Commun. Math. Phys. 102, 337–347 (1985)

Communications in Mathematical Physics

Superconformal Current Algebras and Their Unitary Representations

Victor G. Kac and Ivan T. Todorov*

Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

Abstract. A natural supersymmetric extension $\widehat{(dG)}_\kappa$ is defined of the current (= affine Kac–Moody Lie) algebra $\widehat{dG}$; it corresponds to a superconformal and chiral invariant 2-dimensional quantum field theory (QFT), and hence appears as an ingredient in superstring models. All unitary irreducible positive energy representations of $\widehat{(dG)}_\kappa$ are constructed. They extend to unitary representations of the semidirect sum $S_\kappa(G)$ of $\widehat{(dG)}_\kappa$ with the superconformal algebra of Neveu–Schwarz, for $\kappa = \frac{1}{2}$, or of Ramond, for $\kappa = 0$.

0. Introduction

The semidirect sum of the Virasoro algebra W_c and the algebra $\widehat{dG}$ of left (or right) currents for a compact Lie group G arises naturally in both conformal invariant 2-dimensional QET models [1–3] and in the general study of infinite dimensional Lie algebras [4–7] (see also [8, 9]). Its supersymmetric extension which is implicit in recent work on superstrings [10–12] also admits a local field interpretation (partly exploited in [13, 14] as a development of the QFT approach of [15]).

The objective of this note is two-fold: (a) to set a mathematical framework in which the supercurrent and string superalgebras arise naturally; (b) to classify all hermitian (= unitary) positive energy representations of these algebras. A remark is also included, concerning the unitarity of the discrete series of representations of the super Virasoro algebra (with central charge $c < \frac{3}{2}$).

In the theory of infinite dimensional Lie algebras a chiral current algebra $\widehat{dG}$ (called an *affine Kac–Moody algebra*) arises as a central extension of the *loop algebra* $\widetilde{dG}$ generated by tensor products of elements of the finite dimensional Lie algebra dG with Laurent polynomials of a complex variable t. The supersymmetric extensions

* On leave of absence from the Institute for Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences, BG-1184 Sofia, Bulgaria

$(\widehat{dG})_\kappa$ discussed in this paper are obtained by simply adding a Grassmann variable θ to the argument of the polynomials. We construct a "minimal representation" of the arising algebra which allows us to reduce the representation theory of $(\widehat{dG})_\kappa$ (and of its super Virasoro extension $S_\kappa(G)$) to the known classification of unitary highest weight irreducible representations (UHWIRs) of $\widehat{dG}$.

1. Superconformal Current Algebras

Let G be a compact Lie group and dG be its Lie algebra equipped with the (negative definite) Killing form (x,y). The *super loop algebra* $\widetilde{dG}$ is defined as

$$\widetilde{dG} = dG \otimes_{\mathbb{R}} \mathbb{C}[t,t^{-1};\theta], \quad t\in\mathbb{C}^\times(=\{t\in\mathbb{C}; t\neq 0\}), \quad \theta^2=0, \tag{1.1}$$

regarded as an infinite Lie superalgebra with bracket

$$[x\otimes P(t,t^{-1};\theta), \quad y\otimes Q(t,t^{-1};\theta)] = [x,y]\otimes P(t,t^{-1};\theta)Q(t,t^{-1};\theta), \tag{1.2}$$

where P and Q are any (linear in θ) polynomials and $[x,y]$ is the Lie bracket of dG. We introduce a $\frac{1}{2}\mathbb{Z}$-*gradation* on $\widetilde{dG}$ setting

$$\deg dG = 0, \quad \deg t = 1, \quad \deg\theta = \kappa\in\tfrac{1}{2}\mathbb{Z}; \tag{1.3}$$

the corresponding graded algebra will be denoted by $(\widetilde{dG})_\kappa$. The general even central extension $(\widehat{dG})_\kappa$ of $(\widetilde{dG})_\kappa$ is obtained by adding a cocycle

$$\psi(x\otimes P(t,t^{-1};\theta), \quad y\otimes Q(t,t^{-1};\theta)) = (x,y)f((dP)Q), \tag{1.4a}$$

to the right-hand side of (1.2), where f is a linear functional on 1-forms that vanishes on exact and on odd (in θ) forms:

$$f(P_0dt - P_1\theta d\theta + P_2 d\theta - P_3\theta dt) = \oint_{|t|=1} (\alpha P_0 + \beta t^{2\kappa-1}P_1)\frac{dt}{2\pi i}, \tag{1.4b}$$

where $P_k = P_k(t,t^{-1})$ are polynomials and we assume that α and β are positive numbers[1]. (The powers are chosen in such a way that $\deg\psi = 0$.)

