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Inequalities for the Schatten p-Norm. III

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Abstract. We present some inequalities for the Schatten p-norm of operators on a Hilbert space. It is shown, among other things, that if A is an operator such that $\operatorname{Re} A \geqq a \geqq 0$, then for any operator X, $\|AX+XA^*\|_p \geqq 2a\|X\|_p$. Also, for any two operators A and B, $\||A|-|B|\|_2^2 + \||A^*|-|B^*|\|_2^2 \leqq 2\|A-B\|_2^2$.

In their investigation on the quasi-equivalence of quasi-free states of canonical commutation relations, Araki and Yamagami [1] proved that for any two bounded linear operators A and B on a Hilbert space H, $\||A|-|B|\|_2 \leqq 2^{1/2}\|A-B\|_2$. Also, in working on the approach to an equilibrium in harmonic chain or the elementary excitation spectrum of a random ferromagnet, as mentioned in [3], one may encounter the following useful inequality due to van Hemmen and Ando [3, Lemma 3.1]. If X is a compact operator and A is an operator such that $A \geqq a \geqq 0$, then $\|AX+XA\|_p \geqq 2a\|X\|_p$. This inequality is related to the one proved by the author in [4, Theorem 3]. The inequality in [4, Theorem 3] is equivalent to that $\|AX+XA^*\|_p \geqq a\|X\|_p$ for any operator X, whenever $\dfrac{A+A^*}{2} \geqq a \geqq 0$. But as seen from the proof if X is assumed to be self-adjoint (or even seminormal), then $\|AX+XA^*\|_p \geqq 2a\|X\|_p$.

It is the object of this note to present the best possible extension of this result by removing the restriction on X. We will prove a general theorem which gives the above mentioned inequalities in [3 and 4] as corollaries. The technique developed for this purpose proves to be useful also in extending the Araki and Yamagami result and it is likely to have further applications.

An operator means a bounded linear operator on a separable, complex Hilbert space H. Let $B(H)$ denote the algebra of all bounded linear operators acting on H. Let $K(H)$ denote the ideal of compact operators on H. For any compact operator A, let $s_1(A), s_2(A), \ldots$ be the eigenvalues of $|A| = (A^*A)^{1/2}$ in decreasing order and repeated according to multiplicity. A compact operator A is said to be in the Schatten p-class C_p $(1 \leqq p < \infty)$, if $\sum_i s_i(A)^p < \infty$. The Schatten p-norm of A is defined by $\|A\|_p = \left(\sum_i s_i(A)^p\right)^{1/p}$. This norm makes C_p into a Banach space. Hence C_1 is the trace class and C_2 is the Hilbert-Schmidt class. It is reasonable to let C_∞ denote the ideal of compact operators $K(H)$, and $\|\cdot\|_\infty$ stand for the usual operator norm.

If $A \in C_p$ $(1 \leqq p < \infty)$ and $\{e_i\}$ is any orthonormal set in H, then $\|A\|_p^p \geqq \sum_i |(Ae_i, e_i)|^p$. More generally, if $\{E_i\}$ is a family of orthogonal projections satisfying $E_i E_j = \delta_{ij} E_i$, then

$$\|A\|_p^p \geqq \sum_i \|E_i A E_i\|_p^p = \left\|\sum_i E_i A E_i\right\|_p^p,$$

and for $p > 1$ equality will hold if and only if $A = \sum_i E_i A E_i$. Moreover, if $\sum_i E_i = 1$ and $p = 2$, then $\|A\|_2^2 = \sum_{i,j} \|E_i A E_j\|_2^2$. We refer to [2] for further properties of the Schatten p-classes.

It has been shown in [5, Theorem 8] that if $A, B \in B(H)$ with $A + B \geqq c \geqq 0$, then for any self-adjoint operator X, $\|AX + XB\|_p \geqq c\|X\|_p$ for $1 \leqq p \leqq \infty$. This result admits the following considerable generalization. First we need a key lemma.

Lemma. *If* A, $B \in B(H)$ *and* $T = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ *is defined on* $H \oplus H$, *then* $|T| = \begin{pmatrix} |B| & 0 \\ 0 & |A| \end{pmatrix}$. *Moreover,* $\|T\|_p^p = \|A\|_p^p + \|B\|_p^p$ *for* $1 \leqq p < \infty$ *and* $\|T\| = \max(\|A\|, \|B\|)$.

Proof. Since $T^*T = \begin{pmatrix} B^*B & 0 \\ 0 & A^*A \end{pmatrix}$, it follows by the uniqueness of the square root of a positive operator that $|T| = \begin{pmatrix} |B| & 0 \\ 0 & |A| \end{pmatrix}$. Since $\|T\|_p = \||T|\|_p$ $(1 \leqq p \leqq \infty)$, the second assertion now follows from the basic properties of the C_p norm.

Now we are in a position to prove our main result.

