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Determinants, Torsion, and Strings

Daniel S. Freed[1]

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

Abstract. We apply the results of [BF1, BF2] on determinants of Dirac operators to String Theory. For the bosonic string we recover the "holomorphic factorization" of Belavin and Knizhik. Witten's global anomaly formula is used to give sufficient conditions for anomaly cancellation in the heterotic string (for arbitrary background spacetimes). To prove the latter result we develop certain torsion invariants related to characteristic classes of vector bundles and to index theory.

String Theory has spawned a vigorous interaction between mathematics and physics. This intermingling of two quite separate intuitions is fruitful for both disciplines. Of particular value for mathematicians are the concrete examples generated and discussed by physicists. The purpose of this paper is to consider the relationship of some examples to the circle of ideas surrounding the Atiyah-Singer Index Theorem.

Atiyah and Singer first demonstrated the connection between determinants (in Quantum Field Theory) and the families index theorem [AS1]. Their concern was with anomalies, which they interpreted as nontrivial topology (over the reals) in the determinant line bundle $\mathscr{L}$. Witten's work on global anomalies [W1] suggested a more refined geometric picture: $\mathscr{L}$ has a natural connection whose curvature and holonomy represent the local and global anomaly, respectively.[2] The mathematical ideas used to construct such a connection are largely due to Quillen [Q1] who, for entirely different reasons, introduced a metric on $\mathscr{L}$. The connection on $\mathscr{L}$ was rigorously constructed in [BF1]. The analytical techniques developed by Bismut in [B] were used in [BF2] to derive the formulae for its curvature and holonomy. We discuss these developments in Sect. 1.

In Sect. 2 we interpret some known results about the bosonic string in terms of the geometry of the determinant line bundle. We emphasize that purely topological methods do not suffice here. In fact, our main purpose in this paper is

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[2] Witten explicitly stated that his work could be interpreted in terms of this connection

to explain examples which demand methods that go beyond the topological index theorem. We consider first the conformal anomaly. In a recent paper Alvarez [A2] explains a formal connection between the conformal anomaly and the index theorem. Although we provide a geometric setting for his observation, we feel that our understanding here is incomplete. In the critical dimension $d=26$, the conformal anomaly vanishes and the theory can be formulated over the Riemann moduli space. There is then a "holomorphic factorization" of the partition function, recently proved by Belavin and Knizhnik [BK] (cf., [CCMR]). That is, the partition function is the norm square of a holomorphic function on moduli space. We use the connection on $\mathscr{L}$ to recover this result. A recent announcement by Manin [M] gives an explicit formula for the partition function.

Our other theme is torsion. Witten's original examples of global gravitational anomalies can be explained in terms of torsion in the determinant line bundle. We pointed this out to Witten, who applied this observation in many contexts [W2, WW]. Witten's conviction that torsion phenomena are important in String Theory inspired us to develop this material further. Hence, in Sect. 3 we give an exposition of the relevant torsion invariants in cohomology, and in Sect. 4 we show how they apply to anomalies in the heterotic string. These anomalies were discussed (in special cases) by Witten in [W2], and we follow his work closely. Our main theorem in Sect. 4 gives sufficient conditions for the cancellation of global anomalies. This theorem applies to arbitrary spacetimes. Although torsion invariants enter to compute the global anomaly, we emphasize that the global anomaly is geometric, not topological. That is, one cannot recover these results by studying just the topology of $\mathscr{L}$; for $\mathscr{L}$ could be topologically trivial but have a nontrivial connection. Because the typical element of the mapping class group has infinite order, and because the topology of spacetime is arbitrary, the holonomy around a typical loop is not computable from the Chern class. Rather, for certain loops it can be expressed in terms of torsion invariants on spacetime. The torsion invariants relevant to the index theorem (hence to anomalies) lie in K-theory, though we work around the K-theory in this paper. They are related to ideas of Dennis Sullivan, and are more or less known to topologists. We defer their consideration to [F1], where we discuss the connection with index theory.

Rather more algebraic topology than we would prefer enters our proof of the global anomaly cancellation. More precisely, the relationship between the torsion invariants in K-theory and the torsion invariants in cohomology involves a *bordism* invariant (Corollary 3.22). Recall that bordism is the generalized homology theory defined roughly by replacing "singular chain" with "manifold" in the definition of ordinary singular homology. In [W3] Witten shows that the global anomaly always vanishes in the 10 dimensional field theory limit of the $E_8 \times E_8$ heterotic string, given the same topological condition we require in Sect. 4. The key ingredient of his proof is a very impressive bordism calculation of Stong. Such bordism computations seem an indispensible ingredient in any discussion of global anomalies.

The impetus to develop these ideas came in part from an effort to explain some formulae of Vafa [V].[3] He derived conditions for modular invariance of the

[3] These have been derived by others as well [DHVW]. My knowledge of the String Theory and literature is far from complete, so my references to it should not be presumed definitive

heterotic string theory formulated on "orbifolds." (His orbifolds are orbit spaces of representations of finite groups.) Modular invariance is equivalent to the absence of global anomalies, and so we were led to look for an interpretation of his formulae in terms of the determinant line bundle. The material in Sect. 4 is a prerequisite for that discussion, which will appear elsewhere [FV].

The examples discussed in this paper provided motivation and guidance for a general theory of determinant line bundles [BF1, BF2, F2] and of torsion invariants [F1]. We hope that our expository efforts in Sects. 1 and 3 form a useful introduction to these ideas.

1. Determinants and Families of Dirac Operators

We begin by reviewing determinants of operators in finite dimensions. Let V be a finite dimensional complex vector space and $\Delta: V \to V$ an endomorphism of V. Then Δ induces an endomorphism

$$\det\Delta : \det V \to \det V$$

on the one dimensional line of totally antisymmetric tensors $\det V = \wedge^{\max} V$, the highest exterior power of V. For $v_1, v_2, \ldots, v_n \in V$,

$$(\det\Delta)(v_1 \wedge v_2 \wedge \ldots \wedge v_n) = \Delta v_1 \wedge \Delta v_2 \wedge \ldots \wedge \Delta v_n . \tag{1.1}$$

Since $\det V$ is a one dimensional vector space, the endomorphism $\det\Delta$ is multiplication by a complex number, which is the product of the eigenvalues of Δ. We usually identify $\det D$ with this complex number.

Suppose now that W is another vector space and $D: V \to W$ a linear map. If $\dim V = \dim W$ we can form the induced map

$$\det D : \det V \to \det W$$

on the highest exterior powers. Without additional choices there is no natural way to identify this with a complex number. Instead we must be content to regard

$$\det D \in (\det V)^* \otimes (\det W)$$

as an element of a complex line. Note that $\det D$ vanishes if D has a kernel and is nonzero otherwise. In fact there is an exact sequence

$$0 \longrightarrow \ker D \longrightarrow V \xrightarrow{D} W \longrightarrow \operatorname{coker} D \longrightarrow 0 , \tag{1.2}$$

and the multiplicative property of the determinant implies a natural isomorphism

$$(\det V)^* \otimes (\det W) \simeq (\det \ker D)^* \otimes (\det \operatorname{coker} D) . \tag{1.3}$$

If $\ker D = \{0\}$ then the right-hand side of (1.3) is canonically identified with C and $\det D$ corresponds to $1 \in C$. If D is not invertible, then $\det D = 0$.

Introduce a parameter space Y and imagine that the operator $D_y: V_y \to W_y$ varies smoothly over $y \in Y$. The determinant construction at $y \in Y$ gives a point in the complex line $(\det V_y)^* \otimes (\det W_y)$. This fits together to yield a section $\det D$ of a line bundle $\mathscr{L} \to Y$. For operators in finite dimensions the line bundle $\mathscr{L}$ is completely determined by the families of vector spaces $\{V_y\}$ and $\{W_y\}$ (which we assume fit together to form vector bundles over Y.) In particular, it is independent of the operators $\{D_y\}$.

We can pass to operators on infinite dimensional spaces by restricting to *Fredholm operators*, which by definition have finite dimensional kernel and cokernel; then the fiber of $\mathscr{L}$ is defined by the right-hand side of (1.3). Notice that the left-hand side no longer makes sense as V and W are infinite dimensional. Also, to rigorously construct the line bundle $\mathscr{L}$ we have to account for the possibility that the dimension of $\ker D_y$ can jump as y varies. (See [Q], for example.) For operators in infinite dimensions the hypothesis $\dim V = \dim W$ is replaced by $\dim \ker D_y = \dim \operatorname{coker} D_y$, i.e., by the hypothesis that the Fredholm operators D_y have index zero. (Henceforth, we write D for D_y.) The line bundle $\mathscr{L}$ exists for arbitrary index, but the section $\det D$ requires the index zero hypothesis. If $\operatorname{index} D \neq 0$, then D is never invertible, so it is natural to set $\det D \equiv 0$.

We are concerned with Dirac operators on a compact manifolds. We also allow operators of Dirac type, which includes the signature operator, Rarita-Schwinger operator, self-dual operator, $\bar{\partial}$ operator on a Kähler manifold, etc. Recall that determinants of Dirac operators arise in fermionic integration. These Dirac operators depend on Bose fields, which are parametrized by a space Y, and the path integral quantization dictates that $\det D$ be integrated over Y. The preceding discussion indicates that $\det D$ comes as a *section* of the determinant line bundle $\mathscr{L} \to Y$. But we can only integrate *functions* over Y. Hence we need to find a global basis for $\mathscr{L}$, that is, a trivialization of $\mathscr{L}$, so as to express $\det D$ as a function on Y. (A global basis is a nonzero section s of $\mathscr{L}$, and with respect to this global basis we can write $\det D = f \cdot s$ for some function f.) The obstruction to finding a trivialization is called the *anomaly*.

This connection between anomalies and the determinant line bundle was pioneered by Atiyah and Singer [AS1]. There is a topological obstruction to trivializing $\mathscr{L}$, namely its integral first Chern class $c_1(\mathscr{L})$, and they showed that it can be computed by the Atiyah-Singer Index Theorem for Families [AS2]. Furthermore, over the reals the first Chern character $ch_1(\mathscr{L})$, which is the image of $c_1(\mathscr{L})$ in real cohomology, is expressed by an explicit cohomological formula. (The torsion information in $c_1(\mathscr{L})$ is only accessible via K-theory [F1].) This topological obstruction is strong enough to detect the anomaly in many situations. Notice, however, that if this topological obstruction $c_1(\mathscr{L})$ vanishes there are many ways of trivializing $\mathscr{L}$, and the function obtained from the section $\det D$ depends strongly on which trivialization is chosen. Extra geometry is needed to fix the trivialization, so as to fix the determinant function. That extra geometry turns out to be a connection $\nabla^{(\mathscr{L})}$ on $\mathscr{L}$. Hence we have the key

Definition 1.4. The *geometric anomaly* is the obstruction to trivializing the connection on $\mathscr{L}$.

In other words, the geometric anomaly is the obstruction to finding a global *flat* nonzero section (see Fig. 1). If one exists it is unique up to a phase on every connected component of Y, and the ratio of the section $\det D$ to the flat section is a function representing the determinant. (The determinant bundle also has a metric $g^{(\mathscr{L})}$ and we take the trivializing flat section to have unit norm.) This connection was constructed in [BF1], following closely ideas of Quillen. We remark that the Green-Schwartz anomaly cancellation [GS] involves a 1-form on Y, which enters geometrically as a modification to the canonical connection $\nabla^{(\mathscr{L})}$ (cf. Sect. 4).

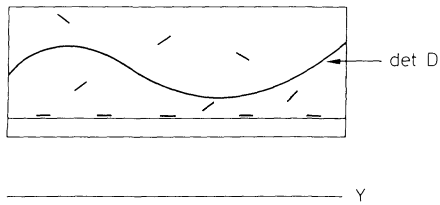

Fig. 1

Before describing the geometric setup abstractly, let us consider a few physical examples.

Example 1.5 (Gravitational Anomalies [AgW, ASZ]*):* Let X be a compact even dimensional spin manifold, say of dimension n, and $\mathrm{Met}(X)$ the space Riemannian metrics on X. The spin structure α on X determines a double cover $\widetilde{GL}(X)$ of the frame bundle $GL(X)$. Now let

$$\mathscr{F} = \{\langle g, f\rangle \in \mathrm{Met}(X) \times \widetilde{GL}(X)\colon \text{the frame } f \text{ is orthonormal in the metric } g\}.$$

Then $\mathscr{F} \to \mathrm{Met}(X) \times X$ is a principal $\mathrm{Spin}(n)$ bundle which restricts on $\{g\} \times X$ to the bundle of spin frames on X for the metric g. Notice that the product $\mathrm{Met}(X) \times X$ carries a partial metric along the fibers of the projection $\mathrm{Met}(X) \times X \to \mathrm{Met}(X)$ onto the first factor. The half-spinor representations $\sigma_{\pm}$ of $\mathrm{Spin}(n)$, applied to the bundle $\mathscr{F}$, yield vector bundles over $\mathrm{Met}(X) \times X$, which restrict on $\{g\} \times X$ to the bundles of positive and negative spinors over X. For each fixed metric the chiral Dirac operator can be defined as usual. Globally we obtain a family of chiral Dirac operators on X parameterized by $\mathrm{Met}(X)$.