Proposition 1. *The most general graded odd and even differentiations* $D^\varepsilon(\varepsilon=1,0)$ *satisfying*

$$D^\varepsilon f(\{d(P_0+\theta P_1)\}Q) := f(\{dD^\varepsilon(P_0+\theta P_1)\}Q) + f(\{d(P_0+(-1)^\varepsilon\theta P_1)\}D^\varepsilon Q) = 0 \tag{1.5}$$

$(P_{0,1} = P_{0,1}(t,t^{-1}), \quad Q = Q_0(t,t^{-1}) + \theta Q_1(t,t^{-1}))$ *are multiples of*

$$D^1_{n+\kappa} = t^n\left(\sqrt{\frac{\beta}{\alpha}}\,t^{2\kappa}\frac{\partial}{\partial\theta} - t\sqrt{\frac{\alpha}{\beta}}\,\theta\frac{\partial}{\partial t}\right), \quad D^1_{n-\kappa} = t^n\left(\sqrt{\frac{\beta}{\alpha}}\frac{\partial}{\partial\theta} - t^{1-2\kappa}\sqrt{\frac{\alpha}{\beta}}\,\theta\frac{\partial}{\partial t}\right), \tag{1.6a}$$

$$D^0_n = \tfrac{1}{2}[D^1_{n-\kappa}, D^1_\kappa]_+ = -t^n\left\{t\frac{\partial}{\partial t} + \left(\frac{n}{2}+\kappa\right)\theta\frac{\partial}{\partial\theta}\right\}(n\in\mathbb{Z}). \tag{1.6b}$$

We shall sketch the *proof* for the odd generators. Setting $D^1 = R_0\theta(\partial/\partial t) +$

---

1 If $\alpha\beta<0$, then the energy operator L_0, constructed below, would have negative spectrum

$R_1(\partial/\partial\theta)$ $(R_{0,1}=R_{0,1}(t,t^{-1}))$, we find

$$f\left(\left\{d\left(R_0\theta\frac{\partial P_0}{\partial t}+R_1P_1\right)\right\}(Q_0+Q_1\theta)+\{d(P_0-P_1\theta)\}\left(R_0\theta\frac{\partial Q_0}{\partial t}+R_1Q_1\right)\right)$$
$$=\oint\left(Q_1\frac{\partial P_0}{\partial t}-P_1\frac{\partial Q_0}{\partial t}\right)(\alpha R_1+\beta t^{2\kappa-1}R_0)\frac{dt}{2\pi i}=0.$$

Since P and Q are arbitrary, it follows that $\alpha R_1+\beta t^{2\kappa-1}R_0=0$. A basis of homogeneous solutions of this equation is given by (1.6a).

Corollary. *The differential operators (1.6) play the role of superconformal generators, since they act on the* 1*-form*

$$\omega_\kappa=t^{2\kappa-1}dt-\frac{\alpha}{\beta}\theta d\theta \tag{1.7}$$

as a multiplication by a function:

$$D^1_{n+\kappa}\omega_\kappa=0,\quad D^0_n\omega_\kappa=-(2\kappa+n)t^n\omega_\kappa. \tag{1.8}$$

With the change of variables $\theta\to(\sqrt{\beta/\alpha})t^{[\kappa]}\theta$ ($[\kappa]$ being the integer part of κ) we can normalize the ratio α/β in (1.6a) and (1.7) to 1 and reduce the class of graded superalgebras under consideration to two cases: $\kappa=\frac{1}{2}$ and $\kappa=0$. The *super Virasoro algebra* SV_κ is defined as the universal central extension of the algebra of differential operators (1.6). For $\kappa=\frac{1}{2}$ we have the *Neveu–Schwarz algebra* [16]; for $\kappa=0$ we obtain the *Ramond algebra* [17]. We denote the semidirect sum of the superalgebra, SV_κ and $(\widehat{dG})_\kappa$ by $S_\kappa(G)$, and call it the *superconformal current algebra.*

Remark. We have derived the superalgebra $S_\kappa(G)$ starting with the superaffine Lie algebra $(\widehat{dG})_\kappa$ and looking for the most general (super-) differentiations that annihilate the cocycle (1.4). Alternatively, we could obtain $S_\kappa(G)$ starting with the super Virasoro algebra SV_κ coupled to an extension of the ordinary (Bose) current algebra, determined from the super Jacobi identities.

2. A Graded Basis of Physical Generators of $S_\kappa(G)$

Let dG be a simple compact Lie algebra of dimension d_G with a basis x_a satisfying

$$(x_a,x_b)=-C_2\delta_{ab},\quad [x_a,x_b]=f_{abc}x_c,\quad a,b,c=1,\dots,d_G; \tag{2.1}$$

here C_2 is the eigenvalue of the Casimir operator for the adjoint representation of G:

$$\left(\sum_{s=1}^{d_G}\sum_{t=1}^{d_G}\right)f_{sat}f_{sbt}=C_2\delta_{ab}(a,b=1,\dots,d_G); \tag{2.2}$$

if x_1,x_2,x_3 span an su(2) subalgebra, then $f_{123}=1$. We define a graded "physical" basis of the super-extended Kac–Moody Lie algebra $(\widehat{dG})_\kappa$ by

$$Q^a_n=ix_a\otimes t^n, \tag{2.3a}$$

$$h^a_{n+\kappa}=ix_a\otimes t^n\theta. \tag{2.3b}$$

At the price of a possible rescaling of θ as indicated above, we can now write down the following commutation relations for the superalgebra $S_\kappa(G)$:

$$[h^a_{n+\kappa}, h^b_{m-\kappa}]_+ = \frac{\lambda}{2}\delta_{m+n}\delta_{ab}\,(\delta_l \equiv \delta_{l0}), \tag{2.4a}$$

$$[Q^a_n, h^b_{m+\kappa}] = if_{abc}h^c_{n+m+\kappa}, \tag{2.4b}$$

$$[Q^a_n, Q^b_m] = if_{abc}Q^c_{n+m} + \frac{\lambda}{2}n\delta_{n+m}\delta_{ab} \tag{2.4c}$$

