mathscr{C}$, which connects $\partial S(2^{k-1})$ to $\partial S(2^k)$ [in $S(2^k)$] and whose endpoint b lies inside $S(a, \eta 2^{j+k-1}) \supset S(a, 2\sqrt{\eta}2^k)$ on the counter-clockwise side of a on $\partial S(2^k)$. To guarantee (2.27) for $\mathscr{R}$ we must also make sure that there is no $a^*(\mathscr{C}^*)$ on the clockwise side of a in $\partial S(2^k) \cap S(a, 2\sqrt{\eta}2^k)$. Equation (2.26) is a similar requirement for $a(\mathscr{C}(t,k))$ with $\mathscr{C}(t,k)$ disjoint from $\mathscr{C}$. We restrict ourselves to (2.27). Assume that r^* is a vacant connection from $\partial S(2^{k-1})$ to $\partial S(2^k)$ with endpoint a^*. Define E_j^* as the analogue of E_j with "a and occupied" replaced by "a^* and vacant" (and possibly $\mathscr{S}_R$ by one of the other three strips $\mathscr{S}_T, \mathscr{S}_L, \mathscr{S}_B$). Then the occurrence of E_j^* for some j^* with

$$2\eta^{-1/2} \leqq 2^{j^*-1} \leqq \eta^{-3/4}$$

prevents a from lying in the clockwise direction of a^* and inside

$$S(a^*, \eta 2^{j^*+k-1}) \supset S(a^*, 2\sqrt{\eta}2^k).$$

Thus, in this case, a^* cannot lie in $S(a, 2\sqrt{\eta}2^k)$ in the counterclockwise direction of a. It now follows that the left-hand side of (2.29) is bounded by (2.33) plus

> $\hat{P}\{\exists$ an occupied component $\mathscr{C}$ of one of the strips $\mathscr{S}_R, \mathscr{S}_T, \mathscr{S}_L$ or $\mathscr{S}_B$ for whose most clockwise connection from $\partial S(2^{k-1})$ to $\partial S(2^k)$ no E_j with $2\eta^{-1/2} \leqq 2^{j-1} \leqq \eta^{-3/4}$ occurs$\} + \hat{P}\{\exists$ a vacant component $\mathscr{C}^*$ of one of the strips $\mathscr{S}_R, \mathscr{S}_T, \mathscr{S}_L$ or $\mathscr{S}_B$ for whose most clockwise connection from $\partial S(2^{k-1})$ to $\partial S(2^k)$ no E_j^* with $2\eta^{-1/2} \leqq 2^{j-1} \leqq \eta^{-3/4}$ occurs$\}$.

These last probabilities can be made $\leqq \delta/2$ by choosing η small, for the same reasons as given for (2.33). The lemma now follows.

It will be useful to have the following simple generalization of the Harris-FKG inequality.

Lemma 3. *Let A and D be increasing events, and B and E decreasing events. Assume that A, B, D, E depend only on vertices in the finite sets $\mathscr{A}, \mathscr{B}, \mathscr{D}$, and $\mathscr{E}$, respectively. If*

$$\mathscr{A} \cap \mathscr{B} = \mathscr{A} \cap \mathscr{E} = \mathscr{B} \cap \mathscr{D} = \emptyset, \tag{2.34}$$

then

$$P\{A \cap B \mid D \cap E\} \geqq P\{A \cap B\} \tag{2.35}$$

for any measure P according to which all vertices are independently occupied or vacant.

Proof. First, we prove that

$$P\{A \cap D \cap E\} \geqq P\{A\} P\{D \cap E\}, \tag{2.36}$$

by conditioning on the configuration in $\mathscr{E}$. For each choice of such a configuration the Harris-FKG inequality gives $P\{A \cap D | E\} \geqq P\{A|E\} P\{D|E\}$. Since A is independent of E, by (2.34), this is equivalent to (2.36). Essentially, the same argument shows that

$$\begin{aligned} P\{A \cap B \cap D \cap E\} &= P\{B \cap E \mid A \cap D\} P\{A \cap D\} \\ &\geqq P\{B | A \cap D\} P\{E | A \cap D\} P\{A \cap D\} \\ &= P\{B\} P\{A \cap D \cap E\} \quad \text{[by (2.34)]} \\ &\geqq P\{A\} P\{B\} P\{D \cap E\} \quad \text{[by (2.36)]} \\ &\geqq P\{A \cap B\} P\{D \cap E\} \quad \text{(by Harris-FKG)}. \end{aligned}$$

This is equivalent to (2.35). □

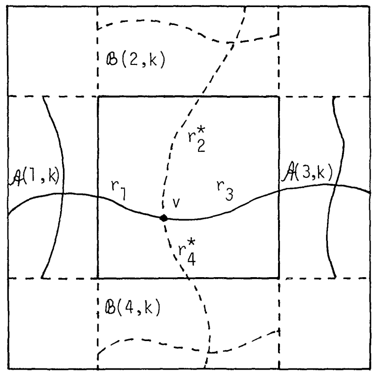

Fig. 8. Illustration of $\Delta(v, S(2^k))$. The inner and outer square are $S(2^{k-1})$ and $S(2^k)$, respectively. The solidly drawn paths are occupied, while the dashed paths are vacant

We come to a fundamental estimate. Let $v \in S(2^{k-1})$. We then define the subevent $\Delta(v, S(2^k))$ of $\Gamma(v, S(2^k))$ as follows (see Fig. 8): $\Delta(v, S(2^k)) := \Gamma(v, S(2^k))$ occurs with the four paths r_1, r_3, r_2^*, r_4^* satisfying the four additional requirements

$$r_1 \cap S(2^k) \backslash \mathring{S}(2^{k-1}) \subset \mathscr{A}(1,k) := [-2^k, -2^{k-1}] \times [-2^{k-1}, 2^{k-1}],$$

$$r_3 \cap S(2^k) \backslash \mathring{S}(2^{k-1}) \subset \mathscr{A}(3,k) := [2^{k-1}, 2^k] \times [-2^{k-1}, 2^{k-1}],$$

$$r_2^* \cap S(2^k) \backslash \mathring{S}(2^{k-1}) \subset \mathscr{B}(2,k) := [-2^{k-1}, 2^{k-1}] \times [2^{k-1}, 2^k],$$

$$r_4^* \cap S(2^k) \backslash S(2^{k-1}) \subset \mathscr{B}(4,k) := [-2^{k-1}, 2^{k-1}] \times [-2^k, -2^{k-1}].$$

In addition, we require that there exist occupied vertical crossings on $\mathscr{G}$ of $\mathscr{A}(i,k)$, $i=1,3$, and vacant horizontal crossings on $\mathscr{G}^*$ of $\mathscr{B}(i,k)$, $i=2,4$.

Lemma 4. *There exists a constant C_0 such that for all $\hat{P}$ satisfying* (2.21) *and all $2^k \leqq L(p, \varepsilon_0)$, we have*

$$\hat{P}\{\Gamma(\mathbf{O}, S(2^k))\} \leqq C_0 \hat{P}\{\Delta(\mathbf{O}, S(2^k))\}. \tag{2.37}$$

Proof. Unfortunately, we have to introduce yet other events. Assume that $\Gamma(\mathbf{O}, S(2^j))$ occurs, so that there exist four paths, r_1, r_3 on $\mathscr{G}$, and r_2^*, r_4^* on $\mathscr{G}^*$, from $\mathbf{O}$ to $\partial S(2^j)$ in $S(2^j)$, with the additional properties listed before. Let r_i' be the last piece of r_i which crosses one of the strips $\mathscr{S}_R$, $\mathscr{S}_T$, $\mathscr{S}_L$ or $\mathscr{S}_B$ and $\mathscr{C}_i$ the associated component $\mathscr{C}(r_i', i)$, $i=1,3$. Similarly, for $\mathscr{C}_2^*$ and $\mathscr{C}_4^*$. Define

> $\Delta(S(2^j), \eta) := \Gamma(\mathbf{O}, S(2^j))$ occurs and the paths r_i, r_{1+i}^* and components $\mathscr{C}_i, \mathscr{C}_{1+i}^*$, $i=1,3$ can be chosen such that each one has an (η, j)-fence.

We begin by proving the existence of some constant $C_4(\eta)$ for which

$$\hat{P}\{\Delta(S(2^j), \eta)\} \leqq C_4(\eta) \hat{P}\{\Delta(\mathbf{O}, S(2^{j+2}))\} \tag{2.38}$$

[again uniformly in $\hat{P}$ satisfying (2.21) and $2^j \leqq L(p, \varepsilon_0)$]. We prove (2.38) by extending the paths r_1, r_3, r_2^*, r_4^*, when $\Delta(S(2^j), \eta)$ occurs, as indicated in Fig. 9. We want to show that such an extension [for which $\Delta(\mathbf{O}, S(2^{j+2})$ occurs] is possible, with a conditional probability bounded away from 0. First partition the perimeter of $S(2^j)$ into approximately $8\eta^{-1}$ disjoint intervals of length $\leqq \eta 2^j$. We can then pick four such intervals I_1, I_3, I_2^*, I_4^* such that

$$\hat{P}\{A(S(2^j),\eta) \text{ occurs with } a_i := a(\mathscr{C}_i) \in I_i \text{ and } a_{1+i}^* := a(\mathscr{C}_{1+i}^*) \in I_{1+i}^*,\ i=1,3\} \geqq C_5 \eta^4 \hat{P}\{A(S(2^j),\eta)\}. \tag{2.39}$$

Define in addition

$\Theta(S(2^j),\eta,I) := \Gamma(\mathbf{O}, S(2^j))$ occurs with $a_i \in I_i$ and $a_{1+i}^* \in I_{1+i}^*$ and paths r_i, r_{1+i}^* with endpoints a_i, a_{i+1}^* and components $\mathscr{C}_i$, $\mathscr{C}_{1+i}^*$ for which (2.26) and (2.27) hold with j in place of $k, i=1,3$.

Also write $A(S(2^j),\eta,I)$ for the event in braces in the left-hand side of (2.39). Note that on $\Gamma(\mathbf{O}, S(2^j))$ the components $\mathscr{C}_1$, $\mathscr{C}_3$, $\mathscr{C}_2^*$, $\mathscr{C}_4^*$ must be pairwise disjoint because r_1, r_3 separate r_2^*, r_4^*. It is important to realize that on $\Theta(S(2^j),\eta,I)$, (2.26) and (2.27) for the specific $\mathscr{C}_1, \mathscr{C}_2^*, \mathscr{C}_3, \mathscr{C}_4^*$ imply that there cannot exist any occupied or closed path from $\mathbf{O}$ to $\partial S(2^j)$ whose corresponding component $\mathscr{C}$ or $\mathscr{C}^*$ in one of the strips $\mathscr{S}_R$, $\mathscr{S}_T$, $\mathscr{S}_L$, $\mathscr{S}_B$ has its point $a(\mathscr{C})$ or $a^*(\mathscr{C}^*)$ in I_i, unless $\mathscr{C} = \mathscr{C}_i$. Similarly, for I_{1+i}^*. In particular, I_1, I_3, I_2^*, I_4^* must all be different. Furthermore, the components $\mathscr{C}_i$, $\mathscr{C}_{1+i}^*$ and their points a_i, a_{1+i}^* are uniquely determined by the configuration in $S(2^j)$, for any configuration in $\Theta(S(2^j),\eta,I)$ (recall that the I's and I^*'s are fixed already). We must even have on $\Theta(S(2^j),\eta,I)$

$$S(a_i, \sqrt{\eta}2^j) \quad \text{and} \quad S(a_{1+i}^*, \sqrt{\eta}2^j), \quad i=1,3 \quad \text{are all disjoint}. \tag{2.40}$$

For any configuration for which $\Theta(S(2^j),\eta,I)$ occurs, $A(S(2^j),\eta,I)$ occurs only if in addition the following events occur for $i=1,3$ ($2\eta < \sqrt{\eta}$ here):

$D_i := \{\exists$ occupied crossing in the short direction of a $2\eta 2^j$ by $\sqrt{\eta}2^j$ rectangle R_i outside $S(2^j)$ but with a_i in the center of its short side; this crossing is connected by an occupied path in $S(a_i, \sqrt{\eta}2^j)$ to $\mathscr{C}_i\}$,

and

$E_{1+i} := \{\exists$ vacant crossing in the short direction of a $2\eta 2^j$ by $\sqrt{\eta}2^j$ rectangle R_{1+i}^* outside $S(2^j)$ but with a_{1+i}^* in the center of its short side; this crossing is connected by a vacant path in $S(a_{1+i}^*, \sqrt{\eta}2^j)$ to $\mathscr{C}_{1+i}^*\}$.

We shall now apply Lemma 3 to the complement of $S(2^j)$, conditionally on the configuration inside $S(2^j)$, whenever this configuration, call it ω_j, lies in $\Theta(S(2^j),\eta,I)$. We make the following choices for the various events. Assume without loss of generality that a_1, a_3, a_2^*, a_4^* are numbered such that they occur in the order a_1, a_2^*, a_3, a_4^* when $\partial S(2^j)$ is traversed clockwise.

