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Localization in General One Dimensional Random Systems, I. Jacobi Matrices

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Abstract. We consider random discrete Schrödinger operators in a strip with a potential $V_\omega(n,\alpha)$ (n a label in $\mathbb{Z}$ and α a finite label "across" the strip) and V_ω an ergodic process. We prove that $H_0 + V_\omega$ has only point spectrum with probability one under two assumptions: (1) The *conditional* distribution of $\{V_\omega(n,\alpha)\}_{n=0,1,\text{all}\,\alpha}$ conditioned on $\{V_\omega\}_{n\neq 0,1;\text{all}\,\alpha}$ has an absolutely continuous component with positive probability. (2) For a.e. E, no Lyaponov exponent is zero.

1. Introduction

This is the second of three papers exploiting ideas of Kotani [11] to understand localization of random Schrödinger operators. In the basic paper of the series with Wolff [20], we combined ideas of Aronszajn [1]–Donoghue [7] and an abstract analog of averaged boundary condition results of Carmona [2]–Kotani [11] to prove localization in the Anderson model (potential given by i.i.d.'s). In this paper, we discuss more general discrete random Schrödinger operators, and in a companion paper with Kotani [12], we discuss the continuum case. Our main result is stated in the abstract, but we would emphasize also our new and, we feel, especially transparent way of going from positive Lyaponov exponents to the critical condition $\int (x-E)^{-2} d\mu_0(x) < \infty$ on spectral measures (see Sects. 3 and 4).

After completing the research we describe here, we learned that Delyon, Levy and Souillard [4–6], also motivated by Kotani [11], had proven results very close to those we find here. They follow Kotani's approach more clodely than we do; in particular, generalized eigenfunction expansions play a crucial role in their arguments, while they do not herein.

We also learned that Fröhlich et al. [21] use their study of localization in ν-dimensions (at large couplings or energy) to study the one-dimensional case at arbitrary coupling and energy.

The fundamental result of Wolff–Simon [20] is the following:

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Theorem 1. *Let A be a selfadjoint operator, and let $P = (\varphi,.)\,\varphi$ be a rank 1 projection. Let $A_\lambda = A + \lambda P$, and let $d\mu_\lambda$ be the spectral measure for A_λ associated to the vector φ. Then $d\mu_\lambda$ is pure point for* a.e. *λ if and only if*

$$\int \frac{d\mu_0(x)}{(x-E)^2} < \infty \tag{1}$$

for a.e. *real E.*

We use the function $F_\lambda(z) = \int (d\mu_\lambda(x)/x - z))$ for Im $z > 0$. Note that by the dominated convergence theorem, if (1) holds, then $F_0(E + i0) \equiv \lim_{\varepsilon \downarrow 0} F_0(E + i\varepsilon)$ exists and is real. An important element in the proof of Theorem 1 is the following:

Theorem 2 (Aronszajn [1]). *Fix $\lambda \neq 0$, and suppose that E is not an eigenvalue of A.* Then the following are equivalent:

(i) *E is an eigenvalue of A_λ*

(ii) (1) *holds and* $F_0(E + i0) = -\lambda^{-1}$.

In [20], we gave a proof of Theorem 2 closely related to that of Aronszajn. In Sect. 2, we provide a variant of Donoghue's proof [7]. Our purpose in doing this is primarily to handle the case $\lambda = \infty$, i.e. we will give a natural meaning to A_∞, and prove Theorem 2 also if $\lambda = \infty$. While the later proofs can go without the case $\lambda = \infty$, the discussion is simpler and more natural if we consider the case $\lambda < \infty$.