The group of diffeomorphisms $\mathrm{Diff}(X)$ acts on $\mathrm{Met}(X) \times X$. The action on a metric g is by pullback and the diffeomorphisms act on X by definition. Restricting to the group of diffeomorphisms $\mathrm{Diff}_\alpha(X)$ which preserve the spin structure α, the action lifts to an action (possibly of a double covering of $\mathrm{Diff}_\alpha(X)$) on $\mathscr{F}$. Taking the quotient we obtain a fibration of manifolds

$$\begin{aligned} &Z = \mathrm{Met}(X) \times X/\mathrm{Diff}_\alpha(X) \\ &\downarrow X \\ &Y = \mathrm{Met}(X)/\mathrm{Diff}_\alpha(X). \end{aligned} \tag{1.6}$$

There is a consistent spin structure along the fibers, so that spinors are defined. The metric along the fibers is preserved by the action of $\mathrm{Diff}_\alpha(X)$, so that the quotient Z also carries a metric along the fibers. Although Z is not globally a product in general, the infinitesimal version of the product structure $\mathrm{Met}(X) \times X$ passes to the quotient; there is a projection $P\colon TZ \to T_{\mathrm{vert}}Z$. This should be thought of as a connection of the fibration of manifolds (1.6) (see Fig. 2).

One difficulty we have not yet mentioned is that in general $\mathrm{Diff}_\alpha(X)$ does not act freely on $\mathrm{Met}(X)$. The isotropy group at $g \in \mathrm{Met}(X)$ is the isometry group of

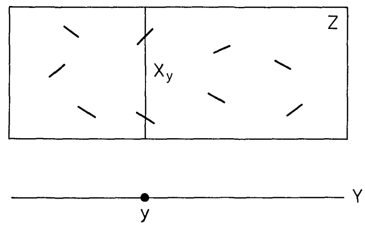

Fig. 2

the metric g, which is a compact Lie group. In the case of major interest to us $X=\Sigma$ is a Riemann surface of genus $\geqq 2$ and this isometry group for metrics of curvature -1 is finite. Thus the quotient Y is an *orbifold* and can be treated by standard methods (cf. the discussion about group actions at the end of this section). As we divide out only by diffeomorphisms preserving a spin structure, Y is a finite cover of the quotient by all diffeomorphisms, the Riemann moduli space.

Example 1.7 (σ-Model [MN]*).* Again let X be a compact even dimensional spin manifold, now with a fixed metric, and suppose that M is an arbitrary manifold. Let $E\to M$ be a Hermitian vector bundle with unitary connection $\nabla^{(E)}$. Consider the mapping space $Y=\mathrm{Map}(X,M)$. For each $\varphi\in Y$ we have a pullback bundle $\varphi^*E\to X$ with connection $\varphi^*\nabla^{(E)}$. The Dirac operator on X couples to this connection (vector potential) to give a Dirac operator on φ^*E-valued spinor fields. This family is parametrized by the map $\varphi\in Y$. Set $Z=Y\times X$ and consider the evaluation map $e: Z\to M$. We obtain the following diagram:

$$\begin{tikzcd} (\mathbf{E},\nabla^{(\mathbf{E})}) \arrow[r] \arrow[d] & (E,\nabla^{(E)}) \arrow[d] \\ Z \arrow[r, "e"] \arrow[d, "X"] & M \\ Y & \end{tikzcd} \quad . \tag{1.8}$$

Here $\mathbf{E}\to Z$ is the bundle e^*E with its pulled back connection $\nabla^{(\mathbf{E})}=e^*\nabla^{(E)}$. Over each $\varphi\in Y$ this restricts to the bundle φ^*E with connection $\varphi^*\nabla^{(E)}$.

The Polyakov formulation of string theory combines these two examples: $X=\Sigma$ is a Riemann surface and the bosonic fields are $\mathrm{Met}(\Sigma)$ and $\mathrm{Map}(\Sigma,M)$.

Next we abstract from these examples the precise geometric setup we need.

Geometric Data 1.9. (1) A smooth fibration of manifolds $\pi: Z\xrightarrow{X} Y$. We suppose $\dim X=n$ is even. To define a family of Dirac operators we need to add the topological hypothesis that the tangent bundle along the fibers $T_{\mathrm{vert}}Z\to Z$ has a fixed spin structure. (This is stronger than assuming that X has a spin structure; the spin structures on the fibers X_y must fit together over the parameter space Y. In Example 1.5 above this hypothesis is guaranteed by dividing out only by diffeomorphisms which preserve the fixed spin structure on X.) If we consider a family of $\bar{\partial}$ operators, say, then the spin structure is irrelevant.

(2) A metric along the fibers, that is, a metric $g^{(T_{\mathrm{vert}}Z)}$ on $T_{\mathrm{vert}}Z$.

(3) A projection $P: TZ \to T_{\mathrm{vert}}Z$. The kernel of P is a distribution of horizontal complements to the vertical tangent spaces, as indicated in Fig. 2.

(4) A complex (virtual) representation ϱ of $\mathrm{Spin}(n)$. This is used to specify the exact combination of Dirac-type operators in the theory. It determines a bundle $V_\varrho \to Z$ to which the basic Dirac operator couples. For example, if $\varrho = 1 - (\sigma_+ + \sigma_-)$, then we have the Dirac operator minus the signature operator. ($\sigma_+ + \sigma_-$ is the total spin representation.)

(5) A complex vector bundle $E \to Z$ with a hermitian metric $g^{(E)}$ and compatible connection $\nabla^{(E)}$. This describes extrinsic data (gauge fields and particles) as in Example 1.7.

The data (2) and (3) in (1.9) determine a connection $\nabla^{(T_{\mathrm{vert}}Z)}$ on the tangent bundle along the fibers of $Z \to Y$. It is the projection of the Levi-Civita connection on Z constructed by choosing an arbitrary metric on Y. All of our constructions are independent of any choice of metric on Y. We denote the curvature of this connection by $\Omega^{(T_{\mathrm{vert}}Z)}$.

The determinant line bundle $\mathscr{L} \to Y$ is constructed from the data (1.9) by patching. Briefly, one handles the jumping kernels by throwing in a finite dimensional space of low "eigenmodes" for the Laplacians D^*D and DD^*. Then (1.2) and (1.3) are used to patch together the determinant lines constructed from these low eigenmode bundles.

Theorem 1.10 [BF1]. *The data* (1.9) *determine functorially a smooth determinant line bundle $\mathscr{L} \to Y$. It carries the Quillen metric $g^{(\mathscr{L})}$ (constructed in* [Q]*) and compatible connection $\nabla^{(\mathscr{L})}$. If the Dirac operators have index zero there is a section* $\det D$ *of* $\mathscr{L}$.

Over the points in Y where D is invertible the section $\det D$ gives a trivialization of $\mathscr{L}$. In terms of that trivialization the Quillen metric is given by

$$\|\det D\|^2_{(\mathscr{L})} = \det D^*D\,. \tag{1.11}$$

The Laplacian $\Delta = D^*D$ is a self-adjoint operator, but on a infinite dimensional space, so that (1.1) cannot be used to define $\det D^*D$. Rather, we use a procedure due to Ray and Singer [RS] – zeta-function regularization. Let $\{\lambda\}$ be the eigenvalues of D^*D, listed according to their multiplicities, and set

$$\zeta(s) = \sum_\lambda \frac{1}{\lambda^s} = \mathrm{Tr}((D^*D)^{-s})\,. \tag{1.12}$$

Then $\zeta(s)$ is finite and holomorphic for $\mathrm{Re}\,s$ sufficiently large, and has a meromorphic continuation which is regular at $s=0$. Define

$$\det D^*D = e^{-\zeta'(0)}\,. \tag{1.13}$$

The connection $\nabla^{(\mathscr{L})}$ on the section $\det D$ is also explicit where D is invertible:

$$\nabla^{(\mathscr{L})}(\det D) = \mathrm{Tr}(\tilde{\nabla} D\, D^{-1})(\det D)\,. \tag{1.14}$$

Here $\tilde{\nabla}$ is the connection $\nabla^{(T_{\mathrm{vert}}Z)}$ operating pointwise, but with a correction term due to the changing volume forms on the fibers of $Z \to Y$. The resulting connection $\tilde{\nabla}$ is unitary on the bundles of L^2 spinor fields. Note that formally $\mathrm{Tr}(\tilde{\nabla}^{(D)}D^{-1}) = \delta \ln \det D$, which is the usual expression in the physics literature. Again we use a zeta function to define the right-hand side of (1.14). Set

$$\omega(s) = \mathrm{Tr}((DD^*)^{-s}\tilde{\nabla}D\, D^{-1}), \tag{1.15}$$

which is finite for $\mathrm{Re}\, s \gg 0$. The meromorphic extension is more complicated than for $\zeta(s)$; in particular, $\omega(s)$ has a pole at $s=0$ in general. We use the regular part to define

$$\mathrm{Tr}(\tilde{\nabla}D\, D^{-1}) = (s\omega(s))'(0). \tag{1.16}$$

These constructions extend to all of $\mathscr{L}$ by patching.

In the preceding paragraph we used the canonical section $\det D$ to trivialize $\mathscr{L}$, where D is invertible. Suppose instead that we work over a region in Y where $\ker D$ and $\operatorname{coker} D$ have constant dimension. We no longer require that D have index zero. Choose smoothly varying bases $\{\varphi_i\}$ for $\ker D$ and $\{\psi_\alpha\}$ for $\ker D^*$. The φ_i and ψ_α are harmonic spinors. Passing to determinants we obtain a nonzero section s of $\mathscr{L}$ [cf. (1.3)]. In terms of this section the Quillen metric is

$$\|s\|^2_{(\mathscr{L})} = \frac{\det(\psi_\alpha, \psi_\beta)}{\det(\phi_i, \phi_j)} \det{}' D^*D\,. \tag{1.17}$$

Here $(\psi_\alpha, \psi_\beta)$ and (ϕ_i, ϕ_j) are the matrices of L^2 inner products of the harmonic spinors. The prime in $\det' D^*D$ denotes the omission of zero modes. This determinant is defined using a ζ-function, as in (1.12) and (1.13), where λ now runs only over nonzero eigenvalues. Similarly, the connection form is given by

$$\nabla^{(\mathscr{L})}(s) = \left\{\sum_\alpha (\tilde{\nabla}\psi_\alpha, \psi_\alpha) - \sum_i (\tilde{\nabla}\varphi_i, \varphi_i) + \mathrm{Tr}'(\tilde{\nabla}D\, D^{-1})\right\} \cdot s\,. \tag{1.18}$$

Tr' is defined as in (1.15) and (1.16), but restricting to the orthogonal complement of $\ker D^*$.

The main theorems in [BF2] determine the curvature and holonomy of $\mathscr{L}$. Of course, the curvature and holonomy are the obstructions to finding a global flat section of $\mathscr{L}$. The curvature formula is related to Bismut's heat equation approach to the index theorem for families [B] and represents the *local geometric anomaly*, while the holonomy formula is Witten's *global geometric anomaly* [W1].

Theorem 1.19 [BF2]. *The curvature of the determinant line bundle $\mathscr{L} \to Y$ is the 2-form*

$$\Omega^{(\mathscr{L})} = \left[2\pi i \int_X \hat{A}(\Omega^{(T_{\mathrm{vert}}T)})\, \mathrm{ch}(\varrho\Omega^{(T_{\mathrm{vert}}Z)})\, \mathrm{ch}(\Omega^{(E)})\right]_{(2)}.$$

$\hat{A}$ and ch are the usual polynomials

$$\hat{A}(\Omega) = \sqrt{\det\left(\frac{\Omega/4\pi}{\sinh \Omega/4\pi}\right)}, \qquad \mathrm{ch}(\Omega) = \mathrm{Tr}\, e^{i\Omega/2\pi}. \tag{1.20}$$

The integrand is a differential form of mixed degree on Z which is integrated over the fibers in the fibration of manifolds $Z \to Y$. The 2-form component of the resulting differential form on Y is the curvature of $\mathscr{L}$. For families of $\bar{\partial}$ operators replace the $\hat{A}$ polynomial by the Todd polynomial. On the level of real cohomology (1.19) is the Atiyah-Singer Index Theorem for families [AS2].