$A_1 := \{\exists$ occupied path s_1 on $\mathscr{G}$ outside $S(2^j)$ starting next to the side of R_1 containing a_1, and ending on the left edge of $S(2^{j+2})$ inside a given corridor K_1 of width $\eta 2^j$; $K_1 \cap S(2^{j+2}) \backslash \mathring{S}(2^{j+1}) \subset \mathscr{A}(1, j+2)$; in addition, there exists an occupied vertical crossing of $\mathscr{A}(1, j+2)\}$;

A_3 is obtained by replacing the index 1 by 3 and the left edge of $S(2^{j+2})$ by its right edge. See the definition of D_i for R_i and the definition of Δ for $\mathscr{A}(i, j+2)$. $A = A_1 \cap A_3$,

$$\mathscr{A} = K_1 \cup K_3 \cup \mathscr{A}(1, j+2) \cup \mathscr{A}(3, j+2).$$

B_2 and B_4 are defined very much like A_1, this time requiring vacant connections from an interval around a_2^* and a_4^* through corridors K_2^* and K_4^* to the top and bottom edge of $S(2^{j+2})$, respectively. In addition, we require a vacant horizontal crossing of $\mathscr{B}(2, j+2)$ for B_2 and of $\mathscr{B}(4, j+2)$ for B_4. We take $B = B_2 \cap B_4$ and

$$\mathscr{B} = K_2^* \cup K_4^* \cup \mathscr{B}(2, j+2) \cup \mathscr{B}(4, j+2).$$

Furthermore, we take $D = D_1 \cap D_3$, $E = E_3 \cap E_4$,

$$\mathscr{D} = S(a_1, \sqrt{\eta} 2^j) \cup S(a_3, \sqrt{\eta} 2^j),$$

$$\mathscr{E} = S(a_2^*, \sqrt{\eta} 2^j) \cup S(a_4^*, \sqrt{\eta} 2^j).$$

The corridors K_1, K_2^*, K_3, K_4^* contain R_1, R_2^*, R_3, and R_4^*, respectively. Furthermore, they have to be chosen disjoint from each other and such that

$$(K_1 \cup K_3) \quad \text{is disjoint from} \quad \mathscr{E} \cup \mathscr{B}(2, j+2) \cup \mathscr{B}(4, j+2),$$

$$(K_2^* \cup K_4^*) \quad \text{is disjoint from} \quad \mathscr{D} \cup \mathscr{A}(1, j+2) \cup \mathscr{A}(3, j+2).$$

Taking into account (2.40) it is not hard to convince oneself that this can be done in the manner indicated in Fig. 9 by making η small (so that the width $\eta 2^j$ of the corridors is small), and keeping the corridors inside $\mathring{S}(2^{j+1})$ until their last leg out to $\partial S(2^{j+2})$. For such a choice of the corridors (2.34) will hold [recall (2.40)]. Therefore, by Lemma 3

$$\begin{aligned} &\hat{P}\{A \cap B \mid \text{configuration in } S(2^j) = \omega_j,\, D \cap E\} \\ &\geq \hat{P}\{A \cap B \mid \text{configuration in } S(2^j) = \omega_j\} \end{aligned} \tag{2.41}$$

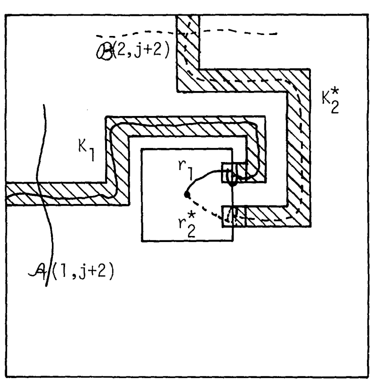

Fig. 9. The inner and outer square are $S(2^j)$ and $S(2^{j+2})$, respectively. Solid (dashed) paths are occupied (vacant). The small squares around a_1 and a_2^* (the endpoints of r_1 and r_2^*) are $S(a_1, \sqrt{\eta} 2^j)$ and $S(a_2^*, \sqrt{\eta} 2^j)$, respectively. The corridors K_1 and K_2^* are hatched; they have width $\eta 2^j$

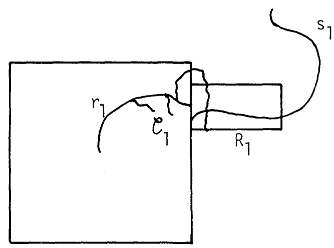

Fig. 10. The large square is $S(2^j)$ and the small rectangle on the right is R_1

for any configuration ω_j in $S(2^j)$ for which $\Theta(S(2^j), \eta, I)$ occurs. Since $\mathscr{A}$ and $\mathscr{B}$ are disjoint from each other and disjoint from $S(2^j)$, the right-hand side of (2.41) equals $\hat{P}\{A \cap B\} = \hat{P}\{A\}\hat{P}\{B\}$, and this in turn is bounded below by some $C_6 = C_6(\eta) > 0$, by virtue of (2.17), (2.18). This, finally, gives for any fixed configuration ω_j in $\Theta(S(2^j), \eta, I)$

$$\hat{P}\{\text{configuration on } S(2^j) \text{ equals } \omega_j,\ A \cap B \cap D \cap E\}$$
$$\geqq C_6 \hat{P}\{\text{configuration in } S(2^j) \text{ equals } \omega_j,\ D \cap E\}. \tag{2.42}$$

It is easy to see (see Fig. 10) that if $A \cap B \cap D \cap E$ occurs, then $\mathscr{C}_i$ is connected to s_i via some paths in $S(a_i, \sqrt{\eta}2^j)$. Thus, $r_i, \mathscr{C}_i$ together with the latter paths in $S(a_i, \sqrt{\eta}2^j)$ and s_i contain a path from $\mathbf{O}$ to $\partial S(2^{j+2})$ which is occupied, except possibly at $\mathbf{O}$, and which stays in $\mathscr{A}(i, j+2)$ outside $\mathring{S}(2^{j+1})$ $(i=1, 3)$. A similar statement holds for r^*_{1+i}, $\mathscr{C}^*_{1+i}$, and s^*_{1+i}. Thus the event on the left-hand side of (2.42) is contained in $\Delta(\mathbf{O}, S(2^{j+2}))$. We also remarked already that given ω_j in $\Theta(S(2^j), \eta, I)$, $\Lambda(S(2^j), \eta, I)$ occurs only if $D \cap E$ occur. Therefore, summation of (2.42) over ω_j in $\Theta(S(2^j), \eta, I)$ gives

$$\hat{P}\{\Delta(\mathbf{O}, S(2^{j+2})\} \geqq C_6 \hat{P}\{\Lambda(S(2^j), \eta, I)\} \geqq C_5 C_6 \eta^4 \hat{P}\{\Lambda(S(2^j), \eta)\}.$$

We finally proved (2.38). Compared with (2.38), it is trivial to prove that

$$\hat{P}\{\Delta(v, S(2^j))\} \leqq C_7 \hat{P}\{\Delta(v, S(2^{j+1}))\} \tag{2.43}$$

for some $1 < C_7 < \infty$ which does *not* depend on j, $\hat{P}$, v or η [again under (2.21), $v \in S(2^{j-1})$, and $2^j \leqq L(p, \varepsilon_0)$]. Equation (2.43) merely requires the extension of four paths from v to $\partial S(2^j)$ to paths up to $\partial S(2^{j+1})$ plus insertion of crossings of $\mathscr{A}(i, j+1)$ and $\mathscr{B}(1+i, j+1)$, $i=1, 3$. Figure 11 illustrates how this can be done for one path. Lemma 3 and (2.17), (2.18) then guarantee that

$$\hat{P}\{\Delta(\mathbf{O}, S(2^{j+1})) \mid \Delta(\mathbf{O}, S(2^j))\}$$

is bounded away from zero, or equivalently, that (2.43) holds.

We shall need below the following inequality, which follows immediately from iterating (2.43):

$$\hat{P}\{\Delta(\mathbf{O}, S(2^j)\} \geqq C_8 C_7^{-j}. \tag{2.44}$$

Now for the proof of (2.37). Fix $0 < \delta \leqq (32 C_7^2)^{-1}$, and then η such that (2.29) holds for this δ. Obviously,

$$\hat{P}\{\Gamma(\mathbf{O}, S(2^k))\} \leqq \hat{P}\{\Gamma(\mathbf{O}, S(2^{k-2}))\}$$
$$= \hat{P}\{\Lambda(S(2^{k-2}), \eta)\} + P\{\Gamma(\mathbf{O}, S(2^{k-2})) \backslash \Lambda(S(2^{k-2}), \eta)\}. \tag{2.45}$$

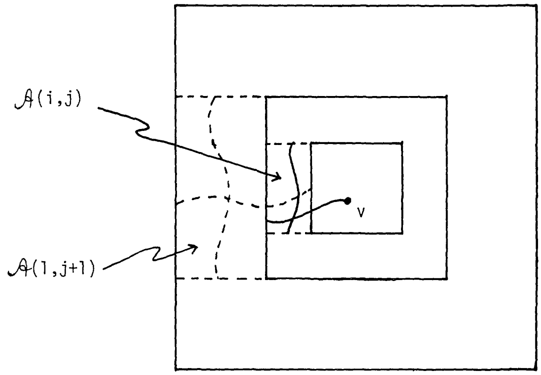

Fig. 11. The three squares are (starting from the inside) $S(2^{j-1})$, $S(2^j)$, and $S(2^{j+1})$. The solidly drawn occupied path exist when $\Delta(v, S(2^j))$ occurs, and when the dashed paths are added we obtain $\Delta(v, S(2^{j+1}))$

If $\Gamma(\mathbf{O}, S(2^{k-2}))\backslash A(S(2^{k-2}), \eta)$ occurs, then there exist four paths r_1, r_2^*, r_3, r_4^* from $\mathbf{O}$ to $\partial S(2^{k-2})$ with the properties listed in the definition of Γ; however, at least one of the components $\mathscr{C}_i$, $\mathscr{C}_{1+i}^*$ does not have an $(\eta, k-2)$-fence. In particular, $\Gamma(\mathbf{O}, S(2^{k-3}))$ still occurs, while either the event in the left-hand side of (2.29) with k replaced by $k-2$ occurs, or a similar event with $\mathscr{S}_R$ replaced by $\mathscr{S}_T$, $\mathscr{S}_L$ or $\mathscr{S}_B$, or occupied replaced by vacant. By virtue of Lemma 2 and symmetry the probability of any of these events occurring is at most 8δ. In addition, these events depend on $S(2^{k-2})\backslash S(2^{k-3})$ only, and therefore are independent of $\Gamma(\mathbf{O}, S(2^{k-3}))$. Therefore, the last term in (2.45) is at most

$$(8\delta)\hat{P}\{\Gamma(\mathbf{O}, S(2^{k-3}))\},$$

so that

$$\hat{P}\{\Gamma(\mathbf{O}, S(2^k))\} \leqq \hat{P}\{A(S(2^{k-2}), \eta)\} + 8\delta\hat{P}\{\Gamma(\mathbf{O}, S(2^{k-3}))\}.$$

We can iterate this argument to obtain

$$\hat{P}\{\Gamma(\mathbf{O}, S(2^k))\} \leqq \sum_{k/2 < j \leqq k-2} (8\delta)^{k-2-j}\hat{P}\{A(S(2^j), \eta)\} + (8\delta)^{-3+k/2}.$$

Now use (2.38), (2.43), and (2.44) to conclude that the right-hand side is at most

$$\sum_{k/2 < j \leqq k-2} (8\delta)^{k-2-j} C_4(\eta)(C_7)^{k-2-j} P\{\Delta(\mathbf{O}, S(2^k))\}$$
$$+ (8\delta)^{-3+k/2} C_8^{-1} C_7^k \hat{P}\{\Delta(\mathbf{O}, S(2^k))\}.$$

Equation (2.37) follows from our choice of δ. □

Corollary 3. *There exist constants $C_1 - C_4$ such that*

$$C_1\hat{P}\{\Delta(v, S(2^{j+2}))\} \leqq P\{\Gamma(v, R)\} \leqq C_2\hat{P}\{\Delta(v, S(2^{j+2}))\}, \tag{2.46}$$

$$C_3\hat{P}\{\Delta(v, S(2^{j+2}))\} \leqq \hat{P}\{\Delta(v, S(2^{j+3}))\} \leqq C_2\hat{P}\{\Delta(v, S(2^{j+2}))\}, \tag{2.47}$$

uniformly for all $\hat{P}$ satisfying (2.21), $v \in S(2^j)$, and any rectangle R for which $S(2^{j+1}) \subset R \subset S(2^{j+3})$, $2^{j+3} \leqq L(p, \varepsilon_0)$. For the measures P_p and $v \in S(2^j)$, $2^{j+3} \leqq L(p, \varepsilon_0)$ we even have

$$C_4 P_p\{\Delta(\mathbf{O}, S(2^{j+2}))\} \leqq P_p\{\Delta(v, S(2^{j+2}))\} \leqq C_2 P_p\{\Delta(\mathbf{O}, S(2^{j+2}))\}. \tag{2.48}$$

Remark. In Lemma 8 we shall obtain a much better version of (2.48).

Proof. From the definitions of Γ and Δ, it is clear that

$$\hat{P}\{\Gamma(v,R_1)\} \geqq \hat{P}\{\Gamma(v,R_2) \quad \text{if} \quad v \in R_1 \subset R_2, \tag{2.49}$$

and for any l

$$\hat{P}\{\Delta(v,S(2^l))\} \leqq \hat{P}\{\Gamma(v,S(2^l))\}, \qquad v \in S(2^{l-1}). \tag{2.50}$$

If we write $\hat{P}^v$ for the measure $\hat{P}$ translated by v, then $S(2^j) \subset R - v \subset S(2^{j+4})$, and the monotonicity properties (2.49), (2.50) show that

$$\begin{aligned}
\hat{P}\{\Gamma(v,R)\} &= \hat{P}^v\{\Gamma(\mathbf{O},R-v)\} \\
&\geqq \hat{P}^v\{\Gamma(\mathbf{O},S(2^{j+4}))\} \geqq \hat{P}^v\{\Delta(\mathbf{O},S(2^{j+4}))\} \\
&\geqq C_7^{-4}\hat{P}^v\{\Delta(\mathbf{O},S(2^j))\} \qquad [\text{by (2.43)}] \\
&\geqq (C_7^4 C_0)^{-1}\hat{P}^v\{\Gamma(\mathbf{O},S(2^j))\} \quad [\text{by (2.37) for } \hat{P}^v] \\
&= (C_7^4 C_0)^{-1}\hat{P}\{\Gamma(v,S(v,2^j))\} \\
&\geqq (C_7^4 C_0)^{-1}\hat{P}\{\Gamma(v,S(2^{j+2}))\} \\
&\geqq (C_7^4 C_0)^{-1}\hat{P}\{\Delta(v,S(2^{j+2}))\}.
\end{aligned}$$

This proves the left-hand inequality in (2.46). For the right-hand inequality we write

$$\hat{P}\{\Gamma(v,R))\} \leqq \hat{P}\{\Gamma(v,S(2^{j+1}))\} \leqq C_0\hat{P}\{\Delta(v,S(2^{j+1}))\} \leqq C_0 C_7 \hat{P}\{\Delta(v,S(2^{j+2}))\},$$

using (2.49), (2.37), and (2.43) in succession.

The left-hand inequality in (2.47) is just (2.43). For the right-hand inequality use (2.50) and (2.46):

$$\hat{P}\{\Delta(v,S(2^{j+3}))\} \leqq \hat{P}\{\Gamma(v,S(2^{j+3}))\} \leqq C_2\hat{P}\{\Delta(v,S(2^{j+2}))\}.$$

Finally, for (2.48) we use $P_p^v = P_p$ to get

$$\begin{aligned}
P_p\{\Delta(\mathbf{O},S(2^{j+2}))\} &\leqq P_p\{\Gamma(\mathbf{O},S(2^{j+2}))\} = P_p\{\Gamma(v,S(v,2^{j+2}))\} \\
&\leqq C_2 P_p\{\Delta(v,S(2^{j+2}))\} \quad [\text{by (2.46)}],
\end{aligned}$$

and

$$\begin{aligned}
P_p\{\Delta(v,S(2^{j+2}))\} &\leqq P_p\{\Gamma(v,S(2^{j+2}))\} = P_p\{\Gamma(\mathbf{O},S(-v,2^{j+2}))\} \\
&\leqq C_2 P_p\{\Delta(\mathbf{O},S(2^{j+2}))\} \quad [\text{by (2.46)}]. \quad \square
\end{aligned}$$

Next we need an analogue of Lemma 4 for connections from ∂R to the outside of R, rather than to a point in R. For a rectangle $R \subset S(2^k)$ with interior $\mathring{R}$, define

> $\tilde{\Gamma}(S(2^k),R) = \{\exists$ two occupied paths r_1 and r_3 on $\mathscr{G}$ from the left and right edge of $\partial S(2^k)$, respectively, to ∂R, and two vacant paths r_2^* and r_4^* on $\mathscr{G}^*$ from the top and bottom edge of $\partial S(2^k)$, respectively, to ∂R; these paths are pairwise disjoint, and expect for their endpoint on ∂R lie in $S(2^k)\backslash R$; $r_1 \cup r_3$ separates r_2^* from r_4^* in $S(2^k)\backslash \mathring{R}\}$.