In Sect. 3, we turn to the study of operators on $l^2(\mathbb{Z})$ given by $H_\omega = H_0 + V_\omega$,

$$(H_0 u)(n) = u(n+1) + u(n-1), \quad (V_\omega u)(n) = V_\omega(n) u(n),$$

where $V(n)$ is a stationary ergodic process. We will prove:

Theorem 3. *Let $d\eta_{v(j \neq 0,1)}(v_0, v_1)$ be the conditional joint distribution on v_0, v_1 conditioned on fixing $V_\omega(j)$ to be v_j for $j \neq 0, 1$. Suppose that with positive probability, $d\eta$ has an absolutely continuous (with respect to $dv_0 dv_1$) component. Then for a.e. ω, H_ω has only point spectrum.*

In Sect. 4, we consider operators in a strip. Explicitly, let $\mathscr{H} = \ell^2(\mathbb{Z}, \mathbb{C}^d)$ and let $u(n;\alpha), \alpha = 1, \dots, d$ denote the components of $u(n)$. Let $V_\omega(n;\alpha)$ be an $\mathbb{R}^m$-valued ergodic process, let A be a fixed real symmetric $m \times m$ matrix, and let

$$(H_0 u)(n;\alpha) = u(n+1;\alpha) + u(n-1;\alpha) + \sum_\beta A_{\alpha\beta} u(n;\beta),$$

$$(V_\omega u)(n;\alpha) = V_\omega(n;\alpha) u(n;\alpha).$$

In the usual way, for E real we can form a $2d \times 2d$ transfer matrix $T_n(E)$ and define Lyaponov exponents $\gamma_1 \geqq \dots \geqq \gamma_{2d}$ with $\gamma_j = -\gamma_{2d-j+1}$, since T is symplectic. We will prove:

Theorem 4. *Suppose that*

(a) *For* a.e. *$E, \gamma_d(E) > 0$.*

(b) *The conditional expectation $d\eta_{v(j \neq 0,1)}(v_0, v_1)$ has an absolutely continuous component (relative to $d^d v_0 d^d v_1$) with positive probability. Then for* a.e. *ω, H_ω has dense point spectrum.*

The analog of hypothesis (a) is not needed in Theorem 3, since the discrete version [19] of Kotani theory [10] implies that $\gamma(E) > 0$ for a.e. E (since the hypothesis implies that V is a nondeterministic process). It is quite likely that (b) implies (a) in great generality, but since no one has developed Kotani theory on the strip, there is no proof.

2. Donoghue's Proof of Aronszajn's Theorem

We want to use a variant of the method that Donoghue gave to prove Theorem 2, that allow us to prove a generalization of it that includes the value of $\lambda = \infty$. We must begin by defining A_∞.

Lemma 2.1. *Let $P = (\varphi, \cdot)\varphi$ be a rank 1 projection, and let A be selfadjoint. Let $B = (1-P)A(1-P)$ on $(1-P)\mathscr{H}$, explicitly*

$$D(B) = D(A) \cap \mathrm{Ran}(1-P),$$
$$B\varphi = (1-P)A\varphi \quad \text{if } \varphi \in D(B).$$

Then B is density defined and selfadjoint on $\mathrm{Ran}(1-P)$.

Remark. In applications $\varphi \in D(A)$, in which case the proof is easy. But since the general result is true, we give it.

Proof. Let $\psi = (A+i)^{-1}\varphi \in D(A)$, so $(\varphi, \psi) = (\varphi, (A+i)^{-1}\varphi) \neq 0$, since $\mathrm{Im}(A+i)^{-1}$ is strictly positive. Any $\eta \in D(A)$ can be written:

$$(\eta - c\psi) + c\psi, \quad c = (\varphi, \eta)/(\varphi, \psi),$$

so $D(A) = D(B) \dot{\oplus} [\psi]$ (algebraic and topological but not orthogonal direct sum). Let $\eta \in (1-P)\mathscr{H}$, and let $\eta_n \in D(A)$ converge to η. Then $c_n = (\varphi, \eta_n)/(\varphi, \psi) \to (\varphi, \eta)/(\varphi, \psi) = 0$, so $\eta_n - c_n\psi$ converges to η and B is densely defined. It is easy to see that B is symmetric.