The holonomy formula is more complicated to state. Let $\gamma: S^1 \to Y$ be a loop. By pullback we obtain a geometric family of Dirac operators parametrized by S^1. Thus there is an $(n+1)$-manifold P fibered over S^1, a metric and spin structure along the fibers, etc. Introduce an arbitrary metric $g^{(S^1)}$ and endow S^1 with its trivial spin structure. Then P acquires a metric and spin structure and so a self-adjoint Dirac operator A (coupled to $V_Q \otimes E$). Since our constructions are independent of the metric on the base, we must scale away the choice of metric on the circle. Replace $g^{(S^1)}$ in the preceding by $g^{(S^1)}/\varepsilon^2$ for a parameter ε, and let A_ε denote the Dirac operator for the scaled metric. Set

$$\eta_\varepsilon = \eta\text{-invariant of } A_\varepsilon, \qquad h_\varepsilon = \dim\ker A_\varepsilon, \qquad \xi_\varepsilon = \tfrac{1}{2}(\eta_\varepsilon + h_\varepsilon).$$

The η-invariant is a spectral invariant defined by analytic continuation as the value of

$$\sum_{\lambda \in \operatorname{spec}(A_\varepsilon)\setminus\{0\}} \frac{\operatorname{sgn}\lambda}{|\lambda|^s}$$

at $s=0$ [APS]. An easy argument shows $\xi_\varepsilon \pmod 1$ is continuous in ε.

Theorem 1.21 [W1, BF2]. *The holonomy of $\nabla^{(\mathscr{L})}$ around γ is*

$$\operatorname{hol}(\gamma) = (-1)^{\operatorname{index} D} \lim_{\varepsilon \to 0} e^{-2\pi i \xi}.$$

Here index D is the numerical index of the Dirac operator on X (for any fixed value of the parameter on the circle).

There will be many applications of these formulae in succeeding sections. We simply remark at this stage that η-invariants are difficult to compute directly unless the metric has some special symmetry. We will be able to compute the holonomy in examples by topological methods involving torsion.

Any geometry added to the basic geometric data (1.9) is reflected by extra structure in the determinant line bundle. We will have occasion to use group actions and holomorphic structures.

Proposition 1.22 [F2]. *Let G be a group acting on Z which preserves all of the data (fibration π, spin structure, $g^{(T_{\mathrm{vert}}Z)}$, projection P, $g^{(E)}$, $\nabla^{(E)}$) in (1.9). Furthermore, we assume that a lift of the action to spinors is given. Then G acts on the determinant line bundle $\mathscr{L}$ preserving its Quillen metric $g^{(\mathscr{L})}$ and connection $\nabla^{(\mathscr{L})}$.*

If G acts freely we can equally work on the quotient family of operators. In general, though, we resort to equivariant constructions – equivariant bundles with connection, equivariant cohomology, equivariant K-theory, etc. This arises twice in String Theory. The moduli space of Riemann surfaces has an orbifold structure, so we can treat it using the equivariant geometry of the mapping class group on the Teichmüller space. On the other hand, the spacetime M is sometimes taken to be an orbifold [DHVW, V] which also requires equivariant techniques [FV].

Consider now the $\bar{\partial}$ operator on a Riemann surface. Because we are in dimension two it is elliptic. (For a higher dimensional complex manifold it is the operator $\bar{\partial}+\bar{\partial}^*$ which is elliptic.) Quillen [Q] considers the family of $\bar{\partial}$ operators obtained by varying the holomorphic structure on some extrinsic vector bundle E. The determinant line bundle $\mathscr{L}$ then has a holomorphic structure, and one must verify that the connection $\nabla^{(\mathscr{L})}$, the construction of which is manifestly not holomorphic [note the adjoint in (1.15)], is the unique holomorphic connection compatible with the metric $g^{(\mathscr{L})}$. This was done in [BF1, Theorem 1.21]. In string theory the holomorphic structure on the Riemann surface also varies. The following extension of [BF1, Theorem 1.21] guarantees that the connection on $\mathscr{L}$ is holomorphic in this case.

Proposition 1.23 [F2]. *Suppose in* (1.9) *that X is a Riemann surface and the representation ϱ is chosen so that D is the $\bar{\partial}$ operator, possibly coupled to a holomorphic bundle. Let $Z \xrightarrow{X} Y$ be a holomorphic fibration of Riemann surfaces. Assume that the projection P is complex. Finally, if there is an extrinsic Hermitian bundle E we take it to have a holomorphic structure and unique compatible holomorphic connection. Then the determinant line bundle $\mathscr{L}$ has a holomorphic structure, and $\nabla^{(\mathscr{L})}$ is the holomorphic connection compatible with the Quillen metric. Furthermore, if* $\operatorname{index} D=0$ *then the natural section is holomorphic.*

In Sect. 4 we shall need to know how the ξ-invariant varies with the differential geometric parameters (metrics and connections). The appropriate formula is contained in the work of Atiyah-Patodi-Singer, though the precise form that we use can be found in [BF2, Theorem 2.10]. Consider a geometric family of *odd* dimensional spin manifolds X, which is defined exactly as in (1.9). Attached to each manifold X_y is a self-adjoint Dirac operator, and so an ξ-invariant ξ_y. This is a function on the parameter space with values in R/Z. Its differential is given by the usual index formula.

Proposition 1.24. *The variation of ξ_y is the 1-form*

$$d\xi_y=\left[\int_X \hat{A}(\Omega^{(T_{\mathrm{vert}}Z)})\,\mathrm{ch}(\varrho\Omega^{(T_{\mathrm{vert}}Z)})\,\mathrm{ch}(\Omega^{(E)})\right]_{(1)} \quad (\mathrm{mod}\,1).$$

2. The Bosonic String

The determinants which arise in the bosonic string illustrate the ideas of Sect. 1. We consider the Polyakov formulation of the bosonic string in a flat Euclidean background $\mathbb{R}^d$. As usual, all metrics are positive definite. Our first goal is to give a geometric description of the conformal anomaly. In the critical dimension $d=26$ there is the "holomorphic factorization" of Belavin and Knizhnik – the partition function is the norm square of a holomorphic function on moduli space. We derive it using the connection on the determinant line bundle. As we have nothing to add to other aspects of the theory, we will be extremely brief in our exposition.

Let Σ be a Riemann surface of genus $\geqq 2$. The cases of genus 0 and 1 can be treated similarly, and for simplicity we omit the modifications needed to treat these cases. The partition function (for fixed genus) is

$$Z=\int_{g\in \mathrm{Met}(\Sigma)}[dg]\int_{\varphi\in \mathrm{Map}(\Sigma,\mathbb{R}^d)}[d\varphi]\exp\left(-\frac{1}{2}\int_\Sigma (d\varphi,d\varphi)_g\right). \tag{2.1}$$

The action $\frac{1}{2}\int_\Sigma (d\varphi, d\varphi)_g$ has many symmetry groups: the translation group of $\mathbb{R}^d$, the diffeomorphism group of Σ, and the group $C^\infty_+(\Sigma)$ of positive functions on Σ, acting as conformal rescalings of the metric. The measures $[dg]$, $[d\varphi]$ in this expression are the natural *formal* measures divided by the volume (of the orbits) of the symmetry groups. Now the integral over φ is a standard Gaussian, and so

$$Z = \int_{\mathrm{Met}(\Sigma)} [dg] \left(\frac{\det' \frac{1}{2}\Delta_g}{(1,1)_g}\right)^{-d/2}. \tag{2.2}$$

Here $\det' \Delta_g$ is the regularized product of the nonzero eigenvalues of the Laplacian Δ_g on functions. The factor $(1,1)_g$, which is the total volume of the metric g, occurs because of the translation symmetry. By a standard change of variables which we will not duplicate here (see [DP] for a nice exposition), the partition function is reexpressed as

$$Z = \int_{\mathrm{Met}(\Sigma)} [dg]' \left(\frac{\det' \frac{1}{2}\Delta_g}{(1,1)_g}\right)^{-d/2} \left(\frac{\det' \bar\partial_L^* \bar\partial_L}{\det(\phi_i, \phi_j)}\right). \tag{2.3}$$

Now $[dg]'$ is a new formal measure on $\mathrm{Met}(\Sigma)$, which reflects the product structure $\mathrm{Met}(\Sigma) = \mathrm{Conf}(\Sigma) \times C^\infty_+(\Sigma)$. $\mathrm{Conf}(\Sigma)$ denote the space of conformal structures on Σ; it is the quotient $\mathrm{Met}(\Sigma)/C^\infty_+(\Sigma)$. The second determinant is the Jacobian from the change of variables. The operator $\bar\partial_L$ (sometimes denoted P_1 in the physics literature) is the $\bar\partial$ operator coupled to the holomorphic tangent bundle $L \to \Sigma$. Here we use the fact that a metric on Σ determines a complex structure. Of course, the second determinant in (2.3) also depends on the metric. The $\{\phi_i\}$ are a basis of holomorphic quadratic differentials (which represent deformations on the Teichmüller space). We choose the ϕ_i to be invariant under $C^\infty_+(\Sigma)$. Note that by Serre duality $\ker \bar\partial_L^*$ is dual to the space of holomorphic quadratic differentials. Also, the measure $[dg]'$ depends on this choice of basis. Our purpose is to focus on the determinants in (2.3), not on the derivation of the measure, so we defer to [DP] for details.

We ask whether the product of determinants in (2.3) passes to the quotient $\mathrm{Conf}(\Sigma) = \mathrm{Met}(\Sigma)/C^\infty_+(\Sigma)$, i.e., whether it is invariant under conformal rescalings of the metric. Any variation is called a *conformal anomaly*, and the precise formula has been computed directly [Fr, A1]. Indeed, this is the calculation which fixes the dimension $d = 26$, as this is the only dimension in which the conformal anomaly vanishes. In [A2] Alvarez observed that formally the families index theorem yields the same result. As the group $C^\infty_+(\Sigma)$ is contractible, one cannot attach purely topological significance to his computation. Rather, we interpret his observation in terms of the geometry of determinant line bundles. His calculation is then the curvature of a connection, not a Chern class.

In the language of (1.9) take $Y = \mathrm{Met}(\Sigma)$ and $Z = \mathrm{Met}(\Sigma) \times \Sigma$ with the natural vertical metric and projection. No spin structure is necessary as we consider $\bar\partial$ operators. The representation ϱ is chosen so that $V_\varrho = -d/2 + L$, where $-d/2$ is the negative of the trivial $d/2$-dimensional bundle and $L \to \mathrm{Met}(\Sigma) \times \Sigma$ is the holomorphic tangent bundle to Σ. (The spacetime M is assumed to have even dimension.) This data describes the family of operators $\bar\partial_{-d/2+L}$ on the Riemann surface, parametrized by the metric. Let $\mathscr{L} \to \mathrm{Met}(\Sigma)$ be the determinant line bundle for this family. Although these operators have nonzero index, the kernel and cokernel have constant dimension as the metric varies. In fact, it is an easy consequence of the Kodaira vanishing theorem (or Weitzenböck formula) that $\ker \bar{\partial}_L = 0$. Furthermore, $\ker \bar{\partial}$ is the space of holomorphic functions on Σ, which always consists only of the constant functions. By Serre duality we can identify $\ker \bar{\partial}^*$ with the dual of the space of holomorphic 1-forms. Let $\{\omega_\alpha\}$ be a basis of holomorphic 1-forms locally on $\mathrm{Met}(\Sigma)$. This data determines a section s of $\mathscr{L}$ whose norm square is given by (1.17):

$$\|s\|^2_{(\mathscr{L})} = \left(\frac{\det' \bar{\partial}^* \bar{\partial}}{(1,1)_g \cdot \det(\omega_\alpha, \omega_\beta)} \right)^{-d/2} \left(\frac{\det' \bar{\partial}_L^* \bar{\partial}_L}{\det(\phi_i, \phi_j)} \right). \tag{2.4}$$

The ω_α and ϕ_i occur in the denominator because the Serre duality mentioned above. Since $\frac{1}{2}\Delta_g = \bar{\partial}^* \bar{\partial}$ we see that $\|s\|^2$ differs from the product of determinants in (2.3) only by the factor $[\det(\omega_\alpha, \omega_\beta)]^{-d/2}$. But as we can choose ω_α to be invariant under $C_+^\infty(\Sigma)$, since the $\bar{\partial}$ operator only depends on the underlying conformal structure, and since the L^2 inner product on 1-forms is conformally invariant, this discrepancy is irrelevant to the computation of the conformal anomaly. In other words, the conformal anomaly vanishes precisely when the function $\|s\|^2$ on $\mathrm{Met}(\Sigma)$ is invariant under the action of $C_+^\infty(\Sigma)$.

There is a natural lift of the action of $C_+^\infty(\Sigma)$ to $\mathrm{Met}(\Sigma) \times \Sigma$ – the conformal rescalings act trivially on Σ – which induces an action on the determinant line bundle $\mathscr{L}$. We claim that s is invariant under this action. For the section s is essentially the $\bar{\partial}$ operator, and the complex structure of Σ is unchanged by rescaling the metric. The other ingredients in s – the ω_α, ϕ_i, and constant functions – are explicitly chosen to be conformally invariant. It follows that the partition function is conformally invariant if $g^{(\mathscr{L})}$ is preserved by $C_+^\infty(\Sigma)$. In Sect. 1 we introduced a unitary connection $\nabla^{(\mathscr{L})}$ on $\mathscr{L}$. Since a metric connection determines the metric through its holonomy, it suffices to prove that $\nabla^{(\mathscr{L})}$ is invariant. For this we use the following

Lemma 2.5. *Let $\mathscr{L} \to Y$ be a line bundle with connection and $\mathscr{G}$ a connected Lie group acting freely on $\mathscr{L}$ and Y. Denote the connection 1-form (on the corresponding principal $\mathbb{C}^*$ bundle) by $\omega^{(\mathscr{L})}$ and its curvature by $\Omega^{(\mathscr{L})}$. Suppose that for each X in the Lie algebra of $\mathscr{G}$,*

$$\text{(i)}\ \iota_X \omega^{(\mathscr{L})} = 0, \qquad \text{(ii)}\ \iota_X \Omega^{(\mathscr{L})} = 0.$$

Then the connection passes to the quotient bundle $\mathscr{L}/\mathscr{G} \to Y/\mathscr{G}$.