(the coefficients α and β in (1.4b) are related to the central charge λ by $\alpha = \beta = \lambda/2C_2$);

$$[h^a_{m+\kappa}, L_n] = \left(m + \kappa + \frac{n}{2}\right)h^a_{m+n+\kappa}, \tag{2.5a}$$

$$[Q^a_m, L_n] = mQ^a_{m+n}, \tag{2.5b}$$

$$[h^a_{m+\kappa}, G_{n-\kappa}]_+ = Q^a_{m+n}, \tag{2.5c}$$

$$[Q^a_m, G_{n+\kappa}] = mh^a_{m+n+\kappa}; \tag{2.5d}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m}, \tag{2.6a}$$

$$[G_{m+\kappa}, L_n] = \left(m + \kappa - \frac{n}{2}\right)G_{m+n+\kappa}, \tag{2.6b}$$

$$[G_{m+\kappa}, G_{n-\kappa}]_+ = 2L_{m+n} + \frac{c}{3}\{(\kappa+m)^2 - \tfrac{1}{4}\}\delta_{m+n}, \tag{2.6c}$$

$$m, n = 0, \pm 1, \pm 2, \ldots; \quad \kappa = 0 \quad \text{or} \quad \tfrac{1}{2}.$$

We notice that only for $\kappa = \frac{1}{2}$ does the algebra (2.6) contain the 5-dimensional superconformal algebra of the circle, generated by $L_0, L_{\pm 1}$ and $G_{\pm 1/2}$.

The superconformal current algebra can be defined in a similar way for an abelian symmetry group $G = U(1)$. In general, it is the direct sum of various G-superalgebras with identified centres.

3. Field Theoretic Interpretation. Hermitian, Positive Energy Representations

Two-dimensional (conformally) compactified Minkowski space is the torus $S^1 \times S^1(/\mathbb{Z}_2)$. The variables $(z, w) \in S^1 \times S^1$ are related to the light-cone variables $\xi = x^1 - x^0, \eta = x^1 + x^0$ by the inverse stereo-graphic projection

$$z = \frac{\xi + i}{1 + i\xi}\left(\xi = \frac{z - i}{1 - iz}\right) \text{etc.} \quad (\xi \in \mathbb{R} \Leftrightarrow |z| = 1). \tag{3.1}$$

The two independent components of the (conserved, symmetric, traceless) conformal stress-energy tensor,

$$T(z) = \tfrac{1}{2}\{T_{10}(z, w) - T_{00}(z, w)\}, \tag{3.2a}$$

$$\bar{T}(w) = \tfrac{1}{2}\{T_{10}(z, w) + T_{00}(z, w)\}, \tag{3.2b}$$

are related to the generators L_n and $\bar{L}_n$ of two (commuting) copies of the Virasoro algebra by [18, 3]

$$T(z)=\sum_{n\in\mathbb{Z}}\frac{L_n}{z^{n+2}},\quad \bar{T}(w)=\sum_{n\in\mathbb{Z}}\frac{\bar{L}_n}{w^{n+2}}. \tag{3.3}$$

Similarly, the left conserved current has a Laurent expansion with coefficients Q_n^a (2.3a):

$$J_a(z)=\tfrac{1}{2}(J_a^0(z,w)+J_a^1(z,w))=\sum_{n\in\mathbb{Z}}\frac{Q_n^a}{z^{n+1}}. \tag{3.4}$$

The corresponding Fermi fields

$$G(z)=\sum_{n\in\mathbb{Z}}\frac{G_{n+\kappa}}{z^{n+\kappa+3/2}}, \tag{3.5a}$$

$$H_a(z)=\sum_{n\in\mathbb{Z}}\frac{h_{n+\kappa}^a}{z^{n+\kappa+1/2}} \tag{3.5b}$$

are single-valued on S^1 in the Neveu–Schwarz case only. In the Ramond case (in which $\kappa=0$, and hence $G(e^{2\pi i}z)=-G(z)$ etc.) they can be regarded as (operator valued) functions on the double cover of the circle.