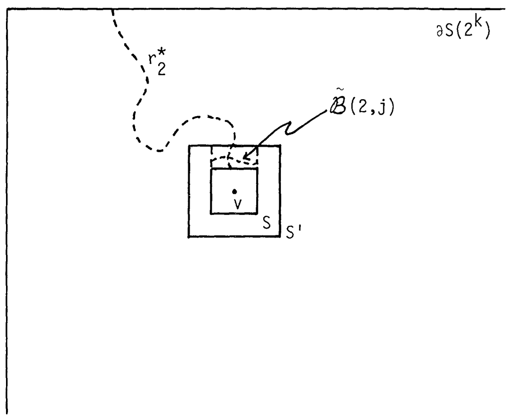

Fig. 12. Two of the required paths for $\tilde{\Delta}(S(2^k), S)$ with $S=S(v, 2^j)$

We define the analogue of Δ for the present situation only if R is a square of the form $S(v, 2^j) \subset S(2^{k-2})$. In this case let $S' = S(v, 2^{j+1})$, and for the analogues of $\mathscr{A}$ and $\mathscr{B}$ in the definition of Δ take the following strips in $S' \setminus \mathring{S}(r, 2^j)$:

$$\tilde{\mathscr{A}}(1,j) = [v(1) - 2^{j+1}, v(1) - 2^j] \times [v(2) - 2^j, v(2) + 2^j],$$
$$\tilde{\mathscr{A}}(3,j) = [v(1) + 2^j, v(1) + 2^{j+1}] \times [v(2) - 2^j, v(2) + 2^j],$$
$$\tilde{\mathscr{B}}(2,j) = [v(1) - 2^j, v(1) + 2^j] \times [v(2) + 2^j, v(2) + 2^{j+1}],$$
$$\tilde{\mathscr{B}}(4,j) = [v(1) - 2^j, v(1) + 2^j] \times [v(2) - 2^{j+1}, v(2) - 2^j].$$

The proper analogue for $\Delta(v, S(2^j))$ is now

> $\tilde{\Delta}(S(2^k), S(v, 2^j)) := \tilde{\Gamma}(S(2^k), S(v, 2^j)$ occurs and the four paths r_i, r^*_{1+i}, $i=1,3$, are such that $r_i \cap (S' \setminus \mathring{S}(v, 2^j)) \subset \tilde{\mathscr{A}}(i,j)$ and $r^*_{1+i} \cap (S' \setminus \mathring{S}(v, 2^j)) \subset \tilde{\mathscr{B}}(1+i, j)$, $i=1,3$; in addition, there exist occupied vertical crossings on $\mathscr{G}$ of $\tilde{\mathscr{A}}(i,j)$ and vacant horizontal crossings on $\mathscr{G}^*$ of $\tilde{\mathscr{B}}(1+i,j)$, $i=1,3$.

Figure 12 illustrates the definition in part.

Lemma 5. *There exists a constant* C_0 *such that for all* $\hat{P}$ *satisfying* (2.21) *and all* $2^k \leqq L(p, \varepsilon_0)$, $S(v, 2^j) \subset S(2^{k-2})$ *one has*

$$\hat{P}\{\tilde{\Gamma}(S(2^k), S(v, 2^j))\} \leqq C_0 \hat{P}\{\tilde{\Delta}(S(2^k), S(v, 2^j))\}. \tag{2.51}$$

Proof. We only state explicitly what we mean by an (η, j) fence for r_1, an occupied path on $\mathscr{G}$ from the left edge of $S(2^k)$ to $\partial S(v, 2^j)$. As before this depends on the location of the endpoint of r_1; for the sake of argument we take it again on the right edge of $S(v, 2^j)$. Now we take for the analogue of $\mathscr{S}_R$, $\tilde{\mathscr{S}}_R$, the right half of the annulus $S(v, 2^{j+1} - 1) \setminus \mathring{S}(v, 2^j)$ (see Fig. 13). We break $\partial \tilde{\mathscr{S}}_R$ up into the following four pieces:

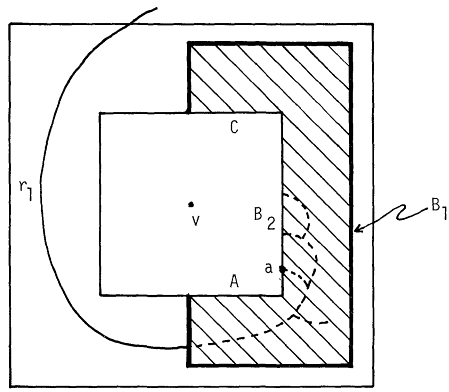

Fig. 13. The inner and outer square are $S(v, 2^j)$ *and* $S' = S(v, 2^{j+1})$, respectively. $\tilde{\mathscr{S}}_R$ is hatched; the part B_1 of its boundary is boldly drawn. r' and some pieces of $\mathscr{C}$ are dashed

$B_1 :=$ the union of right half of $\partial S(v, 2^{j+1}-1)$ from $(v(1), v(2)+2^{j+1}-1)$ to $(v(1), v(2)-2^{j+1}+1)$, with the two vertical segments $\{v(1)\} \times [v(2)-2^{j+1}+1,\ v(2)-2^j]$ and $\{v(1)\} \times [v(2)+2^j,\ v(2)+2^{j+1}-1]$.

$$A = [v(1),\ v(1)+2^j] \times \{v(2)-2^j\},$$

$$B_2 = \{v(1)+2^j\} \times [v(2)-2^j,\ v(2)+2^j],$$

$$C = [v(1),\ v(1)+2^j] \times \{v(2)+2^j\}.$$

Note that $A \cup B_2 \cup C$ is just the right half of $\partial S(v, 2^j)$. For r' we take the piece of r_1 from its last intersection with B_1 till its endpoint on the right edge of $S(v, 2^j)$, B_2. $\mathscr{C} = \mathscr{C}(r_1, j)$ is defined as the occupied component of r' in $\tilde{\mathscr{S}}_R$.

r' is a crosscut of $\mathring{\tilde{\mathscr{S}}}_R$ connecting B_1 with B_2. Again any such crosscut $\tilde{r}$ divides $\mathring{\tilde{\mathscr{S}}}_R \backslash \tilde{r}$ into two components $\tilde{\mathscr{S}}^-(\tilde{r})$ and $\tilde{\mathscr{S}}^+(\tilde{r})$, which have A and C as part of their boundary, respectively. In $\mathscr{C}$ there exists a crosscut $\mathscr{R}$ from B_1 to B_2 for which $\tilde{\mathscr{S}}^-(\mathscr{R})$ is minimal, by virtue of Proposition 2.3 in [13]. $a = a(\mathscr{C})$ is taken to be the endpoint of $\mathscr{R}$ on B_2. We say that r_1 or $\mathscr{C}$ has an *(η, j)-fence* if all three of the following conditions hold:

if t is any occupied path on $\mathscr{G}$ from $\partial S(2^k)$ to $\partial S(v, 2^j)$ which lies in $\mathring{S}(2^k) \backslash S(v, 2^j)$, except for its endpoints, and with corresponding component $\mathscr{C}(t, j)$, and if $\mathscr{C}(t, j) \cap \mathscr{C}(r_1, j) = \emptyset$, then $|a(\mathscr{C}(t, j)) - a(\mathscr{C}(r_1, j))| > 2\sqrt{\eta}\, 2^j$, (2.52)

if r^* is any vacant path on $\mathscr{G}^*$ from $\partial S(2^k)$ to $\partial S(v, 2^j)$ which lies in $\mathring{S}(2^k) \backslash S(v, 2^j)$, except for its endpoints, and with the corresponding vacant component $\mathscr{C}^*(r^*, j)$, then $|a^*(\mathscr{C}^*(r^*, j)) - a(\mathscr{C}(r_1, j))| > 2\sqrt{\eta}\, 2^j$, (2.53)

there exists an occupied vertical crossing s on $\mathscr{G}$ of the rectangle $[a(1) - \eta 2^j,\ a(1)] \times [a(2) - \sqrt{\eta}\, 2^j,\ a(2) + \sqrt{\eta}\, 2^j] \cap S(v, 2^j)$, which is connected to $\mathscr{C}(r_1, k)$ by an occupied path on $\mathscr{G}$ in $S(a, \sqrt{\eta}\, 2^j)$. (2.54)

In (2.54) $a = a(\mathscr{C}(r_1, j))$, while in (2.53) $\mathscr{C}^*$ and a^* are, as before, defined by changing "occupied and $\mathscr{G}$" to "vacant and $\mathscr{G}^*$" on the appropriate places in the definition of $\mathscr{C}$ and a.

We can now copy the proofs of Lemmas 2 and 4 without essential changes to obtain Lemma 5. Only instead of (2.45) we work "outwards" instead of inwards, i.e., we replace (2.45) by

$$\begin{aligned}\hat{P}\{\tilde{\Gamma}(S(2^k), S(v, 2^j)\} &\leqq \hat{P}\{\tilde{\Gamma}(S(2^k), S(v, 2^{j+2}))\} \\ &= \hat{P}\{\tilde{\Lambda}(S(v, 2^{j+2}), \eta)\} + \hat{P}\{\tilde{\Gamma}(S(2^k), S(v, 2^{j+2}))\backslash \tilde{\Lambda}(S(v, 2^{j+2}), \eta)\}\end{aligned}$$

with an obvious analogue $\tilde{\Lambda}$ of Λ. □

Lemma 6. *There exist constants C_1 and C_2 such that*

$$\begin{aligned}C_1 \hat{P}\{\Gamma(v, S(2^k))\} &\leqq \hat{P}\{\tilde{\Gamma}(S(2^k), S(v, 2^j))\} \hat{P}\{\Gamma(v, S(v, 2^j))\} \\ &\leqq C_2 \hat{P}\{\Delta(v, S(2^k))\},\end{aligned} \tag{2.55}$$

uniformly in $\hat{P}$ satisfying (2.21) *and* $v \in S(2^{k-3})$, $S(v, 2^j) \subset S(2^{k-2})$, $2^k \leqq L(p, \varepsilon_0)$.

Proof. By (2.46) and (2.43),

$$\hat{P}\{\Gamma(v, S(2^k))\} \leqq C_2 \hat{P}\{\Delta(v, S(2^{k-1}))\} \leqq C_2 C_7 \hat{P}\{\Delta(v, S(2^k))\}.$$

Now $\Delta(v, S(2^k))$ requires the existence of two appropriate paths r_1 and r_3 from v to the left and right edge of $S(2^k)$, respectively, and two other paths r_2^* and r_4^* to the top and bottom edge, respectively. It is easy to see that these paths all have to reach $\partial S(v, 2^{j-1})$ first [so that $\Gamma(v, S(v, 2^{j-1}))$ occurs] and then continue to connect $\partial S(v, 2^j)$ with $\partial S(2^k)$ [so that also $\tilde{\Gamma}(S(2^k), S(v, 2^j))$ occurs]. Therefore,

$$\begin{aligned}\hat{P}\{\Delta(v, S(2^k))\} &\leqq \hat{P}\{\Gamma(v, S(v, 2^{j-1})) \text{ and } \tilde{\Gamma}(S(2^k), S(v, 2^j))\} \\ &= P\{\Gamma(v, S(v, 2^{j-1}))\} P\{\tilde{\Gamma}(S(2^k, S(v, 2^j))\},\end{aligned}$$

since $\Gamma(v, S(v, 2^{j-1}))$ and $\Gamma(S(2^k), S(v, 2^j))$ depend on vertices in $S(v, 2^{j-1}))$ and the complement of $S(v, 2^j)$, respectively. The left-hand inequality of (2.55) now follows from

$$\hat{P}\{\Gamma(v, S(v, 2^{j-1}))\} \leqq C_2 C_1^{-1} \hat{P}\{\Gamma(v, S(v, 2^j))\}, \tag{2.56}$$

which, in turn, follows for $j \leqq k$ from two applications of (2.46) [once with $R = S(v, 2^{j-1})$ and once with $R = S(v, 2^j)$, and $\hat{P}$ replaced by $\hat{P}^v$].

As for the right-hand inequality in (2.55), observe that if $\tilde{\Lambda}(S(2^k), (S(v, 2^j))$ occurs, then there exists an occupied path r_1' on $\mathscr{G}$ from the left edge of $S(v, 2^j)$ to the left edge of $S(v, 2^k)$. $r_1' \cap (S(v, 2^{j+1}) \backslash \mathring{S}(v, 2^j)) \subset \tilde{\mathscr{A}}(1, j)$, and in addition there exists an occupied vertical crossing s_1' on $\mathscr{G}$ of $\tilde{\mathscr{A}}(1, j)$. Note that, s_1' necessarily intersects r_1' in $\tilde{\mathscr{A}}(1, j)$. If also $\Delta(v, S(v, 2^{j-1}))$ [this is the translate by v of $\Delta(\mathbf{O}, S(2^{j-1}))$ occurs], then there exists a path r_1'' in $S(v, 2^{j-1})$ from v to the left edge of $S(v, 2^{j-1})$ which stays in $v + \mathscr{A}(1, j-1)$ outside $\mathring{S}(v, 2^{j-2})$; r_1'' is occupied, except possibly in v. Finally, there exists an occupied vertical crossing s_1'' of $v + \mathscr{A}(1, j-1)$, which necessarily intersects r_1'' in $v + \mathscr{A}(1, j-1)$. Now recall that $\tilde{\mathscr{A}}(1, j) = v + \mathscr{A}(1, j+1)$ (see Fig. 14). Therefore, an occupied horizontal crossing c_1 on $\mathscr{G}$ of $v + [-2^{j+1}, -2^{j-2}] \times (-2^{j-2}, 2^{j-2})$ will connect s_1' and s_1''. If such a c_1 exists, then

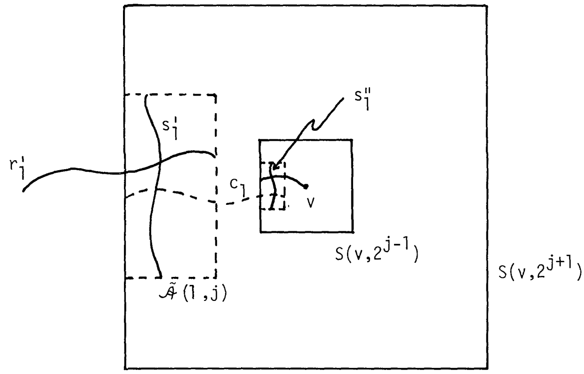

Fig. 14. Connecting r_1'' to r_1' by the dashed path c_1

there exists a path r_1 from v to the left edge of $S(2^k)$ which consists of pieces of r_1'', s_1'', c_1, s_1', and r_1'. Since

$$\hat{P}\{\exists \text{ occupied horizontal crossing on } \mathcal{G} \text{ of } v+[-2^{j+1}, -2^{j-2}]\times(-2^{j-2}, 2^{j-2})\} \geqq \delta_5$$

[by (2.17)] we obtain from Lemma 3 that

$$\hat{P}\{\exists \text{ path } r_1 \text{ on } \mathcal{G} \text{ from } v \text{ to the left edge of } S(2^k) \text{ which is occupied except possibly at } v|\tilde{\Delta}(S(2^k), S(v,2^j)) \text{ and } \Delta(v, S(v, 2^{j-1}))\} \geqq \delta_5 .$$

A slight extension of the above argument also allows us to construct a path r_3 from v to the right edge of $S(2^k)$ and two paths r_2^* and r_4^* to the top and bottom edges of $S(2^k)$, respectively. This yields [again by Lemma 3 and (2.17), (2.18)],

$$\hat{P}\{\Delta(v, S(2^k)) \,|\, \tilde{\Delta}(S(2^k), S(v, 2^j)) \text{ and } \Delta(v, S(v, 2^{j-1}))\} \geqq \delta_5^4 .$$

Finally, then

$$\hat{P}\{\Delta(v, S(2^k))\} \geqq \delta_5^4 \hat{P}\{\tilde{\Delta}(S(2^k), S(v, 2^j))\} \hat{P}\{\Delta(v, S(v, 2^{j-1}))\}, \tag{2.57}$$

which implies the second inequality of (2.55) [by (2.51) and (2.37) for $\hat{P}^v$]. □

The next lemma is the promised weakened version of (2.13).