Theorem 1. *If* A, $B \in B(H)$ *with* $A + B \geqq c \geqq 0$, *then for any* $X \in B(H)$,

$$\|AX + XB\|_p^p + \|AX^* + X^*B\|_p^p \geqq 2c^p \|X\|_p^p$$

for $1 \leqq p < \infty$ *and* $\max(\|AX + XB\|, \|AX^* + X^*B\|) \geqq c\|X\|$.

Proof. On $H \oplus H$, let $T = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, $S = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$, and $Y = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}$. Then $T + S \geqq c \geqq 0$ and Y is self-adjoint. Now

$$TY + YS = \begin{pmatrix} 0 & AX + XB \\ AX^* + X^*B & 0 \end{pmatrix},$$

and so by the lemma we have

$$\|TY + YS\|_p^p = \|AX + XB\|_p^p + \|AX^* + X^*B\|_p^p$$

for $1 \leqq p < \infty$ and

$$\|TY + YS\| = \max(\|AX + XB\|, \|AX^* + X^*B\|).$$

Also, $\|Y\|_p^p = 2\|X\|_p^p$ for $1 \leqq p < \infty$, and $\|Y\| = \|X\|$ since $\|X^*\|_p = \|X\|_p$ for $1 \leqq p \leqq \infty$. Hence, Theorem 8 in [5] applied to the operators T, S, and Y yields

$$\|AX + XB\|_p^p + \|AX^* + X^*B\|_p^p \geqq 2c^p \|X\|_p^p$$

for $1 \leqq p < \infty$ and

$$\max(\|AX + XB\|, \|AX^* + X^*B\|) \geqq c\|X\|$$

as required.

Corollary 1. *Let* $A \in B(H)$ *with* $\operatorname{Re} A = \dfrac{A + A^*}{2} \geqq a \geqq 0$. *Then* $\|AX + XA^*\|_p \geqq 2a\|X\|_p$ *for all* $X \in B(H)$, *and* $1 \leqq p \leqq \infty$.

Proof. Since $A + A^* \geqq 2a$, the result follows from Theorem 1 applied to A and A^* with the observation that $\|AX + XA^*\|_p = \|AX^* + X^*A^*\|_p$ for $1 \leqq p \leqq \infty$. Here $c = 2a$.

It is now obvious that Corollary 1 extends the corresponding inequality of van Hemmen and Ando [3, Lemma 3.2]. Applying Corollary 1 to the operator $-iA$, gives the following improvement of Theorem 3 in [4].

Corollary 2. *Let* $A \in B(H)$ *with* $\operatorname{Im} A = \dfrac{A - A^*}{2i} \geqq a \geqq 0$. *Then* $\|AX - XA^*\|_p \geqq 2a\|X\|_p$ *for all* $X \in B(H)$, *and* $1 \leqq p \leqq \infty$.

Next we consider the Araki-Yamagami inequality. Namely, for A, $B \in B(H)$, $\||A| - |B|\|_2 \leqq 2^{1/2}\|A - B\|_2$. It has been remarked in [1] that if A and B are restricted to be self-adjoint operators, then $\||A| - |B|\|_2 \leqq \|A - B\|_2$. This result has been recently generalized by the author to the case where A and B are normal operators [6, Corollary 3].

A considerably briefer proof of a generalized Araki-Yamagami inequality is now presented.

Theorem 2. *If* A, $B \in B(H)$, *then*

$$\||A| - |B|\|_2^2 + \||A^*| - |B^*|\|_2^2 \leqq 2\|A - B\|_2^2 .$$

Proof. On $H \oplus H$, let $T = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$, and $S = \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}$. Then T and S are self-adjoint. Applying the Araki-Yamagami remark to the operators T and S, we obtain that $\||T|\| - |S|\|_2 \leqq \|T - S\|_2$. But

$$|T| - |S| = \begin{pmatrix} |A^*| - |B^*| & 0 \\ 0 & |A| - |B| \end{pmatrix}$$

by the lemma, and

$$T - S = \begin{pmatrix} 0 & A - B \\ A^* - B^* & 0 \end{pmatrix}.$$

Since

$$\||T| - |S|\|_2^2 = \||A| - |B|\|_2^2 + \||A^*| - |B^*|\|_2^2$$

and

$$\|T-S\|_2^2 = \|A-B\|_2^2 + \|A^*-B^*\|_2^2 = 2\|A-B\|_2^2,$$

it follows that

$$\| |A|-|B| \|_2^2 + \| |A^*|-|B^*| \|_2^2 \leqq 2\|A-B\|_2^2$$

as required.

Theorem 2 enables us to provide the following alternative proof of Corollary 3 in [6].

Corollary 3. *If* N, $M \in B(H)$, *are normal, then*

$$\| |N|-|M| \|_2 \leqq \|N-M\|_2 .$$

Proof. This follows from Theorem 2 and the fact that for a normal operator N we have $|N| = |N^*|$.

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