Introducing the odd (Fermi) superfield

$$F_a(z,\theta)=H_a(z)+\theta J_a(z)z^{1-2\kappa}, \tag{3.6}$$

We can now write down the superconformal (SV_κ-) transformation law (2.5) in the following compact form:

$$[F_a(z,\theta),L_n]=-z_n\left\{z\frac{\partial}{\partial z}+\frac{n}{2}+\kappa+\left(\frac{n}{2}+\kappa\right)\theta\frac{\partial}{\partial\theta}\right\}F_a(z,\theta), \tag{3.7a}$$

$$[F_a(z,\theta),G_{n+\kappa}]_+=z^n\left\{z^{2\kappa}\frac{\partial}{\partial\theta}-\theta\left(z\frac{\partial}{\partial z}+n+2\kappa\right)\right\}F_a(z,\theta). \tag{3.7b}$$

The hermiticity of the fields implies that for a *hermitian* (*unitary*) *representation* of $S_\kappa(G)$ we should have

$$L_n^*=L_{-n},\quad G_\rho^*=G_{-\rho},\quad Q_n^*=Q_{-n},\quad h_\rho^{a*}=h_{-\rho}^a. \tag{3.8}$$

Energy positivity means that the spectrum of L_0 should be non-negative. It follows that there exists a "highest[2] weight" vector $|hw\rangle$ such that, as a consequence of the commutation relations (2.5) and (2.6),

$$L_n|hw\rangle=0=Q_n^a|hw\rangle,\quad G_\rho|hw\rangle=0=h_\rho^a|hw\rangle\quad \text{for } n,\rho>0. \tag{3.9}$$

4. Minimal Unitary Highest Weight Representation of $S_\kappa(G)$

The classification of UHWIRs of $(\widehat{dG})_\kappa$, outlined below, uses in an essential way the "minimal respresentation" of the superconformal current algebra.

2 We stick to the common mathematical terminology. The term "lowest weight" was used in [3]

The *minimal representation* of the Lie superalgebra $S_\kappa(G)$ is constructed in terms of a Fock space ($\mathscr{F}$) realization of the infinite dimensional Clifford algebra (2.4a) (the algebra of the free Fermi field $H_a(z)$ for $\kappa = \frac{1}{2}$) as follows. Let the central charge λ of $\widehat{dG}$ be

$$\lambda = C_2 \left(= \frac{1}{d_G} \operatorname{tr} \vec{T}^2, \quad \text{where } (T_a)_t^s = i f_{\text{sat}} \right) \tag{4.1}$$

($\vec{T}^2 = \sum_{a=1}^{d_G} T_a^2$ standing for the Casimir invariant in the adjoint representation of dG—cf. (2.2).) We set

$$\begin{aligned} Q_n^a &= \frac{i}{C_2} f_{\text{sat}} \sum_{m \in \mathbb{Z}} :h^s_{\kappa - m} h^t_{n+m-\kappa}: \\ &= \frac{i}{2C_2} f_{\text{sat}} \left(\sum_{m \geqq 1} + \sum_{m \geqq -n} \right) (h^s_{\kappa - m} h^t_{n+m-\kappa} - h^t_{\kappa - m} h^s_{n+m-\kappa}) \end{aligned} \tag{4.2}$$

(the last equation serving as the definition of the normal product in the first line),

$$\begin{aligned} G_{n+\kappa} &= \frac{2}{3C_2} \sum_{m \in \mathbb{Z}} :\vec{Q}_{-m} \vec{h}_{m+n+\kappa}: \\ &= \frac{1}{3C_2} \left(\sum_{m \geqq 1} + \sum_{m \geqq -n} \right) (\vec{Q}_{-m} \vec{h}_{m+n+\kappa} + \vec{h}_{\kappa - m} \vec{Q}_{m+n}) \end{aligned} \tag{4.3}$$

($\vec{Q}_k \vec{h}_\rho = \sum_{s=1}^{d_G} Q_k^s h_\rho^s$ is the AdG-invariant inner product). Finally, L_n is evaluated from (2.6):

$$\begin{aligned} L_n &= \tfrac{1}{2}[G_{n-\kappa}, G_\kappa]_+ = \frac{1}{6C_2} \left[G_{n-\kappa}, \left(\sum_{m \geqq 1} + \sum_{m \geqq 0} \right) (\vec{Q}_{-m} \vec{h}_{m+\kappa} + \vec{h}_{\kappa - m} \vec{Q}_m) \right]_+ \\ &= \frac{1}{3C_2} \left\{ \left(\sum_{m \geqq 1} + \sum_{m \geqq -n} \right) \vec{Q}_{-m} \vec{Q}_{m+n} \right. \\ &\quad \left. + \left(\sum_{m \geqq 2\kappa} m + \sum_{m \geqq 2\kappa - n} (m + n - 2\kappa) \right) \vec{h}_{\kappa - m} \vec{h}_{n+m-\kappa} \right\} \quad \text{for } n \neq 0, \end{aligned} \tag{4.4a}$$

$$\begin{aligned} L_0 = \tfrac{1}{2}[L_1, L_{-1}] &= \frac{1}{3C_2} \left\{ \vec{Q}_0^2 + 2 \sum_{m \geqq 1} (\vec{Q}_{-m} \vec{Q}_m + (m - \kappa) \vec{h}_{\kappa - m} \vec{h}_{m-\kappa}) \right. \\ &\quad \left. + (\tfrac{1}{2} - \kappa)^2 \vec{h}_{-\kappa} \vec{h}_\kappa \right\}. \end{aligned} \tag{4.4b}$$