The lemma is a simple application of the Cartan formula for the Lie derivative.

Alvarez's index calculation [A2] is the verification of condition (ii) for the bosonic string.

Proposition 2.6. *The curvature $\Omega^{(\mathscr{L})}$ of the determinant line bundle for the $\bar{\partial}_{-d/2+L}$ family vanishes if $d = 26$.*

Proof. We apply Theorem 1.19. Let

$$x = \frac{i}{2\pi}\, \Omega^{(L)} \in \Omega^2(\mathrm{Met}(\Sigma) \times \Sigma)$$

be the curvature of the holomorphic tangent bundle L. The index density for the $\bar{\partial}$ operator is the Todd genus

$$\mathrm{Todd}(\Omega^{(L)}) = 1 + \frac{x}{2} + \frac{x^2}{12} + \dots . \tag{2.7}$$

The Chern character of the auxiliary bundle $-d/2+L$ is

$$\mathrm{ch}(\Omega^{(-d/2+L)}) = (1-d/2) + x + \frac{x^2}{2} + \dots . \tag{2.8}$$

Then $\Omega^{(\mathscr{L})}$ is $2\pi i$ times the integral over Σ of the product of (2.7) and (2.8). Since

$$\left[\left(1+\frac{x}{2}+\frac{x^2}{12}\right)\left((1-d/2)+x+\frac{x^2}{2}\right)\right]_{(4)} = \frac{26-d}{24}\,x^2,$$

we obtain

$$\Omega^{(\mathscr{L})} = i\left(\frac{d-26}{48\pi}\right)\int_{\Sigma}(\Omega^{(L)})^2, \tag{2.9}$$

which vanishes identically if $d=26$.

The vanishing of $\Omega^{(\mathscr{L})}$ in 26 dimensions is universal – it holds for any family of Riemann surfaces.

Proposition 2.6 is condition (ii) in Lemma 2.5. To prove the vanishing of the conformal anomaly in $d=26$, it remains to verify condition (i). Unfortunately, this is where our understanding falls short. Direct verification would involve computing the connection form in (1.18), contracted along the orbits of $C^{\infty}_{+}(\Sigma)$. But this is the original, direct calculation of the variation of $\|s\|^2$, which we are trying to avoid. Roughly speaking, the connection from $\omega^{(\mathscr{L})}$ is the first derivative of $\|s\|^2$ and the curvature $\Omega^{(\mathscr{L})}$ is the second derivative. Our goal is to use some geometric principle to show that in this particular situation it suffices to compute second derivatives. One possible geometric principle is holomorphicity: If s is a holomorphic section of a flat Hermitian holomorphic line bundle, then the vanishing of the curvature $\partial\bar{\partial}\log\|s\|^2$ implies that $\log\|s\|^2$ is harmonic. Indeed, it is possible to embed $\mathrm{Met}(\Sigma)$ in a complex space and $\mathscr{L}$ in a holomorphic line bundle, with the rescaling action extending appropriately. However, we are unable to prove that the harmonic function $\log\|s\|^2$ is constant. For now we must content ourselves with a geometric *interpretation* of the conformal anomaly calculation, though we hoped for a geometric *derivation*.

Restrict to the critical dimension $d=26$. Because the conformal anomaly vanishes, we now have a line bundle $\mathscr{L}\to\mathrm{Conf}(\Sigma)$ together with a metric, connection, and nonvanishing section s. The next step is to divide out by the action of the connected component $\mathrm{Diff}_0(\Sigma)$ of the diffeomorphism group. We pause first to review the basic geometry of the quotient $\mathrm{Teich}(\Sigma)=\mathrm{Conf}(\Sigma)/\mathrm{Diff}_0(\Sigma)$, the *Teichmüller space*.

A conformal structure on Σ determines a $*$-operator on 1-forms (since the $*$-operator of a Riemannian metric is conformally invariant in the middle dimension), and $*^2=-1$. Thus $*$ defines an almost complex structure on Σ, and for trivial dimensional reasons this structure is integrable. In fact, $\mathrm{Conf}(\Sigma)$ is exactly the space of complex structures on Σ. Now $\mathrm{Conf}(\Sigma)$ itself is a complex manifold. For the tangent space at $*\in\mathrm{Conf}(\Sigma)$ consists of real endomorphisms A of the real tangent bundle to Σ with $*A+A*=0$. An almost complex structure is defined by sending A to $*A$. One can verify that the torsion tensor for this almost complex structure vanishes, so that $\mathrm{Conf}(\Sigma)$ is a complex manifold. (We will not deal with the technicalities of infinite dimensional manifolds here; see [FT1] for details.) The product space $\mathrm{Conf}(\Sigma)\times\Sigma$ carries a natural complex structure. On the first factor it is the structure just discussed, and on $\{*\}\times\Sigma$ it restricts to the complex structure $*$ on Σ. Furthermore, the projection $\mathrm{Conf}(\Sigma)\times\Sigma\to\mathrm{Conf}(\Sigma)$ is clearly holomorphic. Now the obvious action of $\mathrm{Diff}_0(\Sigma)$ on these spaces is free and preserves the complex structures. Hence the quotient

$$\begin{array}{l} Z=\mathrm{Conf}(\Sigma)\times\Sigma/\mathrm{Diff}_0(\Sigma) \\ \downarrow^{\Sigma} \\ Y=\mathrm{Conf}(\Sigma)/\mathrm{Diff}_0(\Sigma)=\mathrm{Teich}(\Sigma) \end{array} \tag{2.10}$$

is a holomorphic fibration of Riemann surfaces. Z is called the *universal (Teichmüller) curve*. As in (1.6) the projection $T(\mathrm{Conf}(\Sigma)\times\Sigma)\to T\Sigma$ passes to a projection $P: TZ\to T_{\mathrm{vert}}Z$ on the quotient. It is clear that P is a complex mapping.

The uniformization theorem provides a Riemannian view of these spaces. Recall that uniformization is a section of $\mathrm{Met}(\Sigma)\to\mathrm{Conf}(\Sigma)$ which assigns to each $*\in\mathrm{Conf}(\Sigma)$ a metric of constant curvature -1. The action of $\mathrm{Diff}_0(\Sigma)$ preserves the space $\mathrm{Met}_{-1}(\Sigma)$ of all such metrics, and so (2.10) can be identified with

$$\begin{array}{l} Z=\mathrm{Met}_{-1}(\Sigma)\times\Sigma/\mathrm{Diff}_0(\Sigma) \\ \downarrow^{\Sigma} \\ Y=\mathrm{Met}_{-1}(\Sigma)/\mathrm{Diff}_0(\Sigma)=\mathrm{Teich}(\Sigma) \end{array}. \tag{2.11}$$

As in (1.6) the universal curve Z inherits a partial metric $g^{(T_{\mathrm{vert}}Z)}$ and projection $P: TZ\to T_{\mathrm{vert}}Z$. Clearly this projection agrees with the one obtained in (2.10). Also, the metric $g^{(T_{\mathrm{vert}}Z)}$ is Kähler on each fiber Σ_y (since any Riemannian metric on a Riemann surface is Kähler). Hence the holomorphic picture (2.10) and Riemannian picture (2.11) are compatible, in the sense of Proposition 1.21.

As our constructions are independent of any metric on the base $Y=\mathrm{Teich}(\Sigma)$, we have not introduced one. We remark in passing, though, that the natural metric on $\mathrm{Met}_{-1}(\Sigma)$ descends to the Teichmüller space, and the lift to $\mathrm{Met}_{-1}(\Sigma)\times\Sigma$ descends to the universal curve. These turn out to be compatible with the complex picture – they are Kähler metrics [FT2]. On $\mathrm{Teich}(\Sigma)$ this is called the *Weil-Petersson metric*.

We can carry along the holomorphic tangent bundle to Σ in these constructions, and end up with a holomorphic Hermitian line bundle $L\to Z$.

Let us resume our discussion of the bosonic string. At this stage we have a line bundle $\mathscr{L}\to\mathrm{Conf}(\Sigma)$ with metric $g^{(\mathscr{L})}$, connection $\nabla^{(\mathscr{L})}$, and nonvanishing section s. The action of $\mathrm{Diff}_0(\Sigma)$ on $\mathrm{Conf}(\Sigma)$ lifts to an action on $\mathrm{Conf}(\Sigma)\times\Sigma$, as in Example 1.5. All of our geometric data is invariant under $\mathrm{Diff}_0(\Sigma)$, so by Proposition 1.22 an action is induced on $\mathscr{L}$. The action preserves $g^{(\mathscr{L})}$ and $\nabla^{(\mathscr{L})}$. Furthermore, it is a standard fact in Riemann surface theory that $\mathrm{Diff}_0(\Sigma)$ acts freely on $\mathrm{Conf}(\Sigma)$, since we assume that the genus $\geqq 2$. (In the physics literature one states, "there are no conformal Killing vectors on Σ.") So there is an induced line bundle $\mathscr{L}\to\mathrm{Teich}$ on the quotient with metric $g^{(\mathscr{L})}$ and connection $\nabla^{(\mathscr{L})}$. Alternatively, we can start directly with the geometric data implicit in (2.10), (2.11) and the associated $\bar{\partial}_{-13+L}$ family. The resulting determinant line bundle, Quillen metric, and connection agree with the previous ones obtained on the quotient. This is the content of the functioriality statement in Theorem 1.10, and it is clear from the constructions. The section s of $\mathscr{L}\to\mathrm{Conf}(\Sigma)$ passes to a section of the quotient $\mathscr{L}\to\mathrm{Teich}(\Sigma)$ if we take care to choose the local bases $\{\phi_i\}$ and $\{\omega_\alpha\}$ to be $\mathrm{Diff}_0(\Sigma)$-invariant. We can go further. The space of holomorphic 1-forms varies holomorphically with the complex structure, so forms a holomorphic vector bundle over $\mathrm{Teich}(\Sigma)$. We require that ω_α be local holomorphic sections. Similar remarks apply to the holomorphic quadratic differentials. Now Proposition 1.23 immediately implies

Proposition 2.12. *The bundle $\mathscr{L}\to\mathrm{Teich}(\Sigma)$ is holomorphic and $\nabla^{(\mathscr{L})}$ is the unique holomorphic connection compatible with the metric $g^{(\mathscr{L})}$. Furthermore, s is a holomorphic section.*

The bosonic string has no anomalies in the sense described in Sect. 1; there is no obstruction to defining the determinants which arise. Indeed, we have realized the partition function as $\|s\|^2$ for a section s of $\mathscr{L}\to\mathrm{Teich}(\Sigma)$. Now we ask whether $\|s\|^2$ is also the norm square of a holomorphic *function*.

Proposition 2.13 [BK, CCMR]. *In the critical dimension $d=26$ the partition function for the bosonic string is the norm square of a holomorphic function F on* $\mathrm{Teich}(\Sigma)$.

Proof. It suffices to produce a holomorphic section of $\mathscr{L}\to\mathrm{Teich}(\Sigma)$ with unit norm. For this we observe that the curvature of $\nabla^{(\mathscr{L})}$ vanishes by the previous calculation (2.6). Then since $\mathrm{Teich}(\Sigma)$ is simply connected there is a global flat section s_0 of unit norm, unique up to an overall phase. This section is also holomorphic, as $\nabla^{(\mathscr{L})}$ is a holomorphic connection. The desired holomorphic function is $F=s/s_0$.

Proposition 2.13 is the "holomorphic factorization" referred to in the introduction.

3. Torsion

Physicists are most familiar with cohomology via differential forms and the de Rham theory. A smooth manifold M has an exterior derivative operator d which fits into the de Rham complex. The deviation of this complex from exactness defines the de Rham cohomology of M. In each dimension this cohomology is a real vector space. Topologists, on the other hand, use singular chains and cochains to define homology and cohomology, which then appear as abelian groups. The fundamental theorem of de Rham states that the tensor product of these cohomology groups with the real numbers gives the de Rham cohomology groups. However, *torsion* information is lost in this process. Recall that an element g of an abelian group G is said to be torsion if some multiple $k\cdot g$ (k an integer) vanishes. The torsion elements in G form a subgroup $\mathrm{Tor}\,G$, and there is an exact sequence

$$0\to\mathrm{Tor}\,G\to G\to\mathrm{Free}\,G\to 0\,. \tag{3.1}$$

Note that G projects onto its free part, but there is no God-given projection from G to $\operatorname{Tor} G$.