Let $\eta \in D(B^*) \subset (1-P)\mathscr{H}$, and let $\gamma = (B^* - i)\eta$. A straightforward calculation shows that $\eta \in D(A^*)$ with

$$(A^* - i)\eta = \gamma - \langle \psi, \varphi \rangle^{-1} \langle \psi, \gamma \rangle \varphi,$$

so $n \in D(A)$, and thus $D(B^*) \subset D(B)$ and hence B is selfadjoint. ∎

We *define* $A_\infty = (1-P)A(1-P)$ on $(1-P)\mathscr{H}$. We note that $A_\lambda \to A_\infty$ in generalized strong resolvent sense as $\lambda \to \infty$, i.e. for z nonreal, $(A_\lambda - z)^{-1}\zeta \to (B-z)^{-1}(1-P)\zeta$ as $\lambda \to \infty$. For if $\zeta = \varphi$,

$$\|(A + \lambda P - z)^{-1}\varphi\|^2 = 2(\mathrm{Im}\, z)^{-1}\, \mathrm{Im}(\varphi, (A + \lambda P - z)^{-1}\varphi)$$
$$= 2(\mathrm{Im}\, z)^{-1}[F_0(z)/1 + \lambda F_0(z)] \to 0,$$

so $(A + \lambda P - z)^{-1}P \to 0$. Thus, if $\zeta = (B-z)\eta$ (with $P\eta = 0$), then

$$(A + \lambda P - z)^{-1}\zeta = (A + \lambda P - z)^{-1}(A + \lambda P - z)\eta$$
$$- (A + \lambda P - z)^{-1}PA\eta \to \eta = (B-z)^{-1}\zeta.$$

Theorem 2.2. *Fix $\lambda \neq 0$ (but $\lambda = \infty$ allowed). Suppose that E is not an eigenvalue of A. Then the following are equivalent:*

(a) *E is an eigenvalue of A_λ.*

(b) *$\int d\mu_0(x)/(x-E)^2 < \infty$ and $F_0(x + i0) = -\lambda^{-1}$.*

Remark. We emphasize once more that (1) implies $F_0(x+i0)$ exists and is real. Thus E is an eigenvalue of some A_λ if and only if (1) holds.

Proof (patterned after Donoghue [7]). Suppose first that $\lambda < \infty$. E is an eigenvalue if and only if $(A-E)\eta = -\lambda P\eta$ has a solution $\eta = 0$. Since E is not an eigenvalue of A, $P\eta \neq 0$, i.e. $(\varphi, \eta) \neq 0$. Normalize η so $-\lambda(\varphi, \eta) = 1$. Thus E is an eigenvalue of A_λ if and only if

$$(A-E)\eta = \varphi \tag{2a}$$

has a solution and

$$\lambda^{-1} = -(\varphi, \eta). \tag{2b}$$

We claim this remains true for $\lambda = \infty$. For $(A_\infty - E)\eta = 0$ if and only if $(\eta, \varphi) = 0$ and $(A-E)\eta = P(A-E)\eta$ (which cannot be zero since E is not an eigenvalue of A).

Thus we have proven that E is an eigenvalue of A_λ if and only if (2) has a solution. Pass to a spectral representation so $\mathscr{H} = L^2(R, d\mu_0)$, A is multiplication by x and $\varphi(x) = 1$. Thus (2a) becomes $(x-E)\eta(x) = 1$ or $\eta(x) = (x-E)^{-1}$. Thus (2a) has a solution if and only if $\eta \in L^2$ if and only if (1) holds. Equation (2b) says that $-\lambda^{-1} = \lim_{\varepsilon \downarrow 0} (\varphi, (A - E + i\varepsilon)^{-1}\varphi)$. ■

3. Localization in One Dimension

Our main goal in this section is to prove Theorem 3. We begin with a result that is connected only with $\gamma(E) > 0$. As noted in [20], this result (or one essentially as good) can also be proven from the work of Ishii [9] or Deift–Simon [3].

Theorem 3.1. *Let* $V(n)$ *be a* (deterministic) *potential for a one-dimensional Jacobi matrix,* H. *Suppose that*

(i) E *is not an eigenvalue of* H.