Proposition 2. *The canonical anticommutation relations* (CARs) (*2.4a*) (*with* $\lambda = C_2$) *and Eqs.* (*4.2–4*) *imply the supercommutation relations* (*2.4–6*) *with central charges*

$$\lambda = C_2, \quad c = \frac{d_G}{2}. \tag{4.5}$$

The *proof* of this statement is straightforward. For instance, having verified (2.4), we find:

$$[Q_m^a, G_\rho] = \frac{m}{3} h^a_{m+\rho} + \frac{2i}{3C_2} f_{abc} \sum_{k=1}^{m} [Q^c_{m-k}, h^b_{\rho+k}]$$
$$= \frac{m}{3}\left(h^a_{m+\rho} + \frac{2}{C_2} f_{abc} f_{bcs} h^s_{m+\rho} \right) = m h^a_{m+\rho}.$$

The minimal UHWIR of $S_\kappa(G)$ is thus defined by the corresponding CAR representation, which has different characteristics for $\kappa = 0$ and $\kappa = \frac{1}{2}$. For $\kappa = \frac{1}{2}$ we have the standard Fock representation of (2.4a) with vacuum vector $|0\rangle$ satisfying

$$h^a_\rho |0\rangle = 0 \quad \text{for } \rho \geqq \tfrac{1}{2}, \quad \text{so that } Q^a_n|0\rangle = 0 \quad \text{for } n \geqq 0,$$
$$L_n|0\rangle = 0 \quad \text{for } n \geqq -1. \tag{4.6}$$

For $\kappa = 0$ we define a Ramond-type highest weight vector $|R(G)\rangle$ satisfying

$$h^a_n |R(G)\rangle = 0 \quad \text{for } n \geqq 1, \quad \vec{z}\vec{h}_0 |R(G)\rangle = 0 \quad \text{for } z \in Z_-, \tag{4.7}$$

where Z_- is a fixed maximal ($[d_G/2]$-dimensional) isotropic subspace of $\mathbb{C}^{d_G}$ that is closed under the skew vector multiplication $(\vec{z}_1 \wedge \vec{z}_2)_c = f_{abc} z^a_1 z^b_2$ and gives rise to a subalgebra of dG of elements $\{\vec{z}\vec{q}_0, z \in Z_-\}$ which contains all "raising operators" (for a given Cartan basis). The linear span of the vectors $h^{a_1}_0 \dots h^{a_n}_0 |R(G)\rangle (0 \leqq n \leqq d_G)$ is the representation space for the $2^{[d_G/2]}$-dimensional irreducible representation of the Clifford algebra of $O(d_G)$. It carries a representation of G of highest weight $[1, \dots, 1]$ (see, e.g. [6]) and multiplicity $m_R = 2^{[1/2 d_G] - n_+}$, where n_+ is the number of positive roots of dG ($n_+ = \frac{1}{2}N(N-1)$ for $G = SU(N)$; the representation of G is irreducible, i.e., $m_R = 1$, for $G = SU(2)$ only). Unlike the vacuum, the vector $|R(G)\rangle$ is neither G-nor $SL(2, \mathbb{R})$-invariant, its conformal weight being

$$\Delta_{R(G)} = \frac{C_2[1, \dots, 1]}{3C_2} + \frac{d_G}{48} = \frac{d_G}{16}, \quad ((L_0 - \Delta_{R(G)})|R(G)\rangle = 0), \tag{4.8}$$

where we have used the identity $C_2[1, \dots, 1] = C_2 d_G/8$.

Remark. Whenever the vectors $Q^a_0|hw\rangle$ span irreducible representation of G (i.e. for $\kappa = \frac{1}{2}$, or for $\kappa = 0$ and $G = SU(2)$) the following identity holds for the generators (4.4) of the Virasoro subalgebra:

$$L_n = \frac{1}{2C_2}\left(\sum_{m \geqq 1} + \sum_{m \geqq -n} \right) \vec{Q}_{-m} \vec{Q}_{n+m} \tag{4.9a}$$

$$= \frac{1}{2C_2}\left(\sum_{m \geqq 2\kappa} m + \sum_{m \geqq 2\kappa - n} (m + n - 2\kappa) \right) \vec{h}_{\kappa - m} \vec{h}_{n+m-\kappa} + \frac{\delta_{n,0}}{C_2} (\tfrac{1}{2} - \kappa)^2 \vec{h}_{-\kappa} \vec{h}_\kappa. \tag{4.9b}$$

We notice that Eq. (4.9a) is a graded (discrete-) basis counterpart of the Sugawara formula [19] $T(z) = 1/2C_2 {:}J^2(z){:}$

5. Arbitrary UHWIRs of $\widehat{(dG)}_\kappa$ and $S_\kappa(G)$

We shall distinguish in this section the generators (4.2–4) of the minimal representation of $S_\kappa(G)$ by a superscript °. The following observation is similar to one made by Goddard and Olive [8] (in the context of the Sugawara realization of L_n).