Recently torsion phenomena have entered string theory, mainly through the work of Ed Witten [W1, W2, WW]. Our goal in this section is to develop the torsion invariants in cohomology relevant to Witten's work. The analogous invariants in K-theory also play a role, but we stop short of explaining them here. (We will develop them in [F1].) Still, those invariants enter the discussion through analytic expressions (involving ξ-invariants), and play a role in Sect. 4.

The author faces here the unenviable task of explaining these ideas, which perhaps are not completely standard in mathematics, to an audience partly composed of (brave) physicists, who are presumably unfamiliar with integral cohomology. Fortunately, Witten has kindly provided expositions of some basics [W3, WW], and we encourage the reader to consult these references first. The beautiful book [BT] should also serve as a useful guide. In our present expository account we omit proofs, which are in any case not too difficult.

Let M be a reasonable space. Then for the cohomology groups of M we have by (3.1) an exact sequence

$$0 \to \operatorname{Tor} H^l(M) \to H^l(M) \to \operatorname{Free} H^l(M) \to 0\,. \tag{3.2}$$

[Standard notation identifies $H^l(M) = H^l(M;Z)$ as the integral cohomology.] The *universal coefficient theorem* identifies these groups in terms of homology groups:

$$\operatorname{Tor} H^l(M) \simeq \operatorname{Tor} H_{l-1}(M)\,, \qquad \operatorname{Free} H^l(M) \simeq \operatorname{Free} H_l(M)\,. \tag{3.3}$$

The isomorphisms in (3.3) are noncanonical, that is, involve choices. The canonical version of (3.3) is

$$\begin{aligned} \operatorname{Tor} H^l(M) &\simeq \operatorname{Hom}(\operatorname{Tor} H_{l-1}(M), \mathbb{Q}/\mathbb{Z})\,, \\ \operatorname{Free} H^l(M) &\simeq \operatorname{Hom}(\operatorname{Free} H_l(M), \mathbb{Z})\,. \end{aligned} \tag{3.4}$$

Examples. (1) Consider the real projective plane $\mathbb{RP}^2 = S^2/(\mathbb{Z}/2)$. Since the 2-sphere is simply connected, the fundamental group $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2$. It follows that the first homology group, which is the abelianization of the fundamental group, is $H_1(\mathbb{RP}^2) = \mathbb{Z}/2$. From (3.3) we obtain $H^2(\mathbb{RP}^2) = \mathbb{Z}/2$. The generator is represented by a nontrivial flat line bundle, the quotient of the trivial line bundle over S^2 by the nontrivial $\mathbb{Z}/2$ action covering the antipodal map. The holonomy of the flat connection around the nontrivial loop in $\mathbb{RP}^2$ is multiplication by -1.

(2) The *lens space* $L_{n,k} = S^{2n-1}/(\mathbb{Z}/k)$, where we identify S^{2n-1} as the unit vectors in $\mathbb{C}^n$ and the generator of $\mathbb{Z}/k$ acts as multiplication by $e^{2\pi i/k}$. Then as in (1) we deduce $H^2(L_{n,k}) = \mathbb{Z}/k$. In fact,

$$H^l(L_{n,k}) = \begin{cases} \mathbb{Z}/k\,, & l \text{ even}\,, \\ 0\,, & l \text{ odd}, \ l < 2n-1\,, \\ \mathbb{Z}\,, & l = 2n-1\,, \end{cases} \tag{3.5}$$

This example occurs in [W2, WW].

(3) Simply connected spaces can also exhibit torsion in their integral cohomology. For the Lie group E_8 we have $H^6(E_8) = \mathbb{Z}/2$.

(4) Set $Y = \mathrm{Met}(S^{10})/\mathrm{Diff}(S^{10})$. Then $\pi_1(Y) = \pi_0(\mathrm{Diff}(S^{10})) = \mathbb{Z}/992$. (We gloss over the fact that $\mathrm{Diff}(S^{10})$ does not act freely on $\mathrm{Met}(S^{10})$; strictly speaking, we should work with the equivariant quotient.) It follows that $H^2(Y) = \mathbb{Z}/992$. The determinants considered in [W1] are sections of a line bundle $\mathscr{L} \to Y$, which is determined topologically by $c_1(\mathscr{L}) \in \mathbb{Z}/992$. Witten's global anomaly detects this topological invariant of the determinant line bundle. In general, however, the global anomaly is *not* determined by the topology of $\mathscr{L}$, but rather depends on the connection $\nabla^{(\mathscr{L})}$, as explained in Sect. 1. Only for torsion loops is the global anomaly purely topological.

Fix an integral cohomology class $c \in H^l(M)$. According to (3.2) there is a well-defined free part $c^{\mathrm{free}} \in \mathrm{Free}\, H^l(M) \simeq \mathrm{Hom}(H_l(M), \mathbb{Z})$, which can be detected by evaluating on l-dimensional cycles. In terms of differential forms, c is represented by a closed l-form $\gamma \in \Omega^l(M)$, and if $F: Q \to M$ is a map of a closed l-manifold (or differentiable cycle) Q into M, then Q carries a homology class $[Q] \in H_l(M)$, and

$$\langle [Q], c \rangle = \int_Q F^*(\gamma). \tag{3.6}$$

The angle brackets denote the pairing $H_l(M) \otimes H^l(M) \to \mathbb{Z}$. So (3.6) describes an analytic method of detecting $\mathrm{Free}\, H^l(M)$, as is well-known to physicists.

The torsion is more difficult to detect. Note from (3.2) that the torsion part of c is well-defined only when $c^{\mathrm{free}} = 0$. The key to detecting the integral information in c beyond the rational information is to start with torsion information in homology. Thus fix $p \in \mathrm{Tor}\, H_{l-1}(M)$. We will use p to denote both this homology class and a cycle representing the homology class. Since p is a torsion class, $k \cdot p = 0$ in $H_{l-1}(M)$ for some positive integer k. Choose an l-chain $q \in C_l(M)$ with $\partial q = k \cdot p$ (as chains). Thus $\partial\left(\frac{1}{k} q\right) = p$ is an integral chain, so vanishes (mod 1). Let $\overline{\frac{1}{k} q}$ denote this chain with coefficients reduced (mod 1); then $\overline{\frac{1}{k} q}$ is closed. Hence $\overline{\frac{1}{k} q}$ represents an element of $H_l(M; \mathbb{Z}/k)$. [Note $\mathbb{Z}/k \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z}$; the cyclic group $\mathbb{Z}/k$ is $\left\{\frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}\right\}$. We often write $H_l(M; \mathbb{Q}/\mathbb{Z})$ or $H_l(M; \mathbb{R}/\mathbb{Z})$ even though $\overline{\frac{1}{k} q}$ lives in the more precise group $H_l(M; \mathbb{Z}/k)$.] Finally, we can evaluate the integral class c on $\frac{1}{k} q$ to obtain

$$\left\langle \overline{\frac{1}{k} q}, c \right\rangle \in \mathbb{Z}/k. \tag{3.7}$$

This is the basic torsion invariant in cohomology. Notice that in general it depends on q (not just on p). However, if c is a torsion class then we have the following

Proposition 3.8. *Suppose that $c^{\mathrm{free}} = 0$. Then c is a torsion class, which by* (3.4) *can be identified with an element $c^{\mathrm{tor}} \in \mathrm{Hom}(\mathrm{Tor}\, H_{l-1}(M); \mathbb{Q}/\mathbb{Z})$. Furthermore,*

$$\left\langle \overline{\frac{1}{k} q}, c \right\rangle = \langle p, c^{\mathrm{tor}} \rangle \tag{3.8}$$

under this identification. In particular, this $\mathbb{Z}/k$ invariant depends only on p, not on q.

Dennis Sullivan uses these invariants to study general properties of manifolds.[4] His point of view is that an integral cohomology class can be specified by its $\mathbb{Z}$ periods (3.6) and its $\mathbb{Q}/\mathbb{Z}$ periods (3.7). (This is [MS, Sect. 2].) Our present interest is not in these topological discussions, but rather in analytic expressions for these invariants when c is a characteristic class of a vector bundle.

Consider first a line bundle $L\to M$. Endow L with a Hermitian structure and unitary connection, and denote the curvature 2-form by $\Omega^{(L)}$. Set $\gamma=\frac{\mathrm{i}}{2\pi}\Omega^{(L)}$; then γ is a closed form representing the real cohomology class $c_1(L)^{\mathbb{R}}\in H^2(M;\mathbb{R})$. Now suppose $f:P\to M$ is a loop ($P=S^1$) which represents a torsion element in $H_1(M)$. Then we can find a 2-manifold Q whose boundary ∂Q consists of k disjoint copies of P, and a map $F:Q\to M$ which restricts on each boundary component $(\partial Q)_i$ to the map $f:P\to M$ (see Fig. 3). Consider the expression

$$\frac{i}{2\pi}\ln\operatorname{hol}(P)+\frac{1}{k}\int_Q F^*(\gamma)\qquad(\operatorname{mod}1).\tag{3.9}$$

(The signs are explained by the fact that the log holonomy is *minus* the integral of the connection form around the loop.) By Stokes' theorem k times this expression vanishes. Therefore, it is a topological invariant (taking values in $\mathbb{Z}/k$), and it should come as no surprise that

Proposition 3.10. *The analytic expression in* (3.9) *is* $\left\langle\overline{\frac{1}{k}q},c_1(L)\right\rangle$, *where* $\overline{\frac{1}{k}q}\in H_2(M;\mathbb{Z}/k)$ *is the fundamental class carried by the quotient space* $\bar{Q}$ *obtained from* Q *by identifying the boundary components with* P.

The identification space $\bar{Q}$ is called a $\mathbb{Z}/k$-*manifold* by Sullivan, and plays an important role here in deriving analytic expressions for $\mathbb{Z}/k$ invariants. For flat connections (3.8) and (3.10) combine to show that the holonomy around torsion loops gives the torsion first Chern class.

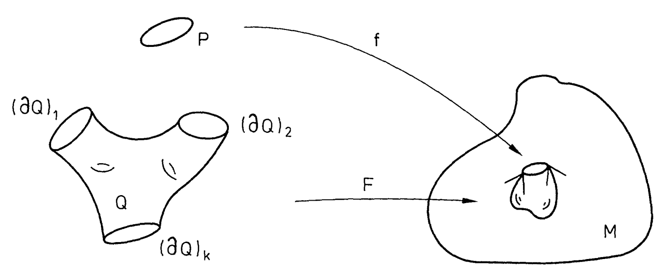

Fig. 3

We generalize to a Hermitian vector bundle $E\to M$ using Chern-Weil Theory. To each unitary connection was assign differential forms $\gamma_i\in\Omega^{2i}(M)$ constructed out of the curvature $\Omega^{(E)}$:

$$\det\left(1+\frac{i}{2\pi}\Omega^{(E)}\right)=1+\gamma_1+\gamma_2+\dots.\tag{3.11}$$

[4] I thank Gunnar Carlsson, Mike Freedman, and Ron Stern for directing me to Sullivan's work

The form γ_i is a de Rham representative for the real Chern class $c_i(E)^{\mathbb{R}} \in H^{2i}(M;\mathbb{R})$. To detect the torsion information in the integral Chern class $c_i(E) \in H^{2i}(M;\mathbb{Z})$ we introduce *Chern-Simons invariants* [CS]. These are differential forms α_{2i-1} which live on the total space of the principal bundle $\pi: P \to M$ associated to E. They are given by explicit formulas in terms of the connection, and have the following properties.

Proposition 3.12 [CS]. (a) $d\alpha_{2i-1} = \pi^* \gamma_{2i}$.

(b) *The Chern-Simons form* α_{2i-1} *determines a map*

$$\bar{\alpha}_{2i-1}: Z_{2i-1}(M) \to \mathbb{R}/\mathbb{Z}$$

on $(2i-1)$*-cycles* $Z_{2i-1}(M)$. *It is given by lifting the cycle to* $\mathscr{P}/U(i-1)$, *integrating* α_{2i-1} *over the lifted cycle, and reducing* (mod 1).

(c) *If* $p \in Z_{2i-1}(M)$ *is the boundary of a* $2i$*-chain* q, *then*

$$\bar{\alpha}_{2i-1}(p) = \int_q \gamma_{2i} \pmod 1. \tag{3.13}$$

Of course, (c) follows from (a) and (b) using Stokes' theorem. We often use manifold representatives P and Q instead of chains, and then we write (3.13) as

$$\int_P \bar{\alpha}_{2i-1} = \int_Q \gamma_{2i} \pmod 1. \tag{3.14}$$

The left-hand side is shorthand notation for the map in [3.12(b)].

Now suppose that $f: P \to M$ is a map of a closed $(2i-1)$-manifold P to M representing a torsion element in $H_{2i-1}(M)$. Suppose also that $F: Q \to M$ extends f as in Fig. 3; the boundary ∂Q consists of k disjoint components each diffeomorphic to P, and F restricts on each component to f. Then the identification space ($\mathbb{Z}/k$-manifold) $\bar{Q}$ carries a fundamental class $\overline{\frac{1}{k}q} \in H_{2i}(M;\mathbb{Z}/k)$ as above.