(ii) *The transfer matrix* $T(n)$ *at energy* E *obeys*

$$\lim_{|n| \to \infty} \frac{1}{|n|} \ln \| T(n) \| = \gamma > 0.$$

Let $d\mu_0$ *be the spectral measure for* H *associated to* γ_0. *Then*

$$\int d\mu_0(x)(x-E)^{-2} < \infty.$$

Proof. By the Osceledec theorem [17, 18], $Hu = Eu$ has solutions $u_\pm$ decaying exponentially at ∞. Suppose first that neither vanishes at $n = 0$ so we can normalize them by $u_\pm(0) = 1$. Define

$$\eta(n) = u_+(n), \quad n \geqq 0,$$
$$= u_-(n), \quad n \leqq 0,$$

so $\eta \in \ell^2$ and

$$(H\eta)(n) = E\eta(n) + c\delta_{n0}, \tag{2}$$

with

$$c = u_+(1) + u_-(-1) + (V(0) - E).$$

Notice that $c \neq 0$ since E is not an eigenvalue of H. Since $\eta(0) = 1$, (2) can be rewritten $(H - cP)\eta = E\eta$ with $P = (\delta_0, \cdot)\delta_0$. Thus E is an eigenvalue of $H - cP$, so (1) holds by Theorem. 2.2.

Next suppose that $u_+(0) = 0$. Then take

$$\begin{aligned} \eta(n) &= u_+(n), \quad n \geqq 0, \\ &= 0, \qquad\quad n \leqq 0, \end{aligned}$$

Let $P = (\delta_0, \cdot)\delta_0$. Then it is easy to see that

$$(1 - P)\eta = \eta \quad (1 - P)(H - E)\eta = 0.$$

Thus E is an eigenvalue of $H_{\lambda = \infty}$, and again (1) holds by Theorem 2.2. ■

Remarks. 1. Formally this analysis is connected with the method used in [20] to prove (1). There we used

$$\int (x - E)^{-2} d\mu_0(x) = \lim_{\varepsilon \downarrow 0} \sum_n |G(n, 0; E + i\varepsilon)|^2,$$

with G the matrix elements of $(H - E - i\varepsilon)^{-1}$. One expects that $\lim_{\varepsilon \downarrow 0} G(n, 0; E + i\varepsilon) = u_+(\max(n, 0))u_-(\min(n, 0))W^{-1} = \eta(n)W^{-1}$ with W the Wronskian of u_+ and u_-. Modulo the interchange of sum and limit, this provides a way of understanding why (1) holds. This leads to the formula:

$$\int (x - E)^{-2} d\mu_0(x) = W^{-2} \sum_{n = -\infty}^{\infty} |\eta(n)|^2.$$

2. Since the above derivation is formal, it might be worth giving a real proof of the last formula. Notice that $W = u_+(1) - u_-(1) = u_+(1) + u_-(-1) + V(0) - E$, which is just the value of c in the proof. Moreover, the weight, Γ, of the point mass at E in the spectral measure for $H + cP$ is just $(\eta(0))^2 \Big/ \sum_{n = -\infty}^{\infty} |\eta(n)|^2 = 1[\Sigma |\eta(n)|^2]^{-1}$. Thus, the above formula comes from the relation found in [20] between Γ, c and $\int (x - E)^{-2} d\mu_0(x)$

3. The proof did not require $\gamma > 0$, but only that $(H - E)u = 0$ had solutions l^2 and $+\infty$ and l^2 at $-\infty$.