Lemma 3. *Let $\tilde{Q}^a_n$ and $\tilde{h}^a_\rho$ be the operators of an arbitrary representation of $(\widehat{dG})_\kappa$. Then the differences*

$$q^a_n = \tilde{Q}^a_n - \hat{Q}^a_n, \text{ where } \hat{Q}^a_n = \frac{i}{C_2} f_{sat} \sum_{m\in\mathbb{Z}} :\tilde{h}^s_{\kappa-m}\tilde{h}^t_{m+n-\kappa}:, \tag{5.1}$$

commute with $\tilde{h}^a_n$ and satisfy the Kac–Moody commutation relations (2.4c):

$$[q^a_n, \tilde{h}^b_\rho] = 0, \quad [q^a_n, q^b_m] = if_{abc} q^c_{n+m} + \frac{n}{2}\lambda(q)\delta_{n+m}\delta_{ab}. \tag{5.2}$$

The *proof* is an immediate consequence of (2.4) and of the commutation relations

$$[\tilde{Q}^a_n, \tilde{h}^b_\rho] = [\hat{Q}^a_n, \tilde{h}^b_\rho] = if_{abc}\tilde{h}^c_{n+\rho}. \tag{5.3}$$

The classification of UHWIRs of both $(\widehat{dG})_\kappa$ and $S_\kappa(G)$ is given by the following result.

Theorem 4. *Given an* UHWIR *of the affine Kac–Moody algebra $\widehat{dG}$ generated by the operators q^a_n acting in a Hilbert space $V_{[\mu]}$ of highest weight vector $|\mu\rangle$, $[\mu] = [\mu_1, \ldots, \mu_r]$ ($r =$ rank G) and central charge $\lambda(q)$, such that*

$$q^a_n|\mu\rangle = 0 \quad \text{for } n \geqq 1, \quad \vec{q}^{\,2}_0|\mu\rangle = C_2[\mu]|\mu\rangle, \tag{5.4}$$

the operators

$$h^a_\rho = \sqrt{\frac{\lambda(q) + C_2}{C_2}}\,\mathring{h}^a_\rho, \quad Q^a_n = \mathring{Q}^a_n + q^a_n \tag{5.5}$$

give rise to an UHWIR *of $(\widehat{dG})_\kappa$ on $\mathscr{F}_\kappa \otimes V_{[\mu]}$, which extends to $S_\kappa(G)$ by*

$$G_{n+\kappa} = \frac{1}{C_2 + \lambda(q)}\left(\sum_{m\geqq 1} + \sum_{m\geqq -n}\right)\{(\tfrac{1}{3}\vec{\mathring{Q}} + \vec{q})_{-m}\vec{h}_{m+n+\kappa} + \vec{h}_{\kappa-m}(\tfrac{1}{3}\vec{\mathring{Q}} + \vec{q})_{m+n}\}, \tag{5.6}$$

$$L_n = \tfrac{1}{2}[G_{n-\kappa}, G_\kappa]_+ \quad \text{for } n \neq 0, \quad L_0 = \tfrac{1}{2}[L_1, L_{-1}]. \tag{5.7}$$

The central charges are

$$\lambda = \lambda(q) + C_2, \quad c = \frac{d_G}{2}\frac{C_2 + 3\lambda(q)}{C_2 + \lambda(q)} = \frac{d_G}{2} + \frac{\lambda(q)d_G}{C_2 + \lambda(q)}; \tag{5.8}$$

the highest weights depend on κ:

$$(\mu_0 + C_2), \mu_1, \ldots, \mu_r; \Delta_{[\mu]}(= \min L_0) = \frac{C_2[\mu]}{C_2 + \lambda(q)} \quad \text{for } \kappa = \tfrac{1}{2}, \tag{5.9a}$$

$$(\mu_0 + 1), \mu_1 + 1, \ldots, \mu_r + 1; \Delta_{[\mu]} = \frac{d_G}{16} + \frac{C_2[\mu]}{C_2 + \lambda(q)} \quad \text{for } \kappa = 0. \tag{5.9b}$$

All the UHWIRs of $(\widehat{dG})_\kappa$ and $S_\kappa(G)$ are constructed in this way.

Proof. The fact that the operators (5.5) generate an UHWIR of $\widehat{dG}$ follows from

Proposition 2 and Lemma 3. The commutation relations (2.5d) are implied by the following corollaries of (2.4):

$$[q^a_m, G_{n+\kappa}] = \frac{m\lambda(q)}{C_2+\lambda(q)} h^a_{m+n+\kappa} + \frac{if_{abc}}{C_2+\lambda(q)}\left(\sum_{k\geqq 1} + \sum_{k\geqq -n}\right) \cdot (q^c_{m-k} h^b_{n+k+\kappa} + h^b_{\kappa-k} q^c_{m+n+k}) \tag{5.10a}$$

$$[\mathring{Q}^a_m, G_{n+\kappa}] = \frac{mC_2}{C_2+\lambda(q)} h^a_{m+n+\kappa} + \frac{if_{abc}}{C_2+\lambda(q)}\left(\sum_{k\geqq 1} + \sum_{k\geqq -n}\right) \cdot (q^b_{-k} h^c_{n+m+k+\kappa} + h^c_{m+\kappa-k} q^b_{n+k}), \tag{5.10b}$$

which are also used in deriving (5.9). The properties of the Virasoro generators (5.7) are a consequence of (2.5c, d) and of the super Jacobi identities. The fact that we get all the UHWIRs of $(\widehat{dG})_\kappa$ follows from Lemma 3.