Lemma 7. *Assume $p_0(v)$ and $p_1(v)$ both satisfy* (2.21) *and $p_1(v) \geqq p_0(v)$ for all v. Let $\hat{P}_t$ be the product measure with*

$$\hat{P}_t\{v \text{ occupied}\} = p_t(v) := t p_1(v) + (1-t) p_0(v) .$$

Then for some constants C_1, $\zeta > 0$ (independent of p, $p_0(\cdot)$, $p_1(\cdot)$).

$$\int_0^1 dt \sum_{v \in S(2^{j-2})} (p_1(v) - p_0(v)) \hat{P}_t\{\Gamma(v, S(2^j))\} \leqq C_1 2^{-\zeta(k-j)} \quad \text{for all} \quad 2^{j+1} \leqq 2^k \leqq L(p, \varepsilon_0) . \tag{2.58}$$

If $p_0(w)=p_c$ *and* $p_1(w)=p_c$ *for all* w, *or vice versa, then for all* $2^{j+1}\leqq 2^k\leqq L(p,\varepsilon_0)$

$$2^{2j}\left|\int_{p_c}^{p} ds P_s\{\Gamma(\mathbf{O},S(2^j))\}\right| \asymp \left|\int_{p_c}^{p} ds \sum_{v\in S(2^{j-2})} P_s\{\Gamma(v,S(2^j))\}\right|$$
$$\leqq C_1 2^{-\zeta(k-j)}|\sigma((2^k,2^k);1,p)-\sigma((2^k,2^k);1,p_c)|\,. \qquad (2.59)$$

Proof. Note that $\hat{P}_t(\Gamma(v,S(2^j))$ depends only on the $p_i(w)$ for $w\in S(2^j)$ and for (2.58) we may therefore change the $p_i(\cdot)$ outside $S(2^j)$ in any way which is compatible with (2.21). We leave $p_i(\cdot)$ unchanged in $\mathring{S}(2^{j+1})$, but change it outside $\mathring{S}(2^{j+1})$ so that it becomes periodic with periods $(0,2^{j+2})$ and $(2^{j+2},0)$. We may therefore assume that

$$p_i(v)=p_i(v+(n_1 2^{j+2},n_2 2^{j+2}))\,,\qquad n_i\in\mathbb{Z}\,.$$

Of course, if $p_0(w)=p_c$, $p_1(w)=p$ for all w, then no change will be performed in the p_i and we have $\hat{P}_t=P_{p(t)}$ with $p(t)=tp+(1-t)p_c$.

Recall that $C(S(2^j))$ denotes the event

$$\{\exists \text{ occupied horizontal crossing on } \mathscr{G} \text{ of } S(2^j)\}\,.$$

With $\hat{P}_t$ fixed as in the last paragraph we shall prove for $j\leqq k-10$, $v\in S(2^{j-2})$,

$$2^{\zeta(k-j)}\hat{P}_t\{\Delta(v,S(2^j))\}\leqq C_9\textstyle\sum^* \hat{P}_t\{v+2^{j+2}\mathbf{n} \text{ is pivotal for } C(S(2^k))\}\,, \qquad (2.60)$$

where $\sum^*$ in (2.60) denotes the sum over the integral vectors $\mathbf{n}=(n_1,n_2)$ with

$$S(2^{j+2}\mathbf{n},2^{j+1})=[n_1 2^{j+2}-2^{j+1},n_1 2^{j+2}+2^{j+1}]$$
$$\times[n_2 2^{j+2}-2^{j+1},n_2 2^{j+2}+2^{j+1}]\subset S(2^{k-3})\,.$$

Equation (2.60) will imply our lemma because by Russo's formula [compare (2.10)],

$$\sum_{v\in S(2^{j-2})}[p_1(v)-p_0(v)]\textstyle\sum^*\hat{P}_t\{v+2^{j+2}\mathbf{n} \text{ is pivotal for } C(S(2^k))\}$$
$$\leqq \frac{d}{dt}\hat{P}_t\{C(S(2^k))\}\,.$$

Thus (2.46) and (2.60) show that the left-hand side of (2.58) is at most

$$C_2C_9 2^{-\zeta(k-j)}[\hat{P}_1\{C(S(2^k))\}-\hat{P}_0\{C(S(2^k))\}]\,. \qquad (2.61)$$

This implies (2.58) if $j\leqq k-10$. For $k-10<j<k$ (2.58) is automatically valid for large enough C_1, because the left-hand side of (2.58) is bounded as can be seen from Russo's formula. Indeed, by (2.46),

$$\sum_{v\in S(2^{j-2})}(p_1(v)-p_0(v))\hat{P}_t\{\Gamma(v,S(2^j))\}$$
$$\leqq C_2\sum_{v\in S(2^{j-2})}(p_1(v)-p_0(v))\hat{P}_t\{\Delta(v,S(2^j))\}$$
$$\leqq C_2\sum_{v\in S(2^{j-2})}(p_1(v)-p_0(v))\hat{P}_t\{v \text{ is pivotal for } C(S(2^j))\}$$
$$\leqq C_2\frac{d}{dt}\hat{P}_t\{C(S(2^j))\} \qquad (2.62)$$

[compare (2.10)].

The equivalence in (2.59) follows for $j<k$ from (2.46) and (2.48) (with j replaced by $j-2$). The inequality in (2.59) follows for $j \leqq (k-11)$ from (2.61) with k replaced by $k-1$ [recall that $\sigma((2^k, 2^k); 1, p) = P_1\{C(S(2^{k-1}))\}$]. For $k-11<j<k$ the inequality in (2.59) again follows, by taking C_1 large enough, from (2.62) with $C(S(2^j))$ replaced by $C(S(2^{k-1}))$. [This requires $k-1-j$ applications of (2.43).]

We turn to the proof of (2.60). For the remainder we take $j \leqq k-10$. As in the second inequality of (2.62), the sum in the right-hand side of (2.60) is at least

$$\begin{aligned}
&\sum{}^* \hat{P}_t\{\Delta(v+2^{j+2}\mathbf{n}, S(2^k))\} \\
&\quad \geqq C_2^{-1} \sum{}^* \hat{P}_t\{\Gamma(v+2^{j+2}\mathbf{n}, S(v+2^{j+2}\mathbf{n}, 2^{j+1}))\} \\
&\qquad \times \hat{P}_t\{\tilde{\Gamma}(S(2^k), S(v+2^{j+2}\mathbf{n}, 2^{j+1}))\} \quad \text{[by (2.55)]} \\
&\quad = C_2^{-1} \hat{P}_t\{\Gamma(v, S(v, 2^{j+1}))\} \\
&\qquad \times \sum{}^* \hat{P}_t\{\tilde{\Gamma}(S(2^k), S(v+2^{j+2}\mathbf{n}, 2^{j+1}))\} \quad \text{(by periodicity)}.
\end{aligned}$$

Since for $v \in S(2^{j-2})$,

$$\hat{P}_t\{\Gamma(v, S(2^{j+1}))\} \geqq C_1 \hat{P}_t\{\Delta(v, S(2^j))\} \quad \text{by (2.46)},$$

we only have to prove

$$\begin{aligned}
&\sum{}^* \hat{P}_t\{\tilde{\Gamma}(S(2^k), S(v+2^{j+2}\mathbf{n}, 2^{j+1}))\} \\
&\quad \geqq C_{10} 2^{\zeta(k-j)}, \quad j \leqq k-10, \quad v \in S(2^{j-2}).
\end{aligned} \tag{2.63}$$

Equation (2.63) can be proved in the same way as Lemma 8.2 of [13]. We shall indicate why this is so, but leave the details to the reader. Let $\mathcal{R}$ denote the lowest occupied horizontal crossing on $\mathcal{G}$ of $S(2^k)$, if it exists.[5] The probability that $\mathcal{R}$ exists and lies entirely in the strip $T := [-2^k, 2^k] \times [-2^{k-4}, 2^{k-4}]$ is bounded away from 0. To see this note that $\mathcal{R}$ will exist and lie in T if there exists a vacant horizontal crossing r^* on $\mathcal{G}^*$ of $[-2^{-k}, 2^k] \times [-2^{k-4}, 0)$ and r^* is connected by a vacant path on $\mathcal{G}^*$ in $S(2^k)$ to the bottom edge of $S(2^k)$ and if in addition there is an occupied crossing on $\mathcal{G}$ of $[-2^k, 2^k] \times [0, 2^{k-4}]$. The $\hat{P}_t$-probability of all these events simultaneously is at least δ_{64}^3 by (2.17), (2.18).

Now assume that $\{\mathcal{R} = r_0\}$ occurs for some path r_0 in T from the left to the right edge of T. $\mathring{S}(2^k) \setminus r_0$ consists then of two components $S^-(r_0)$ and $S^+(r_0)$, with the bottom and top edge of $S(2^k)$ in their boundary, respectively. In this situation each vertex w of r_0 is connected by a path r^* on $\mathcal{G}^*$ to the bottom edge of $S(2^k)$ and with r^* minus its endpoints contained in $S^-(r_0)$, and r^* vacant, except at w. This follows from Proposition 2.2 in [13] [recall that r_0 is the *lowest* occupied crossing of $S(2^k)$]. If R is any rectangle with $w \in \mathring{R} \subset R \subset S(2^k)$, then there exist two occupied paths r_1, r_3 from ∂R to the left and right edge of $\partial S(2^k)$, respectively (these are pieces of $\mathcal{R} = r_0$), and there exists a vacant path from ∂R to the bottom edge of $S(2^k)$ (this is a piece of r^*). Thus, if there is in addition a vacant path from ∂R to the top edge of $S(2^k)$, then $\tilde{\Gamma}(S(2^k), R)$ occurs for the rectangle R.

Now assume that in addition to $\{\mathcal{R} = r_0\}$ there is a path s^* on $\mathcal{G}^*$ from some $a \in r_0$ to the top edge of $S(2^k)$, and that $s^* \setminus \{a\} \subset S^+(r_0)$, and s^* is vacant except at a.

[5] Again we are somewhat cavalier in not bringing in the planar modification $\mathcal{G}_{pl}$ for constructing the lowest crossing (or the left most crossing below)

Then there is a leftmost such path (again by Proposition 2.3 of [13]). Write $\mathcal{S}^*$ for the leftmost such path. Arguing as above, and using the fact that the vertices in $S^+(r_0)$ are independent of $\{\mathcal{R}=r_0\}$ we see that

$$\hat{P}_t\{\mathcal{S}^* \text{ exists and lies in } [0,2^{k-4}]\times[-2^k,2^k]\,|\,\mathcal{R}=r_0\}\geqq \delta_{64}^3$$

(see [13, Lemma 7.4] for similar arguments).

If r_0 is a fixed horizontal crossing of T on $\mathcal{G}$ and $s^*\subset[0,2^{k-4}]\times[-2^k,2^k]$ a fixed path on $\mathcal{G}^*$ from some point a on r_0 to the top edge of $S(2^k)$ in $S^+(r_0)$ (except for the point a), then $S^+(r)\backslash s^*$ again has two components. Write $S'=S'(r_0,s^*)$ for the component which contains the upper right-hand corner, $(2^k,2^k)$, of $S(2^k)$, in its boundary (see Fig. 15).

Denote the piece of r_0 from a to the right edge of $S(2^k)$ by r. If now $S(v+2^{j+2}\mathbf{n},2^{j+1})$ is a square which intersects r, in a point w say, and

$$S(2^{j+2}\mathbf{n},2^{j+1})\subset S(2^{k-3}), \tag{2.64}$$

and if there exists a vacant path t^* on $\mathcal{G}^*$ which lies in S' (except for its endpoints) and connects a point in $S(v+2^{j+2}\mathbf{n},2^{j+1})$ with s^*, then $\tilde{\Gamma}(S(2^k),S(v+2^{j+2}\mathbf{n},2^{j+1})$ occurs. Also (2.64) assures that this square is then counted in $\sum^*$ in (2.60). The existence of such a t^* is independent of the event $\{\mathcal{R}=r_0,\ \mathcal{S}^*=s^*\}$, because by Proposition 2.3 in [13], the occupancies of the vertices in $S'(r_0,s^*)$ are independent of the event $\{\mathcal{R}=r_0,\ \mathcal{S}^*=s^*\}$ (cf. Lemma 7.4 of [13] again). Note that if s^* intersects $S(v+2^{j+2}\mathbf{n},2^{j+1})$ then we don't need t^* at all; $\tilde{\Gamma}(S(2^k),S(v+2^{j+2}\mathbf{n},2^{j+1}))$ automatically occurs in this situation. Also, since we took $v\in S(2^{j-2})$, $j\leqq k-10$, and $a\in(r_0\cap s^*)\subset[-2^{k-4},2^{k-4}]^2$, (2.64) will hold for all $\mathbf{n}$ for which $S(v+\mathbf{n}2^{j+2},2^{j+1})$ intersects $S(a,3.2^l)$, provided we restrict ourselves to

$$j\leqq l-4,\qquad l\leqq k-6. \tag{2.65}$$

Finally, for such an l set $Y(v,\mathbf{n},a,l,r_0,s^*)=1$ if $S(v+2^{j+2}\mathbf{n},2^{j+1})$ intersects r_0 and $S(a,3.2^l)$, and if there exists a vacant path t^* as above, which in addition lies in the square $S(a,3.2^l)$; set $Y(v,\mathbf{n},a,l,r_0,s^*)=0$ otherwise. The condition on the existence of t^* is taken as fulfilled whenever s^* itself intersects $S(v+2^{j+2}\mathbf{n},2^{j+1})$. Define

$$Z(l)=\min\hat{E}_t\left\{\sum_{\mathbf{n}} Y(v,\mathbf{n},a,l,r_0,s^*)\right\}, \tag{2.66}$$

where $\hat{E}_t$ denotes expectation with respect to $\hat{P}_t$ and the min is over all permissible choices of $\hat{P}_t$, periodic with periods $(2^{j+2},0)$ and $(0,2^{j+2})$, $v\in S(2^{j-2})$, $r_0\subset T$, $a\in r_0$,