Proof of Theorem 3. The idea of the proof is the same as in [20]. Fix $\{v_j\}_{j \neq 2}$. By Kotani's theorem, $\gamma(E) > 0$ for a.e. E since the hypothesis of the theorem implies that V_ω is nondeterministic. Thus, for a.e. choice of $\{V(j)\}_{j \neq 2}$, we have for a.e. E Lyaponov behavior with $\gamma > 0$ at both $+\infty$ and $-\infty$. We suppose v_j is such a choice for $V(j)$. For each value of v_1 and v_0, the operator H with $V(n) = v(n)$ has $\int (d\mu_0(x)/(x - E)^2) < \infty$ for a.e. E by the above discussion coupled with Theorem 3.1. Thus, by Theorem 1, for each choice of v_1 and a.e. choice (with respect to Lebesgue measure) of v_0, the cyclic subspace generated by δ_0 has only point masses in its spectral measure. Thus, if v_0, v_1 lie in their a.c. component, both δ_0, δ_1 are sums of eigenfunctions. Since this pair is cyclic for ℓ^2, we see that if v_0, v_1 lie in their a.c. component, H_ω has only point spectrum. Thus, H_ω has only point spectrum with nonzero probability, and so, by general principles [13], with probability 1. ■

The above proof, if looked at in detail, seems almost circular. We show that a.e. E is an eigenvalue of $H_\omega + \lambda P$ for some λ, and then use this to show that $H_\omega + \lambda P$ has only point spectrum! The point is that once E is an eigenvalue for some λ, it will have nothing to do with the singular spectral measure for any other value of λ.

Example 1. If V_ω is a Markov process with an invariant measure $f(x)dx$ and transition *integral* kernel $K(x, y)$ (so $\int K(x, y)f(y)dy = f(x)$), then

$$d\eta_{v(j\neq 0,1)}(v_0, v_1) = N^{-1}K(v_{-1}, v_0)K(v_0, v_1)K(v_1, v_2)dv_0dv_1,$$

and the hypotheses of the theorem hold.

Example 2. Let V_ω be a Gaussian process with covariance

$$\mathrm{Exp}(v_iv_j) = g(i-j).$$

Suppose that g has an inverse matrix h, i.e. $\sum_j h(i-j)g(j-k)$ is absolutely convergent and equals δ_{ik} (as well as the right inverse). Formally

$$d\eta_{v(j\neq 0,1)}(v_0, v_1) = N^{-1}\exp(-A(v))dv_0dv_1,$$

where

$$A(v) = \tfrac{1}{2}h(0)[v_0^2 + v_1] + C_0(v_{j\neq 0,1})v_0 + C_1(v_{j\neq 0,1})v_1 + h(1)v_0v_1$$

with

$$C_a = \sum_{j\neq 0,1} h(a-j)v_j.$$

Since

$$\mathrm{Exp}(|C_a|^2) = \sum_{\substack{j\neq 0,1\\ k\neq 0,1}} h(a-j)g(j-k)h(k-a)$$

is easily seen to be finite, we have $C_0, C_1 < \infty$ for a.e. $\{v_j\}_{j\neq 0,1}$. It is then easy to see that $\exp(-A(v))\in L^1(dv_0dv_1)$, and to justify the formula for $d\eta$ and to verify the hypotheses of Theorem 3.

Example 3. The hypotheses of Theorem 3 suggest the identification of an interesting problem in probability theory. Let $a_m(\omega)$ be i.i.d.'s with some distribution, $d\kappa(\lambda) = F(\lambda)d\lambda$. Let $V_\omega(n) = \sum_m a_m(\omega)f(n-m)$. When does V obey the hypotheses of Theorem 3?

Example 4. Clearly DLR processes (i.e. Gibbs' states for one dimensional lattice systems) with a priori measure a.c. component and finite range interactions will give examples of V_ω's obeying the hypotheses of Theorem 3.