Remarks. A. If an integrable UHWIR of the affine Kac–Moody algebra (with generators q^a_n) is given by its (generalized) highest weight [5, 20] $(\hat{\mu}) = (\mu_0, \dots, \mu_r)$, where all μ_ν are non-negative integers, then its central charge is

$$(\lambda(q) =)\lambda(\hat{\mu}) = \mu_0 + a^\vee_1\mu_1 + \dots + a^\vee_r\mu_r, \tag{5.11}$$

where the positive integers $a^\vee_i$ are the coefficients of the expansion of the highest short root into simple roots (for SU(N), $a^\vee_i = 1$, $i = 1, \dots, N-1$; for E_8, $a^\vee_i = i+1$ for $i = 1, \dots, 5$, $a^\vee_6 = 4$, $a^\vee_7 = 2$, $a^\vee_8 = 3$ see [5], Chapters 4, 6). The *dual Coxeter number* C_2 of Eqs. (2.2) (4.1) is given by

$$C_2 = 1 + a^\vee_1 + \dots + a^\vee_r, \quad \text{so that } C_2[\mathrm{SU}(N)] = N, \quad C_2[E_8] = 30 \tag{5.12}$$

(see Exercise 6.2 of [5] where the label g is used instead of C_2).

B. For $G = \mathrm{SU}(2)$ Eq. (4.5) gives the lower limit of the continuous spectrum ($c \geqq \frac{3}{2}$) of UHWIRs of the Neveu–Schwarz super-algebra found in [21].

6. Discrete series of UHWIRs of SV_κ

As a further application of Theorem 4 we shall prove the unitarity of the discrete series of positive energy representations of the super Virasoro algebra with central charge [14]

$$c_m = \frac{3}{2}\left(1 - \frac{8}{(m+2)(m+4)}\right) \quad \text{for } m = 2, 3, \dots, \tag{6.1}$$

using a construction of Goddard–Kent–Olive [9]. We take $G = \mathrm{SU}(2) \times \mathrm{SU}(2)$ and consider the UHWIR

$$(\mu_0 = m - 2I,\ \mu_1 = 2I) \oplus (2, 0) \quad (2I = 0, 1, \dots, m-2;\ m \geqq 2) \quad \text{if} \quad \kappa = \tfrac{1}{2} \tag{6.2}$$

of the super-algebras

$$(\widehat{dG})_\kappa = (\widehat{\mathrm{su}(2)})_\kappa \oplus (\widehat{\mathrm{su}(2)})_\kappa \text{ and } S_\kappa(G) = S_\kappa(\mathrm{SU}(2)) \oplus S_\kappa(\mathrm{SU}(2)). \tag{6.3}$$

According to (5.8) and (5.11) the values of the central charges for this representation are

$$\lambda_G = m+2, \quad c_G = \frac{3m}{m+2} + \frac{3}{2}. \tag{6.4}$$

On the other hand, for the diagonal SU(2)-subgroup, $H = \mathrm{SU}(2)_{\mathrm{diag}}$, we have

$$\lambda_H = \lambda_G = m+2, \quad c_H = \frac{d_H \lambda_H}{\lambda_H + C_2} = \frac{3(m+2)}{m+4}. \tag{6.5}$$

An analogue of Lemma 3, established in [8], says that the differences

$$l_n = L_n(G) - L_n(H) \tag{6.6}$$

satisfy the commutation relations (2.6a) with

$$c = c_G - c_H = \frac{3}{2} - \frac{12}{(m+2)(m+4)} = c_m. \tag{6.7}$$

Since $L_n(G)$ and $L_n(H)$ correspond to hermitian representations of the Virasoro algebra realized in the same Hilbert space, then the same is true for l_n. This completes the proof of the above statement.

There also exist unitary representations of SV_κ with central charge $c_1 = \frac{7}{10}$, but the proof of this fact requires a different argument.

Acknowledgements. V. Kac acknowledges partial support by NSF grant MCS-8203739. I Todorov thanks Professor I. E. Segal for his kind hospitality in M.I.T.; financial support from the Department of Mathematics under NSF grant DMS-8303304 and from the Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics through funds provided by the U.S. Department of Energy under contract DE-AC02-76ER03069 is also gratefully acknowledged.