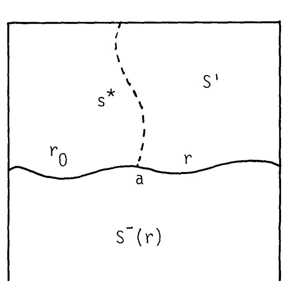

Fig. 15

$S(a, 3.2^l) \subset S(2^k)$, and s^* a vacant connection in $S^+(r)$ from a to the top edge of $S(2^k)$. Note that the last condition is a relaxation of $s^* \subset [0, 2^{k-4}] \times [-2^k, 2^k]$. The stronger condition on s^* is only used to guarantee (2.64) whenever $Y(v, \mathbf{n}, a, l, r_0, s^*) = 1$ with an l satisfying (2.65); for estimating Z it is, however, not useful to restrict s^* so much and relaxing the restrictions on s^* moves the min in (2.66) in the right direction. We have now arrived at the situation of Lemma 8.2 of [13]. In exactly the same way as in that lemma we can now show

$$Z(l) \geqq (1 + C_{11}) Z(l-3), \tag{2.67}$$

as long as l satisfies (2.65). Moreover, $Z(l) > 0$ for all $l \geqq j+4$, since we get a contribution of size 1 to (2.66) from the square $S(v + \mathbf{n}2^{j+2}, 2^{j+1})$ which contains a. It is now immediate from (2.67) by iteration, and the fact that $Z(l)$ increases in l, that

$$\begin{aligned} &\textstyle\sum^* \hat{P}_t\{\tilde{\Gamma}(S(2^k), S(v + 2^{j+2}\mathbf{n}, 2^{j+1}))\} \\ &\quad \geqq \min_{r_0, s^*} \hat{P}\{\mathcal{R} = r_0,\ \mathcal{S}^* = s^*\} \textstyle\sum^* P_t\{Y(v, \mathbf{n}, a, k-6, r_0, s^*) = 1 \mid \mathcal{R} = r_0,\ \mathcal{S}^* = s^*\} \\ &\quad \geqq C_{12} Z(k-6) \geqq C_{13}(1 + C_{11})^{(k-j)/3}. \end{aligned}$$

[The conditioning on $\{\mathcal{R} = r, \mathcal{S}^* = s^*\}$ has no influence, since the Y's are defined in terms of vertices in $S'(r_0, s^*)$, and hence independent of $\{\mathcal{R} = r_0,\ \mathcal{S} = s^*\}$, as observed above.] Thus (2.63) and (2.60) hold. □

It remains to put our lemmas together to complete the *proof of Theorem 1*. Let $n \leqq L(p, s_0)$ and choose the largest k with $2^k \leqq n$. It suffices to prove Theorem 1 with n replaced by 2^k, by virtue of (2.20). Also $\pi(p, 2^k) = \hat{\pi}(1, 2^k)$, and by virtue of (2.22) we may replace $\pi(p_c, 2^k)$ by $\hat{\pi}(0, 2^k)$, where

$$\hat{\pi}(t, 2^k) = \hat{P}_t\{\mathbf{O} \rightsquigarrow \partial S(2^k)\} = \hat{P}_t\{A(2^k)\},$$

and $\hat{P}_t$ is defined by (2.6). In addition, we have (2.8). Integration of (2.8) over t from 0 to 1 yields, by the method of (2.10),

$$\left|\log \frac{\hat{\pi}(1, 2^k)}{\hat{\pi}(0, 2^k)}\right| \leqq C_3 |p - p_c| \left[1 + \sum_R \sum_{\substack{R(v) = R \\ |v| \leqq 2^{k-3}}} \int_0^1 \hat{P}_t\{\Gamma(v, R)\} dt\right]. \tag{2.67}$$

Now, if R is given by (2.4), then the only v's with $R(v) = R$ are the ones satisfying (2.3). Therefore, by (2.46) [after translation by $(-l_1 2^{j-2}, -l_2 2^{j-2})$ and changing j to $j-2$] for any v satisfying (2.3)

$$\hat{P}_t\{\Gamma(v, R)\} \leqq C_2 C_1^{-1} \hat{P}_t\{\Gamma(v, (l_1 2^{j-2},\ l_2 2^{j-2}) + S(2^j))\}.$$

By (2.58) we have

$$\int_0^1 dt |p - p_c| \sum_{v \text{ satisfying } (2.3)} \hat{P}_t\{\Gamma(v, (l_1 2^{j-2}, l_2 2^{j-2}) + S(2^j))\} \leqq C_1 2^{-\zeta(k-j)}.$$

Substitution into (2.67) and using the fact that only l, j with $|l_1|, |l_2| \leqq 17$, $j \leqq k-4$ can occur in (2.3) [see (2.5)] we obtain

$$\left|\log \frac{\hat{\pi}(1, 2^k)}{\hat{\pi}(0, 2^k)}\right| \leqq C_4 \left[1 + \sum_{j \leqq k-4} 2^{-\zeta(k-j)}\right] \leqq C_5.$$

This takes the place of (2.13) and Theorem 1 follows.

3. Proofs of Theorems 2 and 3 and Corollaries 1 and 2

Proof of Theorem 2. The first and last inequality in Theorem 2 are immediate from Theorem 1. The second inequality is obvious from

$$\{\mathbf{O} \rightsquigarrow \partial S(L)\} \subset \{\mathbf{O} \rightsquigarrow \infty\}.$$

Finally, the third inequality was proven in [19, Sect. 4] [see also proof of (1.25) below]. □

Corollary 1 is immediate, if one takes into account that (1.18) implies (1.6) with $\delta = 2\delta_r - 1$ (cf. [14]).

Proof of Theorem 3. We prove (1.25). Clearly, for $L = L(p, \varepsilon_0)$ and any $\varepsilon_1 > 0$,

$$\begin{aligned} E_p\{[\# W]^t;\ \# W < \infty\} &\geqq [\varepsilon_1 L^2 \pi(p, L)]^t P_p\{\mathbf{O} \rightsquigarrow \partial S(L)\} \\ &\quad \times P_p\{\# W \geqq \varepsilon_1 L^2 \pi(p, L) \,|\, \mathbf{O} \rightsquigarrow \partial S(L)\} \\ &\geqq C_1^{t+1} \varepsilon_1^t [L(p, \varepsilon_0)]^{2t} [\pi(p_c, L(p, \varepsilon_0))]^{t+1} \\ &\quad \times P_p\{\# W \geqq \varepsilon_1 L^2 \pi(p, L) \,|\, \mathbf{O} \rightsquigarrow \partial S(L)\}. \end{aligned} \tag{3.1}$$

The proof of (54) in the second part of Theorem 8 of [15] can be copied almost without changes to show that $\varepsilon_1 > 0$ can be chosen to make the last probability in (3.1) at last 1/2. Thus

$$E_p\{[\# W]^t;\ \# W < \infty\} \geqq \tfrac{1}{2} C_1^{t+1} \varepsilon_1^t [L(p, \varepsilon_0)]^{2t} [\pi(p_c, L(p, \varepsilon_0))]^{t+1}.$$

In the opposite direction we first observe that the first part of the proof of Theorem 8 of [15] gives without essential changes that

$$E_p\{[\#(W \cap S(2n))]^t \,|\, \mathbf{O} \rightsquigarrow \partial S(n)\} \leqq C_t [n^2 \pi(p, n)]^t \quad \text{for} \quad n \leqq L(p, \varepsilon_0), \tag{3.2}$$

and

$$\begin{aligned} &E_p\{[\#\text{ of vertices in } S(L) \text{ connected by occupied paths to } \partial S(L)]^t\} \\ &\quad \leqq C_t [L^2 \pi(p, L^2)]^t. \end{aligned} \tag{3.3}$$

As in [14] we introduce the "radius of W"

$$R = R(W) = \max\{|v| : v \in W\},$$

and define ($\mathbf{n} = (n_1, n_2)$, $L = L(p, \varepsilon_0)$),

$$B(\mathbf{n}) = B(\mathbf{n}, p) = [(n_1 - 1)L, (n_1 + 1)L] \times [(n_2 - 1)L, (n_2 + 1)L].$$

Then, still as in [14],

$$\begin{aligned} E_p\{[\# W]^t;\ \# W < \infty\} &\leqq C_{14} + \sum_{1 \leqq 2^k \leqq L} E_p\{[\# W \cap S(2^{k+1}))]^t;\ 2^k \leqq R < 2^{k+1}\} \\ &\quad + E_p\left\{\left[\#(W \cap S(L)) + \sum_{\mathbf{n} \neq 0} (\# W \cap B(\mathbf{n}))\right]^t;\ L < R < \infty\right\}. \end{aligned} \tag{3.4}$$

By (3.2) the first sum on the right-hand side of (3.4) is at most

$$\begin{aligned}\sum_{1\leqq 2^k\leqq L} & P_p\{R\geqq 2^k\}E_p\{[\#(W\cap S(2^{k+1}))]^t\,|\,R\geqq 2^k\}\\ &\leqq C_t\sum_{1\leqq 2^k\leqq L}\pi(p,2^k)[2^{2k}\pi(p,2^k)]^t\\ &\leqq (C_2)^{t+1}C_t\sum_{1\leqq 2^k\leqq L}2^{2kt}[\pi(p_c,2^k)]^{t+1}\end{aligned}\tag{3.5}$$

(by Theorem 1). We claim that for any k_0,

$$\frac{\pi(p_c,2^k)}{\pi(p_c,2^{k_0})}\leqq C_{15}2^{(k_0-k)/2},\qquad k\leqq k_0.\tag{3.6}$$

We postpone the proof of (3.6) till the end of this proof. Once we have (3.6), the right-hand side of (3.5) is seen to be bounded by

$$\begin{aligned}C_{16}&2^{2k_0t}[\pi(p_c,2^{k_0})]^{t+1}\sum_{0\leqq k\leqq k_0}2^{-(k_0-k)(2t-(t+1)/2)}\\ &\leqq C_{17}2^{2k_0t}[\pi(p_c,2^{k_0})]^{t+1},\end{aligned}\tag{3.7}$$

for $t>1/3$, where now k_0 is such that

$$2^{k_0}\leqq L(p,\varepsilon_0)<2^{k_0+1},$$

and C_{17} depends on t only. The second sum in (3.4) has to be treated slightly differently, according as $t\geqq 1$ or $\frac{1}{3}<t<1$. We only consider the former case; the case $t<1$ is easier, since then

$$\left(\sum a_n\right)^t\leqq\sum a_n^t$$

for $a_n\geqq 0$. For $t\geqq 1$, the second sum in the right-hand side of (3.4) is at most

$$\left[(E_p\{[\#W\cap S(L))]^t;R>L\})^{1/t}+\sum_{\mathbf{n}\neq 0}(E_p\{[\#(W\cap B(\mathbf{n}))]^t;R<\infty\})^{1/t}\right]^t.\tag{3.8}$$

By (3.2)

$$\begin{aligned}E_p\{[\#(W\cap S(L))]^t;R>L\}&\leqq P_p\{R\geqq L\}E_p\{[\#(W\cap S(L))]^t\,|\,\mathbf{O}\rightsquigarrow\partial S(L)\}\\ &\leqq\pi(p,L)C_t[L^2\pi(p,L)]^t\leqq C_{18}L^{2t}[\pi(p_c,L)]^{t+1}\end{aligned}\tag{3.9}$$

[since $\{R\geqq L\}=\{\mathbf{O}\rightsquigarrow\partial S(L)\}$, and Theorem 1 for the last inequality]. Furthermore, for $B(\mathbf{n})\cap\mathring{S}(L)=\emptyset$,

$$\begin{aligned}&[\#(W\cap B(\mathbf{n}))]^tI[R<\infty]\leqq[\#\text{ of vertices in }B(\mathbf{n})\text{ connected by occupied paths to }\partial B(\mathbf{n})]^tI[\mathbf{O}\rightsquigarrow\partial S(L/2)]I[\partial S(L)\text{ is connected}\\ &\text{by an occupied path outside }S(L/2)\text{ to }\partial S(1/2(|n_1|+|n_2|-3)L)]\,I[\exists\text{ vacant circuit on }\mathcal{G}^*\text{ surrounding }\mathbf{O}\\ &\text{and some point of }\partial B(\mathbf{n})].\end{aligned}\tag{3.10}$$

If $p<p_c$ then we estimate the expectation of the right-hand side of (3.10) by ignoring the last factor and using the independence of the other factors. This gives by virtue

of (3.3), (2.20), and (2.24)

$$\begin{aligned}&E_p\{[\#(W\cap B(\mathbf{n}))]^t;\ R<\infty\}\\ &\quad\leqq C_t[L^2\pi(p,L)]^t\pi(p,L/2)P_p\{\partial S(L)\rightsquigarrow\partial S((|n_1|+|n_2|-3)L/2)\}\\ &\quad\leqq C_{19}L^{2t}[\pi(p_c,L)]^{t+1}\exp-C_7(|n_1|+|n_2|)/2.\end{aligned}\tag{3.11}$$

Similarly, if $p>p_c$ we ignore the third factor in the right-hand side of (3.10) to obtain, by using the Harris-FKG inequality,

$$\begin{aligned}&E_p\{\#(W\cap B(\mathbf{n}))]^t;R<\infty\}\leqq E_p\{[\#\text{ of vertices in } B(\mathbf{n})\text{ connected}\\ &\text{by occupied paths to } \partial B(n)]^t I[\mathbf{O}\rightsquigarrow\partial S(L/2]\}P_p\{\exists\text{ vacant circuit}\\ &\text{on } \mathcal{G}^*\text{ surrounding } \mathbf{O}\text{ and some point of } \partial B(\mathbf{n})\}\\ &\leqq C_{20}L^{2t}[\pi(p_c,L)]^{t+1}\cdot\sum_{j=-\infty}^{+\infty}P_p\{\exists\text{ vacant paths on } \mathcal{G}^*\text{ from}\\ &(jL,0)+[-L,L]^2\text{ to } \partial((jL,0)+S(jL))\text{ and to the complement}\\ &\text{of } S((|n_1|+|n_2|-3)L/2)\}.\end{aligned}\tag{3.12}$$

In the last inequality we used the fact that any circuit surrounding the origin must intersect the x-axis in some square $[(j-1)L,(j+1)L]\times[-L,L]=(jL,0)+[-L,L]^2$, as well as the y-axis. By (2.25) the right-hand side of (3.12) is for $p>p_c$ at most

$$C_{21}L^{2t}[\pi(p_c,L)]^{t+1}(|n_1|+|n_2|)\exp-\tfrac{1}{4}C_7(|n_1|+|n_2|).\tag{3.13}$$

Putting together the estimates (3.9)–(3.13) we see that (3.8) is at most

$$C_{22}(t)L^{2t}[\pi(p_c,L)]^{t+1}.$$

Finally, in (3.7),

$$\pi(p_c,2^{k_0})\asymp\pi(p_c,L)$$

by virtue of (2.20). Therefore,

$$E_p\{[\#W]^t;\ \#W<\infty\}\leqq C_{23}(t)[1+L^{2t}[\pi(p_c,L)]^{t+1})$$

with $L=L(p,\varepsilon_0)$; the "one" in the right-hand side may be ignored, since $\pi(p_c,L)\geqq CL^{-1/2}$ (see [28, (3.15), or (3.6) above]).