Example 5. If δ_0 and δ_2 are sums of eigenvectors of H_ω, then so is δ_1, since $(H_\omega - V(1)\mathbb{1})\delta_1 = \delta_0 + \delta_2$. Thus, Theorem 3 has an analog where $\{v_j\}_{j\neq 0,2}$ are fixed. For example, if $\tilde{V}_\omega$ is the process $\tilde{V}(n) = 0$ if n is even and the odd V's are i.i.d.'s with a.c. distribution and V_ω is the suspension of $\tilde{V}_\omega$, i.e. with probability $\tfrac{1}{2}$, $V_\omega(n) = \tilde{V}_\omega(n)$ and with probability $\tfrac{1}{2}$, $V_\omega(n) = \tilde{V}_\omega(n-1)$, then H_ω has dense point spectrum by this alternative theorem. Another example where the alternative theorem is applicable is the Markov process where, if $V(0)=0$, then $V(1)=0$ with probability $\frac{1}{2}$ and is uniformly distributed in $[1, 2]$ with probability $\frac{1}{2}$ and if $V(0)\in[1,2]$, then $V(1)=0$ with probability 1 (with invariant measure $\frac{2}{3}\delta_0+\frac{1}{3}\chi_{[1,2]}dx$).

4. Localization on the Strip

Localization for the discrete Schrödinger operator in a one-dimensional strip with a potential given by i.i.d.'s with some extra hypotheses on the density has been announced and sketched by Goldsheid [8]. A complete proof has been given by LaCroix [14, 15]. In this section, we will prove a more general result. We begin with a deterministic theorem. Let $\mathscr{H}=\ell^2(\mathbb{Z},\mathbb{C}^d)$, i.e. wave functions have values in $\mathbb{C}^d$. We will think of the distinguished basis in $\mathbb{C}^d$ to write the components of $u(n)\in\mathbb{C}^d$ as $\{u(n;\alpha)\}_{\alpha=1,\dots,d}$, so $\mathscr{H}$ is thought of as $l^2(\mathbb{Z}\times\{1,\dots,d\})$.

A "potential" will be a set of real $d\times d$ matrices, $W(n)$, indexed by $n\in\mathbb{Z}$ so that

$$(Hu)(n)=u(n+1)+u(n-1)+W(n)u(n).$$

The example to bear in mind is where $\{1,\dots,d\}$ has some symmetric set, S, of "neighboring pairs" $S=\{(\alpha,\beta)\}$ (e.g. α,β are neighbors if $|\alpha-\beta|=1$, which is a strip approximating Z^2) and

$$(W(n)u)(n;\alpha)=\sum_{\{B|(\alpha,\beta)\in S\}}u(n;\beta)+V(n;\alpha)u(n;\alpha)$$

with V the "real" potential. When one has thin set up, one can define a transfer matrix in Sp(d), the $2d\times 2d$ symplectic matrices by $T(n)=A(n)\dots A(1)$ and

$$A(j)=\begin{bmatrix} E1-W(j) & -1 \\ 1 & 0\end{bmatrix}$$

with 1 the $d\times d$ identity matrix. We say there is Lyaponov behavior at $+\infty$ and an energy E if

$$\lim_{n\to\infty}\frac{1}{n}\ln\|\Lambda^p(T(n))\|=\gamma_1+\dots+\gamma_p$$

exists where Λ^p is the alternating function and this relation defines γ_j. There is a similar definition at $-\infty$. By the symplectic nature of $T,\gamma_1\geqq\dots\geqq\gamma_{2d}$ and $\gamma_j=-\gamma_{2d-j+1}$.

By the subadditive ergodic theorem, if W is an ergodic process, then for each fixed E_0 one has Lyaponov behavior (with equal γ_j at $+\infty$ and $-\infty$ and γ_j independent of ω) for a.e. ω and so for a.e. ω for a.e. E.

It is a trifle easier not to worry about some of the possibilities $\lambda=\infty$, so we will rule out a countable set in proving the analog of Theorem 3.1. Let $H_\omega^\pm$ denote the operators on $l^2([\pm1,\infty),\mathbb{C}^d)$ with $u(0)=0$ boundary conditions. The analog of Theorem 3.1 is

Theorem 4.1. *Let $W(n)$ be a deterministic potential for a $\mathbb{C}^d$-strip. Suppose that*

(i) *E is not an eigenvalue of H, H^+ or H^-.*

(ii) *H has Lyaponov behavior at $\pm\infty$ and energy E and $\gamma_d>0$.*

Let $d\mu_{0,\alpha}$ be the spectral measures for H associated to $\delta_{n=0,\alpha}$. Then

$$\int d\mu_{0,\alpha}(x)(x-E)^{-2} < \infty \tag{1'}$$

for each α.