Proposition 3.15. $\left\langle \overline{\frac{1}{k}q}, c_i(E) \right\rangle = \frac{1}{k}\int_Q \gamma_{2i} - \int_P \bar{\alpha}_{2i-1} \pmod 1$.

This should be regarded as a $\mathbb{Z}/k$ version of the Chern-Weil Theorem. That the right-hand side of (3.15) is a topological invariant is clear; by (3.14) it is k-torsion. The identification of this invariant with the left-hand side of (3.15) proceeds by passing to the universal bundle and connection. The special case (3.10) follows since $\ln \operatorname{hol}(P) = 2\pi i \int_P \bar{\alpha}_1 \pmod 1$.

Examples. (1) For the nontrivial flat bundle over $\mathbb{RP}^2$, the holonomy around the nontrivial loop is -1, so

$$\frac{i}{2\pi} \ln \operatorname{hol}(P) = \frac{i}{2\pi}(i\pi) = \frac{1}{2} \pmod 1.$$

This reflects the fact that the Chern class $c_1 \in H^2(\mathbb{RP}^2;\mathbb{Z}/2) = \mathbb{Z}/2$ is the generator. The flat line bundle over the lens space is similar.

(2) For the determinant line bundle $\mathscr{L} \to Y$ of a family of Dirac operators, (3.10) provides the link between the torsion in $c_1(\mathscr{L})$ and the global anomaly [F1].

Proposition 3.15 extends to principal bundles with compact structure group and more general characteristic classes.

This entire discussion has an analogue in the index theory for Dirac operators. Roughly speaking, the new point is the presence of denominators. For example, on a closed *spin* 4-manifold Q there is a Dirac operator D whose index is given by the Atiyah-Singer Index Theorem [AS3]:

$$\operatorname{index} D = \left\langle [Q], \frac{1}{24} p_1(Q)^{\text{free}} \right\rangle . \tag{3.16}$$

Hence there is a restriction on the Pontrjagin class of a spin 4-manifold: its free part is divisible by 24. More generally, fix a complex vector bundle $E \to M$, M an arbitrary manifold, and consider a map $F: Q \to M$ of a closed, even dimensional spin manifold Q into M. Then the Dirac operator on Q couples to the pullback bundle $F^*(E)$, and the index theorem asserts

$$\operatorname{index} D_{F^*(E)} = \langle [Q], \hat{A}(Q) \operatorname{ch}(F^*(E)) \rangle . \tag{3.17}$$

For simplicity we have omitted the intrinsic twisting bundle V_Q of (1.9(4)). It can, of course, be absorbed into E. If differential geometric data – a metric on Q, metric and unitary connection on E – are specified, then there is a local formula [ABP]

$$\operatorname{index} D_{F^*(E)} = \int_Q \hat{A}(\Omega^{(Q)}) \operatorname{ch}(\Omega^{(F^*(E))}) . \tag{3.18}$$

The analogue of (3.6) is the index theorem, which equates a K-theory invariant with the expressions in (3.18). Note that the characteristic classes of the "probing manifold" Q enter in the K-theory invariant (3.17), whereas the corresponding cohomology invariant (3.6) only involves the fundamental class of Q.

K-theory contains torsion information, just as cohomology does, and there are torsion invariants in K-theory analogous to (3.7). We will not pursue this here, but simply introduce the analytic invariants involved. We state a proposition relating them to Chern-Simons invariants, which is all we need in this paper. These invariants appear in the next section when we discuss global anomalies for the heterotic string.

As above, let $E \to M$ be a Hermitian vector bundle with connection. Suppose P is a $(2i-1)$-dimensional spin manifold, and Q a $2i$-dimensional spin manifold with ∂Q having k disjoint components, each diffeomorphic to P. Choose a metric on Q which is a product near ∂Q and agrees on each boundary component $(\partial Q)_i$ with a fixed metric on P. Finally, suppose $F: Q \to M$ is a map restricting on each $(\partial Q)_i$ to a fixed map $f: P \to M$. On P we have the self-adjoint Dirac operator $A_{f^*(E)}$ coupled to $f^*(E)$. As in Sect. 1 it has an associated spectral invariant $\xi(A_{f^*(E)})$ in $\mathbb{R}/\mathbb{Z}$. Consider

$$\frac{1}{k} \int_Q \hat{A}(\Omega^{(Q)}) \operatorname{ch}(\Omega^{(F^*(E))}) - \xi(A_{f^*(E)}) \pmod{1} . \tag{3.19}$$

We claim that this expression is k-torsion. For multiplying by k the second term becomes the total ξ-invariant of ∂Q, and now the index theorem for manifolds with boundary [APS] expresses the difference as the index of an elliptic boundary value problem on Q. This index is an integer, and so vanishes (mod 1). Therefore, (3.19) is a topological invariant, independent of the differential-geometric data.

The integrand in (3.19) is the Chern-Weil differential form for a combination of Pontrjagin classes of Q and Chern classes of E, but with denominators. In other words, for some positive integer N,

$$\gamma_{2i} = [N \cdot \hat{A}(\Omega^{(Q)}) \operatorname{ch}(\Omega^{(F^*(E))})]_{(2i)}$$

is the Chern-Weil form for (the image in real cohomology of) an integral characteristic class c. For example, if Q is a spin 4-manifold and we consider the ordinary Dirac operator, then $N=24$ and $c=p_1(Q)$ is the integral first Pontrjagin class. Let $\bar{\alpha}_{2i-1}$ be the Chern-Simons form on M associated to the characteristic class c.

Proposition 3.20. *The difference*

$$N \cdot \xi(A_{f^*(E)}) - \int_P \bar{\alpha}_{2i-1} \qquad (\operatorname{mod} 1) \tag{3.21}$$

is a spin bordism invariant depending only on the (stable) class of $f^*E \to P$ in $\Omega^{\mathrm{spin}}_{2i-1}(\mathbb{Z} \times BU)$. In particular, if P bounds a spin manifold over which f^*E extends, then (3.21) *vanishes.*

The stable class of $f^*E \to P$ is an element of $K(P)$, which determines a homotopy class of maps $P \to \mathbb{Z} \times BU$, and therefore an element $[f^*E] \in \Omega^{\mathrm{spin}}_{2i-1}(\mathbb{Z} \times BU)$. Notice that the manifold M is irrelevant; Proposition 3.20 can be stated purely in terms of P. This relationship between ξ-invariants and Chern-Simons invariants is proved in [APSII, Sect. 4]. [It follows easily from (3.13) and the index theorem for manifolds with boundary.] As remarked in that paper, the ξ-invariant is more delicate than the corresponding Chern-Simons invariant.

A simple example shows that (3.21) does not vanish identically. We take P to be the circle and omit the extrinsic bundle E. The circle carries two spin structures. Denote the trivial one by S^1_{trivial} and the nontrivial one by S^1_α. The disk D^2 has a trivial spin structure, and one can check that $\partial(D^2) = S^1_\alpha$. In fact, the spin bordism class of S^1_{trivial} generates $\Omega^{\mathrm{spin}}_1 = \mathbb{Z}/2$. Now the Dirac operator on S^1 is $\frac{id}{d\theta}$, and we easily compute

$$\xi(S^1_{\mathrm{trivial}}) = \tfrac{1}{2} \qquad (\operatorname{mod} 1),$$

$$\xi(S^1_\alpha) = 0 \qquad (\operatorname{mod} 1).$$

(In both cases the η-invariants vanishes, but only for S^1_{trivial} does the Dirac operator have a kernel.) On the other hand, $\hat{A}$ has no component of degree two, so that $N=1$ and $c=0$. Hence $\bar{\alpha}_1 = 0$ also. For S^1_{trivial}, then, the difference (3.21) equals 1/2. Therefore, (3.21) detects the nontrivial element in $\Omega^{\mathrm{spin}}_1 = \mathbb{Z}/2$.

An immediate consequence of (3.20) is the relationship between the K-theory invariant (3.19) and the cohomology invariant (3.15).

Corollary 3.22. *The difference*

$$N\cdot\left\{\frac{1}{k}\int_Q \hat{A}(\Omega^{(Q)})\operatorname{ch}(\Omega^{(F^*(E))})-\xi(A_{f^*(E)})\right\}-\left\{\frac{1}{k}\int_Q \gamma_{2i}-\int_P \bar{\alpha}_{2i-1}\right\}\quad(\mathrm{mod}\,1)$$

is a spin bordism invariant depending on $[f^*E]\in\Omega_{2i-1}^{\mathrm{spin}}(\mathbb{Z}\times BU)$.

Again, because of the presence of denominators, the K-theory invariant is more delicate than the corresponding cohomology invariant.

4. The Heterotic String

In this section we consider anomalies in the $E_8\times E_8$ heterotic string for a general curved background spacetime. The local anomaly cancellation was first discovered by Green and Schwartz [GS] in the low energy 10 dimensional field theory limit. Of course, this sparked the current excitement in String Theory. In the string theory setting the computation of the local anomaly is somewhat simpler. Global anomalies in the heterotic string have already been treated by Witten in [W2]. One purpose here is to illustrate how the torsion invariants of Sect. 3 enter into global anomaly considerations. We use these invariants to generalize Witten's arguments and thereby give sufficient conditions for global anomaly cancellation. Unfortunately, some variations on the algebraic topology presented in Sect. 3 are required. Again we choose not to burden the exposition with all of the proofs, but leave them instead for the enterprising reader.

Rather than start from the beginning – an action on the space of all metrics – we simply set out the combination of determinants which result from the fermionic integration in the heterotic string. Indeed, we immediately pass to the space of conformal structures $\mathrm{Conf}(\Sigma)$. As spacetime M we take an arbitrary 5 dimensional complex Hermitian manifold, which is usually the product of $\mathbb{C}^2$ with a so-called Calabi-Yau manifold, that is, a 3 dimensional compact manifold with vanishing first Chern class. Let $V_i\to M$, $i=1,2$ be complex rank 8 Hermitian bundles over M and $\nabla^{(V_i)}$ unitary connections.[5] Finally, denote by $\mathrm{Spin\,Str}(\Sigma)$ the set of 2^{2g} spin structures on the Riemann surface Σ of genus $g\geqq 2$. The family of Riemann surfaces relevant to the heterotic string is

$$\begin{aligned}&Z=\mathrm{Conf}(\Sigma)\times\mathrm{Map}(\Sigma,M)\times(\mathrm{Spin\,Str}(\Sigma))^3\times\Sigma/\mathrm{Diff}_0(\Sigma)\\&\downarrow{\scriptstyle\Sigma}\\&Y=\mathrm{Conf}(\Sigma)\times\mathrm{Map}(\Sigma,M)\times(\mathrm{Spin\,Str}(\Sigma))^3/\mathrm{Diff}_0(\Sigma).\end{aligned}\tag{4.1}$$

The connected component $\mathrm{Diff}_0(\Sigma)$ of the diffeomorphism group acts trivially on the spin structures. As in Sect. 2 we use the uniformization theorem to identify $\mathrm{Conf}(\Sigma)$ with $\mathrm{Met}_{-1}(\Sigma)$. Then the vertical tangent bundle in (4.1) has a Riemannian structure. The *complex* vertical tangent bundle is a complex line bundle $L\to Z$ with Hermitian metric. Note that (4.1) is not a holomorphic fibration, since we do not restrict to holomorphic maps $\Sigma\to M$.

[5] This differs slightly from [W, (18)] since we use complex operators – see [SW, (1.1)] for an explanation

We have only factored out the identity component of $\mathrm{Diff}(\Sigma)$, and so must account for the action of the mapping class group $\pi_0 \mathrm{Diff}(\Sigma)$ later. The configuration space Y splits into components corresponding to different triples of spin structures and different homotopy classes of maps $\Sigma \to M$. If we treat all genera simultaneously, then the genus is a further discrete invariant distinguishing components of Y. Recall that when the anomaly vanishes, which will be true for the heterotic string, then the determinant of the Dirac operator is only determined up to a phase on each component of Y.

The natural evaluation map $\mathrm{Map}(\Sigma, M) \times \Sigma \to M$ factors through the action of the diffeomorphism group to give a diagram

$$\begin{array}{ccc} \mathbf{V}_i, \mathbf{TM} & \rightarrow & V_i, TM \\ \downarrow & & \downarrow \\ Z & \xrightarrow{e} & M \\ {\scriptstyle \pi}\downarrow{\scriptstyle \Sigma} & & \\ Y & & \end{array} \tag{4.2}$$

[compare (1.8)]. The pulled back bundles over Z also have Hermitian connections.

The final piece of physical data is a 3-form $H \in \Omega^3(M)$. This 3-form is supposed to satisfy

$$dH = \mathrm{ch}_2(\Omega^{(M)}) - \mathrm{ch}_2(\Omega^{(V_1)}) - \mathrm{ch}_2(\Omega^{(V_2)}). \tag{4.3}$$

The 4-form $\mathrm{ch}_2(\Omega) = \dfrac{-1}{8\pi^2} \mathrm{Tr}\,\Omega^2$ is the second component of the Chern character form. In the physics literature there is also a potential field B (whose field strength is H), but it will not enter into anomaly considerations. Integrating over the fibers of (4.1) we obtain a 1-form

$$\omega = (2\pi i)\pi_* e^* H \in \Omega^1(Y). \tag{4.4}$$

The physical connection on the determinant line bundle $\mathscr{L}$ is the canonical connection $\nabla^{(\mathscr{L})}$ minus this 1-form.