To complete the proof of (1.25) we prove (3.6). Firstly,

$$\begin{aligned}&\pi(p_c,2^{k_0})\geqq C_1\pi(p_c,2^k)\\ &\qquad\times P_{cr}\{\text{occupied path outside } \mathring{S}(2^k)\text{ from } \partial S(2^k)\text{ to } \partial S(2^{k_0})\}\end{aligned}\tag{3.14}$$

for the same reasons as given for (6) in [15]; see Sect. 3 of [15] as well as (2.7) above. Secondly, exactly as in (3.15) of [28] we have

$$\begin{aligned}&0<C_2\leqq P_{cr}\{\exists\text{ occupied horizontal crossing of } [-2^{k_0},2^{k_0}]^2\}\\ &\leqq\sum_{|j|\leqq 2^{k_0-k}}P_{cr}\{\exists\text{ two disjoint occupied horizontal paths outside}\\ &(0,j2^k)+\mathring{S}(2^k)\text{ from } \partial(0,j2^k)+S(2^k)\text{ to the union of the left and}\\ &\text{right edge of } S(2^{k_0})\}\leqq 2(2^{k_0-k}+1)\cdot[P_{cr}\{\exists\text{ occupied path outside}\\ &\mathring{S}(2^k)\text{ from } \partial S(2^k)\text{ to } \partial S(2^{k_0})\}]^2.\end{aligned}$$

In view of (3.14) this proves (3.6) and completes the proof of (1.25).

The reader should now be able to prove (1.26) in a similar way by using

$$P_p\{\mathbf{O} \rightsquigarrow y\} \asymp [\pi(p_c, |y|)]^2 \quad \text{for} \quad |y| \leqq L(p, \varepsilon_0)$$

(cf. [14, Formula (4)]) and

$$\sum_{y \in S(L)} |y|^t [\pi(p_c, |y|)]^2 \asymp L^{t+2} [\pi(p_c, L)]^2$$

[since $n^{1/2}\pi(p_c, n)$ is essentially increasing in n; cf. (3.6) and (2.20)]. □

Corollary 2 is again self-evident (again use [14] as in Corollary 1), with the exception of the inequality $v \geqq (\delta+1)/\delta$. This last inequality can be obtained from $\Delta_2 \geqq 2$ (which is proven in [9, Formula (3)] or from $\gamma \geqq 2(\delta-1)/\delta$ (which is proven in [17]).

4. Symmetry of the Critical Exponents

In this section we prove (1.23) and (1.24). Note that the critical exponents β, γ, η have all been related to v and δ (or δ_r). δ is defined in terms of P_{cr} only. Thus, once we have (1.23), all critical exponents have to be the same on the left ($p \uparrow p_c$) and right ($p \downarrow p_c$), provided they exist, of course. α_+ and α_- have to be equal by virtue of the Sykes-Essam relation; cf. [13, Theorem 9.2]. We formulate the principal step as Lemma 8. The most important corollaries of this lemma, namely (4.4) and (4.5), have appeared in the literature before (see for instance [6, Eq. (14)]). To formulate the lemma we define for $v \in S(n)$,

> $\Omega(v, S(n)) := \{v$ is pivotal for $C(S(n))\} = \{\exists$ paths r_1 and r_3 on $\mathscr{G}$ from v to the left and right edge of $S(n)$, respectively, and paths r_2^* and r_4^* on $\mathscr{G}^*$ from v to the top and bottom edge of $S(n)$, respectively; any two of these paths only have the point v in common; r_i is occupied and r_{1+i}^* is vacant, $i=1,3$, except possibly at $v\}$ (see Fig. 3),

Lemma 8. *For each fixed* $0<\kappa<1$ *there exist constants* $0<C_i=C_i(\kappa)<\infty$, $i=1,2$, *such that*

$$C_1 \leqq \frac{\hat{P}\{\Omega(v, S(n))\}}{P_{cr}\{\Omega(\mathbf{O}, S(n))\}} \leqq C_2 \tag{4.1}$$

uniformly for $\hat{P}$ *satisfying* (2.21) *and*

$$n \leqq L(p, \varepsilon_0) \quad \textit{and} \quad v \in S(\kappa n). \tag{4.2}$$

Before proving Lemma 8 we make some more comments on the proof of (1.23). Note that the definitions immediately give

$$\Delta(v, S(n)) \subset \Omega(v, S(n)) \subset \Gamma(v, S(n)). \tag{4.3}$$

Lemma 4 and Corollary 3 therefore give us a handle on the probability of Ω, and these results, together with (4.1), lead to

$$|\sigma((n,n);1,p) - \sigma((n,n);1,p_c)| \asymp |p-p_c| n^2 P_{cr}\{\Omega(\mathbf{O}, S(n))\}, \quad n \leqq L(p, \varepsilon_0). \tag{4.4}$$

When we take $n = L(p, \varepsilon_0)$ we obtain

$$L^2 P_{cr}\{\Omega(\mathbf{O}, S(L)\} \asymp \frac{1}{|p-p_c|} \tag{4.5}$$

for $L = L(p, \varepsilon_0)$, since we have defined $L(p, \varepsilon_0)$ in such a way that the left-hand side of (4.4) is of order 1 when $n = L(p, \varepsilon_0)$. The result will be that

$$L(p, \varepsilon_0) \asymp L_0(|p-p_c|),$$

where $L_0(\delta)$ is the smallest n for which

$$n^2 P_{cr}\{\Omega(\mathbf{O}, S(n))\} \geqq \frac{1}{\delta}. \tag{4.6}$$

Clearly, this will imply (1.23).

Proof of Lemma 8. This proof is quite similar to that of Theorem 1 but fortunately almost all the work was done already for Theorem 1. Our first step is to show that it suffices to prove (4.1) with $v = \mathbf{O}$ only. To see this assume that v has distance m to $\partial S(n)$ with $m \geqq (1-\kappa)n$. Let l be such that

$$2^l \leqq m < 2^{l+1}, \tag{4.7}$$

and as before, let $\hat{P}^v$ be the translate of $\hat{P}$ by v. Then $S(v, 2^l) \subset S(n)$, and by (4.3) and (2.49),

$$\hat{P}\{\Omega(v, S(n))\} \leqq \hat{P}\{\Gamma(v, S(n))\} \leqq \hat{P}\{\Gamma(v, S(v, 2^l))\} = \hat{P}^v\{\Gamma(\mathbf{O}, S(2^l))\}.$$

Now, if k is such that

$$2^k \leqq n < 2^{k+1},$$

then $k - l \leqq 1 - (\log 2)^{-1} \log(1-\kappa)$. A bounded number of applications of (2.56) therefore shows

$$\hat{P}^v\{\Gamma(\mathbf{O}, S(2^l))\} \leqq C_3 \hat{P}^v\{\Gamma(\mathbf{O}, S(2^k))\}$$

for a C_3 depending on κ. Thus

$$\begin{aligned} \hat{P}\{\Omega(v, S(n))\} &\leqq C_3 \hat{P}^v\{\Gamma(\mathbf{O}, S(2^k))\} \\ &\leqq C_3 C_0 \hat{P}^v\{\Delta(\mathbf{O}, S(2^k))\} \quad \text{[by (2.37)]} \\ &\leqq C_3 C_0 \hat{P}^v\{\Omega(\mathbf{O}, S(2^k))\}. \end{aligned}$$

In the other direction, note first that a small extension of (2.43) gives for l as in (4.7)

$$\hat{P}\{\Delta(v, S(v, 2^l))\} \leqq C_7 \hat{P}\{\Delta(v, S(n))\},$$

this time with C_7 depending on κ. But also, by (4.3), (2.37), and (2.49),

$$\begin{aligned} \hat{P}\{\Omega(v, S(n))\} &\geqq C_7^{-1} \hat{P}\{\Delta(v, S(v, 2^l))\} = C_7^{-1} \hat{P}^v\{\Delta(\mathbf{O}, S(2^l))\} \\ &\geqq (C_7 C_0)^{-1} \hat{P}^v\{\Gamma(\mathbf{O}, S(2^l))\} \geqq (C_7 C_0)^{-1} \hat{P}^v\{\Gamma(\mathbf{O}, S(2^k))\} \\ &\geqq (C_7 C_0)^{-1} \hat{P}^v\{\Omega(\mathbf{O}, S(2^k))\}. \end{aligned}$$

Thus, under (4.2) $\hat{P}\{\Omega(v, S(n))\}$ is uniformly of the same order as $\hat{P}^v\{\Omega(\mathbf{O}, S(2^k))\}$. In particular, for $v=\mathbf{O}$ and $\hat{P}=P_{cr}$, $P_{cr}\{\Omega(\mathbf{O}, S(n))\}$ and $P_{cr}\{\Omega(\mathbf{O}, S(2^k))\}$ are of the same order. Thus (4.1) will be implied by

$$C_4 \leqq \frac{\hat{P}^v\{\Omega(\mathbf{O}, S(2^k))\}}{P_{cr}\{\Omega(\mathbf{O}, S(2^k))\}} \leqq C_5. \tag{4.8}$$

In other words, it suffices to prove (4.1) for $v=\mathbf{O}$ only, and $n=2^k \leqq L(p, \varepsilon_0)$, as long as our estimates are valid uniformly for all $\hat{P}$ satisfying (2.21).

As in Sect. 2 we shall estimate

$$[P_{cr}\{\Omega(\mathbf{O}, S(2^k))\}]^{-1}\hat{P}\{\Omega(\mathbf{O}, S(2^k))\}$$

by first changing the probabilities for a site to be occupied for all w outside $S(2^{k-3})$, and after that apply Russo's formula to estimate the influence of the sites in $S(2^{k-3})$. Let P^* be the measure which chooses w occupied with probability

$$p^*(w) = \begin{cases} p(w) = \hat{P}\{w \text{ is occupied}\} & \text{if } w \notin S(2^{k-3}), \\ p_c & \text{if } w \in S(2^{k-3}). \end{cases}$$

Of course, all sites are independent under P^*. Our second step is to show that

$$C_6 \leqq \frac{P^*\{\Omega(\mathbf{O}, S(2^k))\}}{P_{cr}\{\Omega(\mathbf{O}, S(2^k))\}} \leqq C_7, \tag{4.9}$$

which is the analogue of (2.22). Equation (4.9) follows quickly from previous estimates. For instance,

$$\begin{aligned} P^*\{\Omega(\mathbf{O}, S(2^k))\} &\leqq P^*\{\Gamma(\mathbf{O}, S(2^{k-3}))\} \quad [\text{by (4.3) and (2.49)}] \\ &= P_{cr}\{\Gamma(\mathbf{O}, S(2^{k-3}))\} \\ &\qquad [\text{since } P^* = P_{cr} \text{ when restricted to } S(2^{k-3})] \\ &\leqq C_2^3 C_1^{-3} P_{cr}\{\Gamma(\mathbf{O}, S(2^k))\} \quad [\text{by (2.56)}] \\ &\leqq C_2^3 C_1^{-3} C_0 P_{cr}\{\Omega(\mathbf{O}, S(2^k))\} \quad [\text{by (2.37) and (4.3)}]. \end{aligned}$$

The lower bound in (4.9) is proved by interchanging P_{cr} and P^*.

Now that we have (4.9) we define $\hat{P}_t$ as the measure according to which the sites are independently occupied, with probability

$$p(t, w) = \begin{cases} p(w) & \text{if } w \notin S(2^{k-3}), \\ (1-t)p_c + tp(w) & \text{if } w \in S(2^{k-3}), \end{cases}$$

for the site w. Note that $\hat{P}_1 = \hat{P}$, $\hat{P}_0 = P^*$, and that $\hat{P}_t$ satisfies (2.21) for all $t \in [0, 1]$. We next estimate

$$\frac{d}{dt}\hat{P}_t\{\Omega(\mathbf{O}, S(2^k))\}.$$

Observe that $\Omega(\mathbf{O}, S(2^k)) = A \cap B$, where

> $A = \{\exists$ two paths r_1 and r_3 on $\mathscr{G}$ from $\mathbf{O}$ to the left and right edge of $S(2^k)$, respectively; these paths only have the vertex $\mathbf{O}$ in common and are occupied, except possibly at $\mathbf{O}\}$,

$B = \{\exists$ two paths r_2^* and r_4^* on $\mathscr{G}^*$ from $\mathbf{O}$ to the top and bottom edge of $S(2^k)$, respectively; these paths only have the vertex $\mathbf{O}$ in common and are vacant, except possibly at $\mathbf{O}\}$.

A is increasing and B is decreasing so that by Lemma 1

$$\frac{d}{dt}\hat{P}_t\{(\mathbf{O}, S(2^k))\} = \sum_{w \in S(2^{k-3})} (p(w) - p_c) \cdot [\hat{P}_t\{w \text{ is pivotal for } A \text{ but not for } B, \text{ and } B \text{ occurs}\}] - \sum_{w \in S(2^{k-3})} (p(w) - p_c))[P_t\{w \text{ is pivotal for } B, \text{ but not for } A \text{ and } A \text{ occurs}\}]. \tag{4.10}$$

We only estimate the first sum in the right-hand side; the second sum can be estimated by interchanging the roles of "occupied" and "vacant." It is not difficult to obtain (e.g. from Proposition 2.2 in [13]) that $w \neq \mathbf{O}$ is pivotal for A, but not for B and B occurs, if and only if B occurs as well as the following events $E(w)$ and $E^*(w)$ (see Fig. 16):

$E(w) = \{\exists$ two paths r_1 and r_3 on $\mathscr{G}$ from $\mathbf{O}$ to the left and right edge of $S(2^k)$, respectively; these paths only have the vertex $\mathbf{O}$ in common and one of them contains w; r_1 and r_3 are occupied, except possibly at $\mathbf{O}$ and $w\}$,

$E^*(w) = \{\exists$ two paths r_5^* and r_6^* on $\mathscr{G}^*$ from w to the top and bottom edge of $S(2^k)$, respectively; these paths only have the vertex w in common and do not contain $\mathbf{O}$; r_5^* and r_6^* are vacant except possibly at $w\}$.