Proof. By hypothesis (ii) and the Osceledec theorem [18], there is a d-dimensional set, S_+, of solutions decaying exponentially at $+\infty$. No such solution can vanish at $n=0$, since E is not an eigenvalue of H^+. Thus, the map $u \mapsto u(0)$ from S_+ to $\mathbb{R}^d$ is one–one and so onto. Thus, given an $a \in \mathbb{R}^d$, there is a unique solution $u_+(\cdot;a) \in S_+$ with $u_+(0;a)=a$. Define $M_+a = u_+(1;a)$. Then M_+ maps R^d to R^d.

We claim that M_+ is a symmetric matrix. For, given any a, a', the Wronskian

$$(u_+(n+1,a), u_+(n,a')) - (u_+(n,a), u_+(n+1,a'))$$

is constant. Because of the decay at plus infinity, this Wronskian is zero. Thus for $n=0$:

$$(M_+a, a') - (a, M_+a') = 0.$$

M_- is defined similarly. Given a, define

$$\begin{aligned} \eta(n;a) &= u_+(n;a), \quad n \geqq 0, \\ &= u_-(n;a), \quad n \leqq 0. \end{aligned}$$

Let P be the rank one projection onto $\delta_{n=0,\alpha}$ in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$, and let Q be the rank one projection onto δ_α in $\mathbb{C}^d$. Then it is easy to see that for $\eta = \eta(\cdot, a)$ and $\lambda < \infty$:

$$(H + \lambda P)\eta = E\eta \tag{3}$$

if and only if

$$(M_+ + M_- + \lambda Q)a = Ea. \tag{4}$$

Thus we have reduced the eigenvalue problem (3) under a rank one perturbation in $\ell^2(\mathbb{Z}, \mathbb{C}^d)$ to a rank one eigenvalue problem (4) in $\mathbb{C}^d$! The above calculation was for $\lambda < \infty$, but a check of what the $\lambda = \infty$ result means shows (3) is equivalent to (4) in that case also, i.e.

$$P\eta = 0, \quad (1-P)(H\eta) = E\eta \tag{3'}$$

is equivalent to

$$Qa = 0; \quad (1-Q)(M_+ + M_-)a = Ea.$$

As we note in the lemma below, since $M_+ + M_-$ is symmetric with $\lambda \neq 0$, (4) always has a solution if the $\lambda = 0$ equation has no solution. Thus (3) always has a solution with $\lambda \neq 0$, since E is not an eigenvalue of H. By Theorem 2.2, (1′) holds. ■

Lemma 4.2. *Let A be a finite Hermitian matrix, P a rank one projection and E not an eigenvalue of A. Then for some λ (possibly $\lambda = \infty$), E is an eigenvalue of $A + \lambda P$.*

Proof. By Theorem 2.2, we only need the fact that $B(E)^{-1} \equiv \int d\mu_\varphi(x) \cdot$

$|x-E|^{-2}<\infty$. But if E_i and φ_i are the eigenvalues and eigenvectors, $B(E)^{-1} = \sum_{i=1}^{\dim H} |(\varphi, \varphi_i)|^2 (E-E_i)^{-2}$ is trivially finite. ■

Given Theorem 4.1, the proof of Theorem 3 immediately yields Theorem 4.

Finally, one can ask when hypothesis (a) ($\gamma_d > 0$) holds. If the V's are i.i.d.'s, then Furstenburg's theorem (see [22] for recent literature on this subject) implies that $\gamma_d > 0$ for all E, and we recover Lacroix's result [15]. In fact, in this i.i.d. case, all that one needs is that the density of $V_\omega(0;\alpha)$ has an a.c. component. Ledrappier [16] has proven that $\gamma_1 > 0$ under great generality in the strip, but it remains to be seen if one can prove $\gamma_d > 0$ under such conditions.

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