References

1. Witten, E.: Non-abelian bosonization in two dimensions. Commun. Math. Phys. **92**, 455–472 (1984)
2. Knizhnik, V. G., Zamolodchikov, A. B.: Current algebra and Wess–Zumino model in two dimensions. Nucl. Phys. **B247**, 83–103 (1984)
3. Todorov, I. T.: Current algebra approach to conformal invariant two-dimensional models. Phys. Lett. **153B**, 77–81 (1985); Infinite Lie algebras in 2-dimensional conformal field theory, ISAS Trieste lecture notes 2/85/E.P.
4. Kac, V. G.: Contravariant form for infinite dimensional Lie algebras and superalgebras. Lecture Notes in Physics, Heidelberg, New York, **94** Berlin: Springer, 1979 pp. 441–445; Frenkel, I. B., Kac, V. G.: Basic representations of affine Lie algebras and dual resonance models. Invent. Math. **62**, 23–66 (1980); Frenkel, I. B.: Two constructions of affine Lie algebra representations and Boson–Fermion correspondence in quantum field theory. J. Funct. Anal. **44**, 259–327 (1981)
5. Kac, V. G.: Infinite dimensional Lie algebras: An introduction, Boston: Birkhäuser 1983
6. Kac, V. G., Peterson, D. H.: Spin and wedge representations of infinite dimensional Lie algebras and groups. Proc. Natl. Acad. Sci. USA **78**, 3308–3312 (1981)
7. Kac, V. G., Peterson, D. H.: Infinite dimensional Lie algebras, theta functions and modular forms. Adv. Math. **53**, 125–264 (1984); Goodman, R., Wallach, N. R.: Structure and unitary cocycle representations of loop groups and the group of diffeomorphisms of the circle. J. Reine Angew. Math. **347**, 69–133 (1984), Erratum, ibid. **352**, 220 (1984)
8. Goddard, P., Olive, D.: Kac–Moody algebras, conformal symmetry and critical exponents, Nucl. Phys. **B257** [FS 14], 226–240 (1985)

9. Goddard, P., Kent, A., Olive, D.: Virasoro algebras and coset space models. Phys. Lett. **152B**, 88–93 (1985)
10. Green, M. B., Schwarz, J. H.: Anomaly cancellation in supersymmetric $D = 10$ gauge theory and superstring theory. Phys. Lett. **149B**, 117–122 (1984); Infinity cancellations in SO(32) superstring theory. Phys. Lett. **151B**, 21–25 (1984)
11. Witten, E.: Some properties of O(32) superstrings. Phys. Lett. **149B**, 351–356 (1984)
12. Gross, D. J., Harvey, J. A., Martinec, E., Rohm, R.: Heterotic string. Phys. Rev. Lett. **54**, 502–505 (1985); Heterotic string theory I. The free heterotic string. Princeton: Princeton University Press, preprint (1985)
13. Eichenherr, H.: Minimal operator algebras in superconformal quantum field theory. Phys. Lett. **151B**, 26–30 (1985); Bershadsky, M. A., Knizhnik, V. G., Teitelman, M. G.: Superconformal symmetry in two dimensions. Phys. Lett. **151B**, 31–36 (1985)
14. Friedan, D., Qiu, Z. Shenker, S.: Superconformal invariance in two dimensions and the tricritical Ising model. Phys. Lett. **151B**, 37–43 (1985)
15. Belavin, A. A., Polyakov, A. M., Zamolodchikov, A. B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. **B241**, 333–380 (1984); Infinite conformal symmetry of critical fluctuations in two dimensions. J. Stat. Phys. **34**, 763–774 (1984)
16. Neveu, A., Schwarz, J. H.: Factorizable dual model of pions. Nucl. Phys. **B31**, 86–112 (1971)
17. Ramond, P.: Dual theory for free fermions. Phys. Rev. **D3**, 2415–2418 (1971)
18. Ferrara, S., Gatto, R., Grillo, A.: Conformal algebra in two-space time dimensions and the Thirring model. Nuovo Cim. **12A**, 959–968 (1972); Mansuri, F., Nambu, Y.: Gauge conditions in dual resonance models. Phys. Lett. **39B**, 375–378 (1972); Fubini, S., Hanson, A., Jackiw, R.: New approach to field theory. Phys. Rev. **D7**, 1732–1760 (1973); Lüscher, M., Mack, G.: The energy momentum tensor of critical quantum field theory in $1 + 1$ dimensions (Hamburg 1975) (unpublished)
19. Sugawara, H.: A field theory of currents. Phys. Rev. **170**, 1659–1662 (1968); Sommerfield, C.: Currents as dynamical variables. Phys. Rev. **176**, 2019–2025 (1968); Coleman, S., Gross, D., Jackiw, R.: Fermion avatars of the Sugawara model. Phys. Rev. **180**, 1359–1366 (1969); Bardakci, K., Halpern, M.: New dual quark models. Phys. Rev. **D3**, 2493–2506 (1971)
20. Kac, V. G.: Infinite dimensional Lie algebras and Dedekind's η-function. Funkts. Anal. Prilozh. **8**, 77–78 (1974) [Engl. transl: Funct. Anal. Appl. **8**, 68–70 (1974)]; Garland, H.: The arithmetic theory of loop algebras. J. Algebra **53**, 480–551 (1978); Kac, V. G., Peterson, D. H.: Unitary structure in representations of infinite-dimensional groups and a convexity theorem. Invent. Math. **76**, 1–14 (1984)
21. Kac, V. G.: Some problems on infinite-dimensional Lie algebras and their representations. In: Lecture Notes in Mathematics. Vol. **933**, pp. 117–126, Heidelberg, New York, Berlin: Springer 1982

Communicated by A. Jaffe

Received May 2, 1985

Note added in proof. In a recent paper by Di Vecchia et al., "A supersymmetric Wess–Zumino Lagrangian in two dimensions", Nucl. Phys. **B253**, 701–726 (1985) (which appeared after our paper has been accepted for publication) it is shown that a supersymmetric Wess–Zumino Lagrangian in $1 + 1$ dimensions gives rise to the superalgebra $S_{1/2}(G)$.