It follows from the definition that if R_0 and R_w are disjoint squares in $S(2^k)$ such that $\mathbf{O} \in R_0$, $w \in R_w$, then on $B \cap E(w) \cap E^*(w)$ both $\Gamma(\mathbf{O}, R_0)$ and $\Gamma(w, R_w)$ must occur. If R_0 and R_w are disjoint, then the latter events are in addition independent. This observation already suffices to handle the w with $2^{k-5} < |w| \leqq 2^{k-3}$. Define $l = l(w)$ by

$$2^l < |w| \leqq 2^{l+1}. \tag{4.11}$$

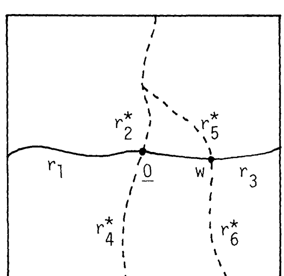

Fig. 16. Illustration of $B \cap E(w) \cap E^*(w)$. Solidly drawn paths are occupied, dashed paths are vacant, except possibly at $\mathbf{O}$ and w. Note that r_2^* and r_5^* do not have to be disjoint; in this figure they partly coincide. r_4^* and r_6^* may also intersect

The w with $|w| \leqq 16$ will be trivial, so that we may assume $l \geqq 4$. We choose $R_0 = S(2^{l-2})$, $R_w = S(w, 2^{l-2})$. We then have for $l = k-5$ or $k-4$

$$\begin{aligned}
&\hat{P}_t\{w \text{ is pivotal for } A \text{ but not for } B, \text{ and } B \text{ occurs}\} \\
&\quad \leqq \hat{P}_t\{\Gamma(\mathbf{O}, S(2^{l-2})\}\hat{P}_t\{\Gamma(w, S(w, 2^{l-2}))\} \\
&\quad \leqq (C_2 C_1^{-1})^7 \hat{P}_t\{\Gamma(\mathbf{O}, S(2^k))\}\hat{P}_t\{\Gamma(w, S(w, 2^{l-2}))\} \quad [\text{by (2.56)}] \\
&\quad \leqq C_3 \hat{P}_t\{\Omega(\mathbf{O}, S(2^k))\}\hat{P}_t\{\Gamma(w, S(w, 2^{l-2}))\} \quad [\text{by (2.37) and (4.3)}]. \qquad (4.12)
\end{aligned}$$

For $16 < |w| \leqq 2^{k-5}$, or $l < k-5$, we have to be slightly more careful. In addition to $\Gamma(\mathbf{O}, R_0)$ and $\Gamma(w, R_w)$ we must also have $\tilde{\Gamma}(S(2^k), S(2^{l+2}))$ if $B \cap E(w) \cap E^*(w)$ occurs [since the paths r_1, r_2^*, r_3, r_4^* connect $\partial S(2^{l+2})$ to $\partial S(2^k)$]. Moreover, $R_0 \cup R_w \subset \mathring{S}(2^{l+2})$ so that $\Gamma(\mathbf{O}, R_0)$ and $\Gamma(w, R_w)$ are independent of $\tilde{\Gamma}(S(2^k), S(2^{l+2}))$. The left-hand side of (4.12) is therefore bounded by

$$\hat{P}_t\{\Gamma(\mathbf{O}, S(2^{l-2}))\}\hat{P}_t\{\tilde{\Gamma}(S(2^k), S(2^{l+2}))\}\hat{P}_t\{\Gamma(w, S(w, 2^{l-2}))\}. \qquad (4.13)$$

By (2.56) and (2.55) the product of the first two factors in (4.13) is at most

$$C_3 \hat{P}_t\{\Delta(\mathbf{O}, S(2^k))\} \leqq C_3 \hat{P}_t\{\Omega(\mathbf{O}, S(2^k))\},$$

so that (4.12) holds for all $w \in S(2^{k-3}) \backslash S(16)$. Putting this into (4.10) we obtain

$$\begin{aligned}
&\frac{d}{dt} \log \hat{P}_t\{\Omega(\mathbf{O}, S(2^k))\} \\
&\quad \leqq C_4 \left[1 + \sum_{l=4}^{k-4} \sum_{2^l < |w| \leqq 2^{l+1}} (p(w) - p_c)\hat{P}_t\{\Gamma(w, S(w, 2^{l-2}))\} \right].
\end{aligned}$$

Finally, for $2^l < |w| \leqq 2^{l+1}$,

$$\begin{aligned}
\hat{P}_t\{\Gamma(w, S(w, S^{l-2}))\} &\leqq C_2^6 C_1^{-6} \hat{P}_t\{\Gamma(w, S(w, 2^{l+4}))\} \quad [\text{by (2.56)}] \\
&\leqq C_2^6 C_1^{-6} \hat{P}_t\{\Gamma(w, S(2^{l+3}))\} \quad [\text{by (2.49)}].
\end{aligned}$$

It follows that

$$\begin{aligned}
\left| \log \frac{\hat{P}\{\Omega(\mathbf{O}, S(2^k))\}}{P^*\{\Omega(\mathbf{O}, S(2^k))\}} \right| &= \left| \int_0^1 \frac{d}{dt} \log \hat{P}_t\{\Omega(\mathbf{O}, S(2^k))\} \right| \\
&\leqq C_5 \left[1 + \sum_{l=4}^{k-4} \int_0^1 \sum_{w \in S(2^{l+1})} |p(w) - p_c| \hat{P}_t\{\Gamma(w, S(2^{l+3}))\} dt \right] \\
&\leqq C_6 [1 + \sum 2^{-\zeta(k-l)}] \leqq C_7 \quad [\text{by (2.58)}].
\end{aligned}$$

Together with (4.9) this proves (4.8), and hence (4.1). □

Lemma 9. *Let $L = L(p, \varepsilon_0)$ and denote by $\hat{P}(\kappa)$ the measure according to which v is occupied with probability*

$$p(\kappa, v) = \begin{cases} p & \text{if } v \notin S(\lfloor \frac{1}{2}\kappa L \rfloor), \\ p_c & \text{if } v \in S(\lfloor \frac{1}{2}\kappa L \rfloor), \end{cases}$$

and with all v independent. Then there exists a $0<\kappa=\kappa(\varepsilon_0)<1$ *such that for all p sufficiently close to* p_c

$$\left|P_p\left\{\exists \text{ occupied horizontal crossing of } S\left(\left[\frac{L}{2}\right]\right)\right\}\right.$$
$$\left.-\hat{P}(\kappa)\left\{\exists \text{ occupied horizontal crossing of } S\left(\left[\frac{L}{2}\right]\right)\right\}\right|$$
$$\geqq \tfrac{1}{8}C_3>0, \tag{4.14}$$

where C_3 *is as in* (2.15).

Proof. For the sake of argument take $p>p_c$ *throughout.* With C_3 as in (2.15) and $L=L(p,\varepsilon_0)$ we have [compare (1.22)]

$$\sigma((L+1,L);1,p_c)\leqq 1-C_3, \quad \sigma((L,L);1,p)\geqq 1-\varepsilon_0\geqq 1-\tfrac{1}{2}C_3, \tag{4.15}$$

so that

$$|\sigma((L+1,L);1,p_c)-\sigma((L,L);1,p)|\geqq \tfrac{1}{2}C_3.$$

To show that this variation in σ is due in part to vertices in "the center of $S(L/2)$" [in $S(\kappa L/2)$, to be more precise] we show first that for any rectangle $[0,a]\times[0,b]$ with

$$\frac{1}{2}\leqq\frac{a}{b}\leqq 2, \quad a,b\leqq 2L(p,\varepsilon_0), \tag{4.16}$$

and $\varepsilon>0$ there exists a $\lambda=\lambda(\varepsilon)>0$ such that

$$\left|\frac{\hat{P}\{C([0,c]\times[0,b])\}}{\hat{P}\{C([0,a]\times[0,b])\}}-1\right|\leqq\varepsilon$$

$$\text{uniformly for } \hat{P} \text{ satisfying (2.21) and for } \left|\frac{c}{a}-1\right|\leqq\lambda. \tag{4.17}$$

Recall that

$$C(R)=\{\exists \text{ occupied horizontal crossing on } \mathcal{G} \text{ of } R\}.$$

(4.17) is rather close to Lemma 3 of [11]. If $a<c$, then clearly

$$C([0,c]\times[0,b])\subset C([0,a]\times[0,b]). \tag{4.18}$$

Thus for $a<c$ we should show

$$\hat{P}\{C([0,a]\times[0,b]) \text{ but not } C([0,c]\times[0,b])\}\leqq\varepsilon\hat{P}\{C([0,a]\times[0,b])\}. \tag{4.19}$$

For this purpose let $\mathcal{R}$ be the lowest occupied crossing of the rectangle $R:=[0,a]\times[0,b]$, and let its right endpoint be (a,H). For the moment we consider only the situation with $H\leqq b/2$. Let r be a fixed horizontal crossing of R with endpoint (a,h) with $h\leqq b/2$. Denote by $R^-(r)$ the lower component of $\mathring{R}\backslash r$ and by $\bar{R}^-(r)$ its closure. Now consider the annuli (see Fig. 17)

$$A_j=T_{j+1}\backslash T_j, \quad \text{where} \quad T_j=[a-3^j,c+3^j]\times[h-3^j,h+3^j]$$

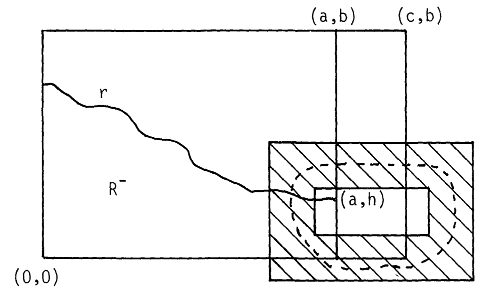

Fig. 17. The hatched region is a typical A_j and the dashed curve is a possible circuit C which connects r to $\{c\} \times \mathbb{R}$

for $3^j \geqq c-a$, $3^{j+1} < b/2$. If $\{\mathscr{R}=r\}$ and any A_j contains a circuit C which surrounds T_j and all of whose sites outside $\bar{R}^-$ are occupied, then a piece of $\mathscr{R}$ plus a piece of C forms an occupied horizontal crossing of $[0,c] \times [0,b]$. Thus, if E_j denotes the event that such a circuit exists in A_j, then

$$\hat{P}\{\text{there is no occupied horizontal crossing of } [0,c] \times [0,b] \mid \mathscr{R}=r\} \\ \leqq \hat{P}\{\text{none of the } E_j \text{ occur} \mid \mathscr{R}=r\}. \tag{4.20}$$

Now given $\mathscr{R}=r$, the E_j only depend on sites outside $\bar{R}^-(r)$, and since $\{\mathscr{R}=r\}$ depends only on sites in $\bar{R}^-(r)$ (compare Proposition 2.3 of [13]) we have

$$P\{E_j \mid \mathscr{R}=r\} \geqq P\{\exists \text{ occupied circuit in } A_j\}.$$

The last probability is at least $\delta > 0$ for some fixed δ, by virtue of (4.16) and (2.19), as long as $3^j \geqq c-a$, $3^{j+1} \leqq b/2$. If we take $c-a \leqq \lambda a$ for some small λ, then we can use all j with $\lambda a \leqq 3^j \leqq a/12$, of which there are $C_2|\log\lambda|$. Since the different A_j are disjoint, the events E_j are conditionally independent, given $\mathscr{R}=r$. It follows that the right-hand side of (4.20) is at most

$$(1-\delta)^{C_2|\log\lambda|} \quad \text{on} \quad \{H \leqq b/2\}.$$

Consequently,

$$\hat{P}\left\{\text{there is no occupied horizontal crossing on } \mathscr{G} \text{ of} \right. \\ \left. [0,c] \times [0,b], \text{ but } \mathscr{R} \text{ exists and } H \leqq \frac{b}{2}\right\} \\ \leqq (1-\delta)^{C_2|\log\lambda|}\sigma((a,b);1,p). \tag{4.21}$$

If $H > b/2$, then the highest occupied crossing of $[0,a] \times [0,b]$ must end above (a,H), and hence in the upper half of the right edge of $[0,a] \times [0,b]$. By interchanging the role of the up and downwards direction we see that the contribution to the left-hand side of (4.19) contained in $\{H > b/2\}$ is also bounded by (4.21). Consequently, if $a \leqq c \leqq a(1+\lambda)$ then

$$1 \geqq \frac{\hat{P}\{C([0,c] \times [0,b])\}}{\hat{P}\{C([0,a] \times [0,b])\}} \geqq 1 - 2(1-\delta)^{C_2|\log\lambda|}.$$

By interchanging the role of a and c one obtains (4.17) also when $a>c$. One may also replace $\hat{P}\{C([0,c]\times[0,b])\}$ by $\hat{P}\{\exists$ vacant crossing on $\mathscr{G}^*$ of $[0,c]\times[0,b]\}$ and $\hat{P}\{C([0,a]\times[0,b])\}$ by the analogous probability. We merely have to change "occupied on $\mathscr{G}$" by "vacant on $\mathscr{G}^*$" everywhere in the proof.

With the help of these inequalities it is now easy to prove the lemma. By[6] (4.18) and by duality analogous to (1.22)

$$\begin{aligned}\hat{P}(\kappa)\left\{C\left(S\left(\frac{L}{2}\right)\right)\right\} &\leqq \hat{P}(\kappa)\left\{C\left(\left[-\frac{1}{2}\kappa L, \frac{1}{2}\kappa L\right]\times\left[-\frac{L}{2},\frac{L}{2}\right]\right)\right\}\\ &\leqq 1-\hat{P}(\kappa)\left\{\exists \text{ vacant vertical crossing}\right.\\ &\qquad \left.\text{on } \mathscr{G}^* \text{ of } \left[-\frac{1}{2}\kappa L+1, \frac{1}{2}\kappa L-1\right]\times\left[-\frac{L}{2},\frac{L}{2}\right]\right\}. \qquad (4.22)\end{aligned}$$

By the analogue of (4.17) for vacant crossings we may choose κ so close to 1 that the last member of (4.22) is at most

$$\begin{aligned}&1+\tfrac{1}{8}C_3-\hat{P}(\kappa)\{\exists \text{ vacant vertical crossing on } \mathscr{G}^* \text{ of}\\ &\quad[-\tfrac{1}{2}\kappa L+1, \tfrac{1}{2}\kappa L-1]\times[-\tfrac{1}{2}\kappa L, \tfrac{1}{2}\kappa L]\}\\ &=1+\tfrac{1}{8}C_3-P_{cr}\{\exists \text{ vacant vertical crossing on } \mathscr{G}^* \text{ of}\\ &\quad[-\tfrac{1}{2}\kappa L+1, \tfrac{1}{2}\kappa L-1]\times[-\tfrac{1}{2}\kappa L, \tfrac{1}{2}\kappa L]\} \qquad (4.23)\end{aligned}$$

[by definition of $\hat{P}(\kappa)$]. We now reverse our steps with P_{cr} instead of $\hat{P}(\kappa)$. By duality as in (1.22), and a monotonicity property analogous to (4.18), (4.23) is at most

$$\begin{aligned}&\frac{1}{8}C_3+P_{cr}\left\{C\left(\left[-\frac{1}{2}\kappa L+1, \frac{1}{2}\kappa L-1\right]\times\left[-\frac{1}{2}\kappa L, \frac{1}{2}\kappa L\right]\right)\right\}\\ &\leqq \frac{1}{8}C_3+P_{cr}\left\{C\left(\left[-\frac{1}{2}\kappa L+1, \frac{1}{2}\kappa L-1\right]\times\left[-\frac{L}{2},\frac{L}{2}\right]\right)\right\}. \qquad (4.24)\end{aligned}$$