With all of the data before us, we can display the combination of fermionic determinants in the heterotic string:

$$(\det D_{\alpha, TM})(\det D_{\beta, V_1})^{-1}(\det D_{\gamma, V_2})^{-1}(\det D_{\alpha, L})^{-1}. \tag{4.5}$$

Here α, β, γ is a triple of spin structures which depends on the component of Y. The chiral Dirac operator D is coupled to an extrinsic bundle in the first three terms, and to the complex tangent bundle of Σ in the last term. In general this combination of Dirac operators does not have zero index, so the determinant line bundle $\mathscr{L} \to Y$ has no canonical section. This will not concern us as we are only interested in calculating the curvature and holonomy of the physical connection $\nabla^{(\mathscr{L})} - \omega$.

Our first result states that the local anomaly vanishes.

Proposition 4.6. *The curvature of* $\nabla^{(\mathscr{L})} - \omega$ *vanishes.*

Proof. This curvature is $\Omega^{(\mathscr{L})} - d\omega$, where $\Omega^{(\mathscr{L})}$ is given by (1.19). Let

$$x = \frac{i}{2\pi}\Omega^{(L)} \in \Omega^2(Z)$$

denote the curvature of the complex tangent bundle L; then

$$\hat{A}(\Omega^{(T_{\mathrm{vert}}Z)}) = 1 - \frac{x^2}{24} + \dots .$$

For convenience we use the formal notation $E = TM - V_1 - V_2$. The contribution of the first three determinants in (4.5) to $\Omega^{(\mathscr{L})}/2\pi i$ is

$$\left[\left(1 - \frac{x^2}{24}\right)\left(\operatorname{rank} E + \mathrm{ch}_1(\Omega^{(E)}) + \mathrm{ch}_2(\Omega^{(E)})\right)\right]_{(4)} = -\frac{\operatorname{rank} E}{24}\, x^2 + \mathrm{ch}_2(\Omega^{(E)}). \tag{4.7}$$

The '4' denotes the component of the differential form of degree 4. The contribution from the last determinant in (4.5) is

$$\left[-\left(1 - \frac{x^2}{24}\right)\left(1 + x + \frac{x^2}{2}\right)\right]_{(4)} = -\frac{11}{24}\, x^2 . \tag{4.8}$$

Combining (4.7) and (4.8), we obtain

$$\frac{\Omega^{(\mathscr{L})}}{2\pi i} = \int_{\Sigma} \frac{-(11 + \operatorname{rank} E)}{24}\, x^2 + \mathrm{ch}_2(\Omega^{(E)}). \tag{4.9}$$

But $\operatorname{rank} E = 5 - 8 - 8 = -11$, and so the first term vanishes. Now by (4.3)

$$\frac{d\omega}{2\pi i} = \int_{\Sigma} dH = \int_{\Sigma} \mathrm{ch}_2(\Omega^{(E)}),$$

which, together with (4.9), shows that the curvature of the physical connection on $\mathscr{L}$ vanishes.

We next consider the global anomaly. Fix spin structures α, β, γ and restrict to the corresponding components of Y. Then our family is equivariant with respect to the subgroup $G_{\alpha,\beta,\gamma} \subset \pi_0 \operatorname{Diff}(\Sigma)$ of the mapping class group fixing the triple of spin structures. Since a Riemann surface can have nontrivial automorphisms, this group does not act freely in general. Therefore, we should pass to equivariant arguments. If the reader is happy with such ideas, then he should use the equivariant quotient $\tilde{Y} = (Y \times EG_{\alpha,\beta,\gamma})/G_{\alpha,\beta,\gamma}$; else, he can view $\tilde{Y}$ as $Y/G_{\alpha,\beta,\gamma}$. The fundamental group $\pi_1 \tilde{Y}$ is a twisted product of $\pi_1 \operatorname{Map}(\Sigma, M)$ and $G_{\alpha,\beta,\gamma}$. Suppose that $S^1 \to \tilde{Y}$ is a nontrivial loop. Let $P \to S^1$ be the induced fibering of Riemann surfaces. Witten's holonomy formula (1.21) demands that we evaluate a combination of ξ-invariants, involving 3 different spin structures. The following two lemmas reduce us to the case $\alpha = \beta = \gamma$.

Lemma 4.10. *Let P be a 3-manifold and $V_i \to P$, $i = 1, 2$, complex vector bundles. Denote by $[V_i] \in K(P)$ the stable class of V_i. Then $[V_1] = [V_2]$ if and only if* $\operatorname{rank} V_1 = \operatorname{rank} V_2$ *and $c_1(V_1) = c_1(V_2)$.*

In other words, stable complex vector bundles over 3-manifolds are classified by rank and first Chern class. The proof follows easily from the following three facts: stable bundles are classified by $\mathbb{Z} \times BU$; the space BU splits topologically as $K(\mathbb{Z}, 2) \times BSU$; the homotopy groups $\pi_i BSU$ vanish for $i \leqq 3$.

Corollary 4.11. *Suppose $V \to P$ has even rank and $c_1(V) \equiv 0 (\mathrm{mod}\, 2)$. Then there exists a bundle $W \to P$ with $[V] = 2[W]$ in $K(P)$.*

Lemma 4.12. *Let P be a compact 3-manifold with two spin structures $\hat{\alpha}$ and $\hat{\beta}$. Fix a Hermitian vector bundle $V \to P$ with unitary connection, and denote by $\xi_{\hat{\alpha}}$ the ξ-invariant for the Dirac operator on P coupled to V. Then*

$$\xi_{\hat{\alpha}} - \xi_{\hat{\beta}} \equiv 0 \, (\mathrm{mod}\, 1)$$

if the rank of V is divisible by 4 and $c_1(V) \equiv 0 \, (\mathrm{mod}\, 2)$.

The proof of this lemma is deferred to an appendix.[6] Lemma 4.12 applies to our 3-manifold P by combining the vertical spin structures α, β (or α, γ) with the trivial spin structure on S^1 to construct spin structures on P. We use the lemma on the second and third determinants in (4.5). The corresponding ξ-invariants arise in the holonomy calculation below. By the lemma we can replace the spin structures β and γ with α. Henceforth, we will restrict our attention to the case where all three spin structures are equal.

Vector bundles E whose first Chern class vanishes modulo 2 are associated to the nontrivial double covering group $\mathrm{U}(n)_2$ of $\mathrm{U}(n)$. These are the complex vector bundles which admit a spin structure. Recall that $\mathrm{U}(n)$ bundles have an integral four dimensional characteristic class $p_1 = (c_1)^2 - 2c_2$. Of course, $\mathrm{U}(n)_2$ bundles can also be regarded as $\mathrm{U}(n)$ bundles, and so carry this characteristic class.

Lemma 4.14. *There is a unique integral characteristic class λ of $\mathrm{U}(n)_2$ bundles with $2\lambda = p_1$. For bundles with $c_1 = 0$ the class $\lambda = -c_2$. The image of λ in real cohomology is the second Chern character class ch_2, which is represented via Chern-Weil Theory by the 4-form $\dfrac{-1}{8\pi^2} \mathrm{Tr}\, \Omega^2$.*

A given $\mathrm{U}(n)$ bundle $E \to M$ has a lift $\tilde{E}$ to a $\mathrm{U}(n)_2$ bundle if and only if $c_1(E) \equiv (\mathrm{mod}\, 2)$. There may be several lifts; differences of lifts are parametrized by $H^1(M; \mathbb{Z}/2)$. However, the characteristic class $\lambda(\tilde{E})$ is independent of the lift. All of these arguments extend to stable complex vector bundles by replacing $\mathrm{U}(n)_2$ with $\mathrm{U}(\infty)_2$.

The exact conditions for the cancellation of the global anomaly are somewhat difficult to state, as we explain below. Therefore, we state a set of sufficient conditions, which are topological restrictions on the spacetime M and extrinsic bundles V_i.

Theorem 4.15. *The following are sufficient conditions for the cancellation of global anomalies:* (i) $c_1(M) \equiv c_1(V_1) \equiv c_1(V_2) \equiv 0 \quad (\mathrm{mod}\, 2)$; (ii) $\lambda(E) = 0$.

However, we may have to adjust H (by a closed 3-form) to cancel all global anomalies.[7]

Remarks. (1) The first condition states that the spacetime M and the extrinsic bundles V_i admit spinors. This seems to be a reasonable physical hypothesis. We

[6] The author thanks Vafa for focusing on an erroneous factor of 2 in the original version. The appendix is written jointly with John Morgan

use it to avoid K-Theory. However, a more careful study may in fact show that (i) is a necessary condition for anomaly cancellation. A consequence of (i) is that $c_1(E) \equiv 0 \pmod 2$, so that $\lambda(E)$ is well-defined. We remind the reader that the stronger hypothesis in (i) – that $c_1 \pmod 2$ vanish for each bundle separately – is used in Lemma 4.12.

(2) The condition on integral characteristic classes is too strong for anomaly cancellation. The torsion information in λ is detected by mappings of 3-manifolds into M, as discussed in Sect. 3. (Because we are in dimension 3 we can use manifolds instead of chains.) However, we are restricted to a spherical class of 3-manifolds – those which fiber over the circle. In general, then, the condition we state above is sufficient, but not necessary (cf. the discussion in [W2, p. 89]). There is another physical constraint beyond anomaly cancellation which may render the condition on integral Pontrjagin classes necessary: unitarity of the theory.[8] This requires a certain "factorization" condition on the Dirac determinants, which has been discussed in other contexts [V, SW]. In particular, the factorization condition places strong restrictions on the phases assigned to the Dirac determinants.

(3) The conditions in Vafa [V] for modular invariance are exactly (i) and (ii), suitably interpreted in the equivariant context.

Proof. Fix a loop $S^1 \to Y$. From (4.2) we obtain a diagram

$$\begin{array}{ccc} \mathbf{E} & \to & E \\ \downarrow & & \downarrow \\ P & \xrightarrow{e} & M \\ \downarrow{\scriptstyle \Sigma} & & \\ S^1 & & \end{array} \tag{4.16}$$

The 3-manifold P carries a spin structure $\hat{\alpha}$ and we can form the self-adjoint Dirac operator corresponding to the combination of operators indicated in (4.5). By Theorem 1.21 the holonomy of the physical connection $\nabla^{(\mathscr{L})} - \omega$ is

$$\exp - 2\pi i \left[\left(\lim_{\varepsilon \to 0} \xi_\varepsilon \right) - \int_P e^* H \right]. \tag{4.17}$$

[Recall the definition of ω in (4.4).] Note that each operator in (4.5) has even index by our hypothesis (i). We first show that ξ_ε is independent of ε.

Lemma 4.18. *The ξ-invariant for our particular combination of Dirac operators is independent of the metric on S^1.*

Proof. We have already described the general formula for the variation of the ξ-invariant in (1.24). The relevant diagram for this case is

$$\begin{array}{c} \mathbf{E} \\ \downarrow \\ \mathrm{Met}(S^1) \times P \\ \downarrow{\scriptstyle P} \\ \mathrm{Met}(S^1) \end{array}$$

[7] In physical terms, this is accomplished by adding a Wess-Zumino term [W2, p. 90]

[8] This was suggested to me by Witten. He also mentioned that Killingback has a different approach to the global anomaly question which sheds light on this issue

and the variation of ξ is a 1-form on $\mathrm{Met}(S^1)$ given by (4.9) as

$$d\xi = \int_P \mathrm{ch}_2(\Omega^{(E)}) \quad (\mathrm{mod}\, 1). \tag{4.19}$$

But $\Omega^{(E)}$ is independent of the metric on the circle, and so lives as a form on P. Hence $\mathrm{ch}_2(\Omega^{(E)})$ is a 4-form on the 3-manifold P, which therefore vanishes.

Of course, we could replace $\mathrm{Met}(S^1)$ by $\mathrm{Met}(P)$ in the preceding. The ξ-invariant does depend on the connection $\varphi^* \nabla^{(E)}$ induced by $\varphi \in \mathrm{Map}(\Sigma, M)$, the variation over a one-parameter family being given by (4.19). Therefore, the combination

$$\int_N e^*H - \xi \quad (\mathrm{mod}\, 1) \tag{4.20}$$

which appears in (4.15) is independent of $\mathrm{Met}(\Sigma)$ and $\mathrm{Map}(\Sigma, M)$, hence is a topological invariant.

Although this invariant (4.20) is defined for general vector bundles E, we will use hypothesis (i) to simplify the discussion, as (4.20) then has a cohomological interpretation. As remarked in (1), this hypothesis implies $c_1(E) \equiv 0\,(\mathrm{mod}\,2)$ so that E carries a four dimensional characteristic class $\lambda(E)$ characterized by (4.14). The point is that for arbitrary complex vector bundles we would need to multiply by 2 before obtaining a *cohomology* invariant.