Applying (4.17) once more for κ close to 1 and L large, we finally see that (4.24) is at most

$$\begin{aligned}&\frac{1}{4}C_3+P_{cr}\left\{C\left(\left[-\frac{1}{2}L-1, \frac{1}{2}L+1\right]\times\left[-\frac{L}{2},\frac{L}{2}\right]\right)\right\}\\ &\leqq \frac{1}{4}C_3+\sigma((L+1,L);1,p_c) \quad \text{(by periodicity)}\\ &\leqq 1-\frac{3}{4}C_3 \quad [\text{by (4.15)}].\end{aligned}$$

Also, by (4.15), for large L

$$P_p\left\{C\left(S\left(\left\lfloor\frac{L}{2}\right\rfloor\right)\right)\right\} \geqq \sigma((L,L);1,p)-\frac{1}{8}C_3 \geqq 1-\frac{5}{8}C_3.$$

This proves (4.14) if $p>p_c$. A similar argument works for $p<p_c$. □

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[6] For simplicity we suppress the largest integer symbol; e.g. we write $\frac{1}{2}\kappa L$ instead of $\lfloor\frac{1}{2}\kappa L\rfloor$

Theorem 4. *There exist constants $C_i = C_i(\varepsilon_0)$, $i=4,5$, such that for small $\delta>0$ and L_0 as in* (4.6)

$$C_4 \leqq \frac{L(p_c+\delta,\varepsilon_0)}{L_0(\delta)} \leqq C_5 \quad \text{and} \quad C_4 \leqq \frac{L(p_c-\delta,\varepsilon_0)}{L_0(\delta)} \leqq C_5 . \tag{4.25}$$

Moreover, (1.24) *holds.*

Proof. We restrict ourselves throughout to the inequalities for the case $p=p_c+\delta>p_c$. We fix $\kappa<1$ so that (4.14) holds and drop the κ from the notation. Thus $\hat{P}$ will stand for $\hat{P}(\kappa)$ in this proof. We take

$$\hat{P}_t = (1-t)\hat{P} + tP_p ,$$

corresponding to

$$P_t(v \text{ is occupied}) = p(t,v) = \begin{cases} p & \text{if } \ v \notin S(\lfloor \tfrac{1}{2}\kappa L \rfloor), \\ tp+(1-t)p_c & \text{if } \ v \in S(\lfloor \tfrac{1}{2}\kappa L \rfloor). \end{cases}$$

To prove the left inequality in (4.25) we start with (4.14) and apply Russo's formula:

$$\frac{1}{8} C_3 \leqq (p-p_c) \int_0^1 dt \sum_{v \in S(\lfloor \frac{1}{2}\kappa L \rfloor)} \hat{P}_t \left\{ v \text{ is pivotal for } C\left(S\left(\left\lfloor \frac{L}{2} \right\rfloor \right) \right) \right\} . \tag{4.26}$$

By virtue of Lemma 8 the summand here is

$$\hat{P}_t \left\{ \Omega\left(v, S\left(\left\lfloor \frac{L}{2} \right\rfloor \right) \right) \right\} \leqq C_2 P_{cr} \left\{ \Omega\left(\mathbf{O}, S\left(\left\lfloor \frac{L}{2} \right\rfloor \right) \right) \right\} .$$

The usual comparisons [(4.3), (2.37), and (2.43); compare Corollary 3] show

$$P_{cr} \left\{ \Omega\left(\mathbf{O}, S\left(\left\lfloor \frac{L}{2} \right\rfloor \right) \right) \right\} \leqq C_3 P_{cr}\{\Omega(\mathbf{O}, S(L))\} , \tag{4.27}$$

so that we obtain from (4.26),

$$\tfrac{1}{8} C_3 \leqq (p-p_c) C_4 L^2 P_{cr}\{\Omega(\mathbf{O}, S(L))\}$$

or

$$L^2 P_{cr}\{\Omega(\mathbf{O}, S(L))\} \geqq \frac{C_5}{p-p_c} \tag{4.28}$$

[recall $L = L(p,\varepsilon_0)$ and C_5 depends on ε_0 only].

We next use (4.27) to derive the following general inequality: If $L_1 \leqq L_2$ and $L_2 = L_2(p_2,\varepsilon_0)$ for some $p_2 > p_c$, then

$$L_2^2 P_{cr}\{\Omega(\mathbf{O}, S(L_2))\} \geqq C_6 \left(\frac{L_2}{L_1} \right)^{\zeta} L_1^2 P_{cr}\{\Omega(\mathbf{O}, S(L_1))\} . \tag{4.29}$$

To prove (4.29) take j,k such that

$$2^j \leqq L_1 < 2^{j+1} \quad \text{and} \quad 2^k \leqq L_2 < 2^{k+1} .$$

Assume first that $j+1 \leqq k$. An application of (2.59) then yields

$$2^{2j} \int_{p_c}^{p_2} ds P_s\{\Gamma(\mathbf{O}, S(2^j))\} \leqq C_7 2^{-\zeta(k-j)} \leqq C_7 2^{\zeta}\left(\frac{L_1}{L_2}\right)^{\zeta}. \tag{4.30}$$

By (4.1), (4.3), and (2.46), respectively, the left-hand side is at least

$$\begin{aligned} 2^{2j} \int_{p_c}^{p_2} ds P_s\{\Omega(\mathbf{O}, S(2^j))\} &\geqq C_1 2^{2j}(p_2-p_c) P_{cr}\{\Omega(\mathbf{O}, S(2^j))\} \\ &\geqq \tfrac{1}{4} C_1 (L_1)^2 (p_2-p_c) P_{cr}\{\Delta(\mathbf{O}, S(2^j))\} \\ &\geqq C_8 L_1^2 (p_2-p_c) P_{cr}\{\Gamma(\mathbf{O}, S(L_1))\} \\ &\geqq C_8 L_1^2 (p_2-p_c) P_{cr}\{\Omega(\mathbf{O}, S(L_1))\}. \end{aligned} \tag{4.31}$$

Combining (4.30), (4.31), and (4.28) for $p=p_2$ we obtain

$$\begin{aligned} \left(\frac{L_2}{L_1}\right)^{\zeta} L_1^2 P_{cr}\{\Omega(\mathbf{O}, S(L_1))\} &\leqq \frac{C_9}{p_2-p_c} \\ &\leqq C_9 C_5^{-1} L_2^2 P_{cr}\{\Omega(\mathbf{O}, S(L_2))\}. \end{aligned} \tag{4.32}$$

Thus (4.29) holds if $j+1 \leqq k$. For $j=k$ (4.29) can be guaranteed by taking C_6 sufficiently large, since then $L_1 \leqq L_2 \leqq 2L_1$ and $P_{cr}\{\Omega(\mathbf{O}, S(L_i))\}$, for $i=1$ and 2 are of the same order by comparisons of the same kind as in the last two inequalities of (4.31).

Note also that the special case $p_2=p$, $L_1=L_2=L:=L(p, \varepsilon_0)$ of (4.32) shows

$$L^2 P_{cr}\{\Omega(\mathbf{O}, S(L))\} \leqq \frac{C_9}{p-p_c}. \tag{4.33}$$

Now let $p-p_c=\delta>0$ and define, as above, $L_0(\delta)$ as the smallest solution of (4.6). Then (4.28) shows

$$L=L(p, \varepsilon_0)=L(p_c+\delta, \varepsilon_0) \geqq L_0(C_5^{-1}\delta). \tag{4.34}$$

Also (4.33) and (4.29) for $L_1=L_0(C_5^{-1}\delta)$, $p_2=p$, $L_2=L$ imply

$$\begin{aligned} \frac{C_9}{\delta} &\geqq L^2 P_{cr}\{\Omega(\mathbf{O}, S(L))\} \geqq C_6 \left(\frac{L}{L_0}\right)^{\zeta} L_0^2 P_{cr}\{\Omega(\mathbf{O}, S(L_0))\} \\ &\geqq C_6 \left(\frac{L}{L_0(C_5^{-1}\delta)}\right)^{\zeta} \frac{C_5}{\delta} \quad [\text{by (4.6) for } C_5^{-1}\delta]. \end{aligned}$$

Thus

$$L(p_c+\delta, \varepsilon_0) \leqq [C_5^{-1} C_6^{-1} C_9]^{1/\zeta} L_0(C_5^{-1}\delta),$$

which together with (4.34) proves

$$L(p_c+\delta, \varepsilon) \asymp L_0(C_5^{-1}\delta). \tag{4.35}$$

Finally, observe that the derivation of (4.35) holds for any C_5 for which (4.28) holds. In particular, we may assume $C_5<1$ and replace C_5 by C_5^2. Equation (4.35) then shows that

$$L_0(C_5^{-2}\delta) \asymp L_0(C_5^{-1}\delta), \quad \delta \downarrow 0.$$

Replacing δ by $C_5\delta$ in the last relation we obtain $L_0(C_5^{-1}\delta) \asymp L_0(\delta)$. The first line of (4.25) therefore follows from (4.35).

Lastly, (1.24) is immediate from (4.25). Indeed, every proof works if ε_0 is reduced; this only changes the various constants. In particular, for $0<\varepsilon_i \leqq \varepsilon_0$, $i=1,2$

$$C_4(\varepsilon_i) \leqq \frac{L(p,\varepsilon_i)}{L_0(|p-p_c|)} \leqq C_5(\varepsilon_i).$$

Since L_0 does not depend on ε_i, this implies (1.24). □

5. Comments to Table of Critical Exponents

The second column of the table at the end of the introduction is copied from [27]; see also [7, 18, 20]. The values on a Bethe tree appear in Appendix 1 of [10] on the line $d=6$; several of these values can also be found in [8, Sect. 3] or can be computed along the lines given in that reference. In particular, on the binary tree, with each bond open with probability p (see [8, p. 429])

$$\begin{aligned} P_p\{W \text{ contains } n \text{ bonds}\} &= P_p\{W \text{ contains } (n+1) \text{ vertices}\} \\ &= \frac{1}{n+1}\binom{2(n+1)}{n} p^n(1-p)^{n+2}. \end{aligned}$$

The analogue of $\Delta(p)$ should presumably be

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\binom{2(n+1)}{n} p^n(1-p)^{n+2}.$$

Direct calculations show that $\Delta''(p)$ is continuous but $\Delta'''(p)$ is bounded with a jump at $p_c=\frac{1}{2}$. Hence $\alpha=-1$.

$\xi(p)$ can be defined exactly as in the introduction, once one decides on the meaning of $|y|$ on the tree. If one thinks of the tree as lying in infinite Euclidean space, with each edge of a new generation being in a new dimension, then the "Euclidean distance" of an n-th generation point to the root of the tree is $\sqrt{n}$. With this choice for distance

$$\sum_y |y|^2 P_p\{w_0 \rightsquigarrow y \text{ and } \# W<\infty\}$$

becomes $\sum nE_p\{\varrho^{Z_n}\}$, where Z_n is the number of n-th generation points connected by occupied paths to the root, and $\varrho = P_p\{\# W<\infty\}$. This can be calculated by standard branching process calculations and yields $\nu=\frac{1}{2}$.

Lastly, we discuss the rigorous bounds in the first column. $\alpha<0$ follows from a small extension of the proof of Theorem 9.4 in [13]. $\beta \leqq 1$ was proven by J. T. Chayes and L. Chayes [4], and the improvement to $\beta<1$ will appear in [16]. $\nu>1$ follows from $\nu \geqq (\delta+1)/\delta$ in Corollary 2 and $\delta<\infty$ (see [13, Theorem 8.2]). The lower bound for γ results from

$$\gamma = 2\nu\frac{\delta-1}{\delta+1} \geqq 2\frac{\delta-1}{\delta} \quad \text{(see Corollary 2)}$$

if one uses $\delta \geqq 5$, so that we merely have to show $\delta \geqq 5$. *The bound $\delta \geqq 5$, and hence the bound $\gamma \geqq 8/5$ have only been proven for bond percolation on $\mathbb{Z}^2$.*

Proof of $\delta \geqq 5$ for Bond Percolation on $\mathbb{Z}^2$. By the results of [14] it suffices to prove $\delta_r \geqq 3$, or the statement

$$\pi(p_c, r) \geqq C_1 n^{-1/3} \tag{5.1}$$

(which does not presuppose the existence of δ_r). The proof of (5.1) is a small improvement on the proof of (3.15) in [28]. The present proof of (5.1) is due to J. van den Berg and slightly simpler than the author's original proof. We saw in the proof of Lemma 7 that there is a strictly positive probability that the lowest occupied horizontal crossing $\mathscr{R}$ of $S(2^k)$ lies in $T = [-2^k, 2^k] \times [-2^{k-4}, 2^{k-4}]$. Moreover, if $\mathscr{R}$ exists and intersects the line $x=0$ in $w \in \{0\} \times [-2^{k-4}, 2^{k-4}]$, then w is connected by occupied paths r_1 and r_3 to the left and right edge of $S(2^k)$, respectively, and by a path r_2^* to the bottom edge of $S(2^k)$. r_2^* is vacant, except at w, and any pair of r_1, r_2^*, r_3 only have the point w in common. Thus

$$\begin{aligned}
0 < C_2 &\leqq \sum_{|n| \leqq 2^{k-4}} P_{cr}\{\mathscr{R} \text{ exists and contains } (0,n)\} \\
&\leqq \sum_{|n| \leqq 2^{k-4}} P_{cr}\{\text{paths } r_1, r_2^*, r_3 \text{ as above connect } (0,n) \text{ to } \partial S(2^k)\} \\
&\leqq \sum_{|n| \leqq 2^{k-4}} P_{cr}\{\exists \text{ vacant path on } \mathscr{G}^* \text{ from a neighbor of } (0,n) \text{ to } \partial S(2^k)\} \\
&\quad \times P_{cr}\{\exists \text{ two disjoint occupied paths from neighbors of } (0,n) \text{ to } \partial S(2^k)\} \\
&\quad \text{(by Harris-FKG inequality)} \\
&\leqq C_3 2^k [\pi(p_c, 2^k)]^3 .
\end{aligned}$$

For the last inequality we used self-duality of bond percolation on $\mathbb{Z}^2$ to estimate

$$\begin{aligned}
&P_{cr}\{\exists \text{ vacant path on } \mathscr{G}^* \text{ from a neighbor of } (0,n) \text{ to } \partial S(2^k)\} \\
&\quad \leqq \pi(p_c, 2^k - 2^{k-4} - 1) \leqq C_4 \pi(p_c, 2^k) \quad \text{[see (2.20)]} .
\end{aligned}$$

The rest is as in (3.15) of [28]. □

Remark. Even for site percolation on $\mathbb{Z}^2$ we have $\delta_r \geqq 2$ and hence $\delta = 2\delta_r - 1 \geqq 3$, by [28, (3.15)] and [14]. Thus in any case $\gamma \geqq 4/3$.

Acknowledgements. The author wishes to thank M. Aizenman, J. and L. Chayes, and R. Durrett for many helpful discussions on critical exponents. The author also thanks the IMA at the University of Minnesota for its hospitality and technical help in producing this paper.

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Communicated by M. E. Fisher

Received June 19, 1986