Lemma 4.21. *Let $\tilde{\alpha}: H_3(M) \to \mathbb{R}/\mathbb{Z}$ be the Chern-Simons invariant associated to the class $\lambda(E)$ and connection $\nabla^{(E)}$. Then*

$$\int_P e^*H - \xi = \int_P (e^*H - \tilde{\alpha}) \quad (\mathrm{mod}\, 1). \tag{4.22}$$

Proof. By Proposition 3.20 it suffices to show that $e^*E \to P$ and $L \to P$ vanish in $\Omega_3^{\mathrm{spin}}(\mathbb{Z} \times BU)$. (We should work with BU_2 instead of BU in the bordism calculation, but these spaces have the same homotopy type.) Recall that L is the holomorphic tangent bundle to the Riemann surface and enters in the last factor of (4.5). First, it is well-known that $\Omega_3^{\mathrm{spin}} = 0$ so that $P = \partial Q$ for some spin 4-manifold Q. Now the Atiyah-Hirzebruch spectral sequence yields an immediate upper bound on $\Omega_3^{\mathrm{spin}}(\mathbb{Z} \times BU)$ as $\oplus_{i+j=3} H_i(\mathbb{Z} \times BU; \Omega_j^{\mathrm{spin}}) = \mathbb{Z}/2$; the possible generator is the Hopf bundle over $S^2 \times S^1_{\mathrm{trivial}}$. Much more sophisticated techniques can be used to show that $\Omega_3^{\mathrm{spin}}(\mathbb{Z} \times BU)$ vanishes,[9] which immediately gives the lemma. Rather than rely on this result, we give a direct argument that $[e^*E]$ and $[L]$ are divisible by 2 in $\Omega_3^{\mathrm{spin}}(\mathbb{Z} \times BU)$. By our simple upper bound, it follows that these elements vanish.

It suffices to prove that $[e^*E]$ and $[L]$ are divisible by 2 in $K(P)$, for which the exact conditions are stated in (4.11). Recall that $E = TM - V_1 - V_2$. Let V denote any of e^*TM, e^*V_1, e^*V_2, and 1_P the trivial bundle of rank one over P. Then $V \oplus 1_P$ has even rank and its first Chern class vanishes (mod 2) by our simplifying hypothesis (i). Therefore, $V \oplus 1_P$ extends to a bundle $U \to Q$.[10] Now 1_Q clearly

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[9] I thank Lionel Schwartz for explaining this to me

[10] To see that all four bundles extend over the *same* 4-manifold Q, we apply our arguments to the class of $e^*TM \times e^*V_1 \times e^*V_2 \times L$ in $\Omega_3^{\mathrm{spin}}((\mathbb{Z} \times BU)^4)$

extends 1_P, so $[U]-[1_Q]$ extends $[V\oplus 1_P]-[1_P]$ in K-theory. Lemma 4.10 implies $[V\oplus 1_P]-[1_P]=[V]$ in $K(P)$, since the bundles on each side of the equality have the same rank and first Chern class. So $[V]=0$ in $\Omega_3^{\mathrm{spin}}(\mathbb{Z}\times BU)$ as desired. A similar argument applies to L. The fact that we divide only by diffeomorphisms which fix the spin structure on Σ implies that the vertical tangent bundle to our family carriers a spin structure. Therefore, the first Chern class of L is divisible by 2. The argument proceeds as before.

Now it is clear that the right-hand side of (4.22) depends only on the class of P in $H_3(M)$, and so defines a linear functional $\lambda(E)^{\mathbb{R}/\mathbb{Z}}: H_3(M)\to\mathbb{R}/\mathbb{Z}$. By the universal coefficient theorem $\lambda(E)^{\mathbb{R}/\mathbb{Z}}$ is an element of $H^3(M;\mathbb{R}/\mathbb{Z})$. Furthermore, there is an exact sequence

$$H^3(M;\mathbb{R}) \xrightarrow{j} H^3(M;\mathbb{R}/\mathbb{Z}) \xrightarrow{\delta} H^4(M;\mathbb{Z}) \longrightarrow H^4(M;\mathbb{R}).$$

Our work in Sect. 3 implies $\delta(\lambda(E)^{\mathbb{R}/\mathbb{Z}})=\lambda(E)\in H^4(M;\mathbb{Z})$. Now our second hypothesis on the topology of E is $\lambda(E)=0$. Over the reals this is already contained in (4.3); the crucial information here is torsion. It follows that for some $v\in H^3(M;\mathbb{R})$ we have $j(v)=\lambda(E)^{\mathbb{R}/\mathbb{Z}}$. Represent v by a closed 3-form $\tilde{v}$ and replace H by $H-\tilde{v}$. Since $d\tilde{v}=0$, Eq. (4.3) is still satisfied. Also, for this new H we have $\lambda(E)^{\mathbb{R}/\mathbb{Z}}=0$. So (4.22) vanishes for all P, in particular for those arising from loops in Y [see (4.16)], which shows that the holonomy (4.17) is trivial. Therefore, the heterotic string has no global anomalies if (i) and (ii) are satisfied.

We emphasize that the relationship of $\lambda(E)^{\mathbb{R}/\mathbb{Z}}$ to our torsion invariants is Proposition 3.8. The torsion in $\lambda(E)$ is simply the restriction of $\lambda(E)^{\mathbb{R}/\mathbb{Z}}: H_3(M)\to\mathbb{R}/\mathbb{Z}$ to the torsion subgroup $\mathrm{Tor}\, H_3(M)$ [cf. (3.4)].

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Appendix

Daniel S. Freed[1] and John W. Morgan[2]

[1] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02138, USA

[2] Department of Mathematics, Columbia University, New York, NY 10027, USA

In this appendix we present a proof of Lemma 4.12.

Lemma 4.12. *Let P be a compact 3-manifold with two spin structures $\hat{\alpha}$ and $\hat{\beta}$. Fix a Hermitian vector bundle $V\to P$ with unitary connection, and denote by $\xi_{\hat{\alpha}}$ the ξ-invariant for the Dirac operator on P coupled to V. Then*

$$\xi_{\hat{\alpha}}-\xi_{\hat{\beta}}\equiv 0 \pmod 1$$

if the rank of V is divisible by 4 and $c_1(V)\equiv 0 \pmod 2$.

Proof. The index theorem for flat bundles [APSIII] gives a topological formula for this difference of ξ-invariants. The two spin structures differ by a flat bundle, which is an element $[\hat{\alpha}-\hat{\beta}]\in K^{-1}(P;\mathbb{Q}/\mathbb{Z})$. Denote the K-Theory class of V by $[V]\in K(P)$. Then the spin structure $\hat{\beta}$ gives rise to a direct image map $p_!: K^{-1}(P;\mathbb{Q}/\mathbb{Z})\to\mathbb{Q}/\mathbb{Z}$, and the index theorem states

$$\xi_{\hat{\alpha}}-\xi_{\hat{\beta}}\equiv p_!([V]\cdot[\hat{\alpha}-\hat{\beta}]) \pmod 1 .$$

The lemma will follow from the stronger assertion that

$$[V]\cdot[\hat{\alpha}-\hat{\beta}]=0 \quad \text{in} \quad K^{-1}(P;\mathbb{Q}/\mathbb{Z}). \tag{A.1}$$

We first recall the definition of $[\hat{\alpha}-\hat{\beta}]\in K^{-1}(P;\mathbb{Q}/\mathbb{Z})$ from [APSII, Sect. 5]. The difference of spin structures determines a homomorphism $\pi_1(P)\to\mathbb{Z}/2$, and therefore a homotopy class of maps $f: P\to B(\mathbb{Z}/2)\sim\mathbb{RP}^\infty$. Define $K^*(\mathbb{RP}^\infty)$ as the inverse limit $\varprojlim_N K^*(\mathbb{RP}^N)$. The reduced groups $\tilde{K}^*(\mathbb{RP}^\infty)\otimes\mathbb{Q}$ vanish, since the rational cohomology of $\mathbb{RP}^\infty$ is trivial. From the K-Theory exact sequence associated to the coefficient sequence $\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}$ we deduce $\tilde{K}^0(\mathbb{RP}^\infty)\cong K^{-1}(\mathbb{RP}^\infty;\,\mathbb{Q}/\mathbb{Z})$. Let $[\zeta]$ be the class of the nontrivial complex line bundle over $\mathbb{RP}^\infty$. Then, by the above isomorphism, $[\zeta]-1$ determines an element of $K^{-1}(\mathbb{RP}^\infty;\,\mathbb{Q}/\mathbb{Z})$, and $[\hat{\alpha}-\hat{\beta}]\in K^{-1}(P;\,\mathbb{Q}/\mathbb{Z})$ is the pullback of this element via f.

Our first observation is that f can be realized by a map into $\mathbb{RP}^4$. This follows from simple obstruction theory – any map from an n-complex to a space Y is homotopic to a map into the n-skeleton of Y.[11] Now $\tilde{K}^0(\mathbb{RP}^4)=\mathbb{Z}/4$ and $K^{-1}(\mathbb{RP}^4)=0$. (See [At, p. 107], for example.) Hence $K^{-1}(\mathbb{RP}^4;\,\mathbb{Q}/\mathbb{Z})=\mathbb{Z}/4$, and so $4[\hat{\alpha}-\hat{\beta}]=0$ in $K^{-1}(P;\mathbb{Q}/\mathbb{Z})$. By the same reasoning, over the 2-skeleton $P^{(2)}$ the difference of spin structures is realized by a map into $\mathbb{RP}^2$. Since $\tilde{K}^0(\mathbb{RP}^2)=\mathbb{Z}/2$ and $K^{-1}(\mathbb{RP}^2)=0$, we have $K^{-1}(\mathbb{RP}^2;\,\mathbb{Q}/\mathbb{Z})=\mathbb{Z}/2$. Therefore, the restriction of $2[\hat{\alpha}-\hat{\beta}]$ to $P^{(2)}$ is zero.

Complex vector bundles on a 3-dimensional space are classified by rank and first Chern class (cf. Lemma 4.10 and Corollary 4.11). Hence $[V]=2[W]$ in $K(P)$ for some bundle W. Note that W has even rank. By the preceding paragraph we see that $2\cdot[\hat{\alpha}-\hat{\beta}]$ vanishes on $P^{(2)}$, so can be lifted to $a\in K^{-1}(P,P^{(2)};\,\mathbb{Q}/\mathbb{Z})$. The following lemma computes the product $[W]\cdot a$.

Lemma A.2. *Let X be any finite n-dimensional CW complex, and $X_0\subset X$ the complement of an open n-cell. Let $[W]$ be an element of $K(X)$, and let G be an abelian group. Then the multiplication map*

$$\cdot[W]: K^*(X,X_0;G)\to K^*(X,X_0;G)$$

is multiplication by the rank of W.

Proof. First consider the case $G=\mathbb{Z}$. By excision and Bott periodicity,

$$K^*(X,X_0)=\tilde{K}^*(X/X_0)=\tilde{K}^*(S^n)=\begin{cases}\mathbb{Z}, & \text{for} \quad *\equiv n \pmod 2,\\ 0, & \text{for} \quad *\not\equiv n \pmod 2.\end{cases}$$

[11] It follows that we can actually push f into $\mathbb{RP}^3$. However, for our present purposes $\mathbb{RP}^4$ suffices and is simpler to use (since $K^{-1}(\mathbb{RP}^4)=0$)

Furthermore, the isomorphism $\tilde{K}^0(S^n)\cong\mathbb{Z}$ is given by the Chern character $\mathrm{ch}_{n/2}$. The desired result now follows from the product formula for the Chern character and from the fact that ch_i vanishes on $K^*(X, X_0)$ for $i<n/2$.

To treat arbitrary G we recall that associated to G is a Moore space M_G with $H_1(M_G)=G$ and all other reduced homology groups zero. The group $K^*(Y, A; G)$ is defined to be $K^*(Y\wedge M_G,\ A\wedge M_G)$. Since $K^*(X, X_0)$ is free, the Künneth formula implies $K^*(X, X_0; G)=K^*(X, X_0)\otimes\tilde{K}^*(M_G)$. This decomposition is natural with respect to multiplication by $[W]\in K^*(X)$, and the lemma follows easily.

Let φ denote the natural map $\varphi: K^{-1}(P, P^{(2)}; \mathbb{Q}/\mathbb{Z})\to K^{-1}(P; \mathbb{Q}/\mathbb{Z})$. Then $\varphi(a)=2[\hat{\alpha}-\hat{\beta}]$ by definition, and $\varphi([W]\cdot a)=[W]\cdot 2[\hat{\alpha}-\hat{\beta}]=[V]\cdot[\hat{\alpha}-\hat{\beta}]$. But the lemma implies $[W]\cdot a=\mathrm{rk}\, W\cdot a$, from which $\varphi([W]\cdot a)=\varphi(\mathrm{rk}\, W\cdot a)$ $=\mathrm{rk}\, V\cdot[\hat{\alpha}-\hat{\beta}]$. This vanishes as the rank of V is divisible by 4.

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