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On The Uniqueness Of Meromorphic Functions Sharing Two Sets

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Abstract

In the paper we employ the notion of weighted sharing of sets to deal with the well known question of Gross and obtain a uniqueness result on meromorphic functions sharing two sets which will improve an earlier result of Lahiri [14].

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1 Introduction, Main Results and Definitions

In this paper by meromorphic functions we will always mean meromorphic functions in the complex plane. We shall use the standard notations of value distribution theory : $T(r,f)$, $m(r,f)$, $N(r,\infty;f)$, $\overline{N}(r,\infty;f),\ldots$ (see [9]). It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For any non-constant meromorphic function $h(z)$ we denote by $S(r,h)$ any quantity satisfying $S(r,h) = o(T(r,h)) \quad (r \longrightarrow \infty, r \notin E)$. For any constant a, we define $\Theta(a;f) = 1 - \limsup\limits_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)}$.

If for some $a \in \mathbb{C}\cup\{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$. If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

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F. Gross was the first to consider the uniqueness of meromorphic functions that share sets of distinct elements instead of values and in 1976 he posed the following question in [7]:

Question A *Can one find two finite sets* S_j $(j=1,2)$ *such that any two non-constant entire functions f and g satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2$ *must be identical ?*

In [7] Gross wrote *If the answer of* Question A *is affirmative it would be interesting to know how large both sets would have to be ?*

Now it is natural to ask the following question [18].

Question B *Can one find two finite sets* S_j $(j=1,2)$ *such that any two non-constant meromorphic functions f and g satisfying* $E_f(S_j)=E_g(S_j)$ *for* $j=1,2$ *must be identical ?*

Nowadays a widely studied topic of the uniqueness theory has been to considering the shared value problems relative to a meromorphic function sharing two sets and at the same time give affirmative answers to *Question B* under weaker hypothesis. {see [1]-[6], [8], [10], [14]-[16], [18]-[25]}.

Dealing with the question of Gross in [5] Fang and Lahiri exhibited a unique range set S with smaller cardinalities than that obtained previously imposing some restrictions on the poles of f and g. They obtained the following result.

Theorem A. [5] Let $S=\{z:z^n+az^{n-1}+b=0\}$ where $n(\geq 7)$ be an integer and a and b be two nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple root. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S)=E_g(S)$ and $E_f(\{\infty\})=E_g(\{\infty\})$ then $f\equiv g$.

In 2001 an idea of gradation of sharing of values and sets known as weighted sharing has been introduced in {[12], [13]} which measure how close a shared value is to being shared CM or to being shared IM. Below we are explaining the notion.

Definition 1.1. [12, 13] Let k be a nonnegative integer or infinity. For $a\in\mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m\leq k$ and $k+1$ times if $m>k$. If $E_k(a;f)=E_k(a;g)$, we say that f,g share the value a with weight k. We write f, g share (a,k) to mean that f, g share the value a with weight k. Clearly if f, g share (a,k) then f,g share (a,p) for any integer p, $0\leq p<k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a,0)$ or (a,∞) respectively.

Definition 1.2. [12] Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $\bigcup_{a\in S}E_k(a;f)$.

With the notion of weighted sharing of sets improving *Theorem A*, Lahiri [14] proved the following theorem.

Theorem B. [14] Let S be defined as in *Theorem A* and $n(\geq 7)$ be an integer. If for two non-constant meromorphic functions f and g, $\Theta(\infty;f)+\Theta(\infty;g)>1$, $E_f(S,2)=E_g(S,2)$ and $E_f(\{\infty\},\infty)=E_g(\{\infty\},\infty)$ then $f\equiv g$.

Suppose that the polynomial $P(w)$ is defined by

$$P(w)=aw^n-n(n-1)w^2+2n(n-2)bw-(n-1)(n-2)b^2 \tag{1.1}$$

where $n \geq 3$ is an integer and a and b are two nonzero complex numbers satisfying $ab^{n-2} \neq 2$. In fact we consider the following rational function

$$R(w) = \frac{aw^n}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.2}$$

where α_1 and α_2 are two distinct roots of

$$n(n-1)w^2 - 2n(n-2)bw + (n-1)(n-2)b^2 = 0.$$

We have from (1.2)

$$R^{'}(w) = \frac{(n-2)aw^{n-1}\,(w-b)^2}{n(n-1)\,(w-\alpha_1)^2\,(w-\alpha_2)^2}. \tag{1.3}$$

From (1.3) we know that $w = 0$ is a root with multiplicity n of the equation $R(w) = 0$ and $w = b$ is a root with multiplicity 3 of the equation $R(w) - c = 0$, where $c = \frac{ab^{n-2}}{2}$.
Then

$$R(w) - c = \frac{a(w-b)^3\,Q_{n-3}(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}, \tag{1.4}$$

where $Q_{n-3}(w)$ is a polynomial of degree $n-3$.

Moreover from (1.1) and (1.2) we have

$$R(w) - 1 = \frac{P(w)}{n(n-1)(w-\alpha_1)(w-\alpha_2)}. \tag{1.5}$$

Noting that $c = \frac{ab^{n-2}}{2} \neq 1$, from (1.3) and (1.5) we have

$$P(w) = aw^n - n(n-1)w^2 + 2n(n-2)bw - (n-1)(n-2)b^2$$

has only simple zeros.

In the paper our prime concern is to improve *Theorem B*. In fact we will show that in our result, for the uniqueness of meromorphic function the conditions over the ramification index ceases to matter at the expense of allowing $n \geq 8$. The following theorem is the main result of the paper.

Theorem 1.3. *Let $S = \{w \mid P(w) = 0\}$, where $P(w)$ is given by (1.1) and $n \geq 7$. Suppose that f and g are two non-constant meromorphic functions satisfying $E_f(S,2) = E_g(S,2)$ and $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$ and $\min\{\Theta_f, \Theta_g\} > 7 + \frac{2}{n-3} - n$ then $f \equiv g$, where $\Theta_f = 4\,\Theta(0;f) + 4\,\Theta(b;f) + \Theta(\infty;f)$ and Θ_g can be similarly defined.*

We are now going to explain the following notations as these are used in the paper.

Definition 1.4. [11] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r,a;f \mid = 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r,a;f \mid \leq m)(N(r,a;f \mid \geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity. $\overline{N}(r,a;f \mid \leq m)$ $(\overline{N}(r,a;f \mid \geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r,a;f \mid < m)$, $N(r,a;f \mid > m)$, $\overline{N}(r,a;f \mid < m)$ *and* $\overline{N}(r,a;f \mid > m)$ are defined analogously.

Definition 1.5. Let f and g be two non-constant meromorphic functions such that f and g share $(1,0)$. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_L(r,1;f)$ the reduced counting function of those 1-points of f and g where $p > q$, by $N_E^{1)}(r,1;f)$ the counting function of those 1-points of f and g where $p = q = 1$, by $\overline{N}_E^{(2}(r,1;f)$ the reduced counting function of those 1-points of f and g where $p = q \geq 2$. In the same way we can define $\overline{N}_L(r,1;g)$, $N_E^{1)}(r,1;g)$, $\overline{N}_E^{(2}(r,1;g)$. In a similar manner we can define $\overline{N}_L(r,a;f)$ and $\overline{N}_L(r,a;g)$ for $a \in \mathbb{C} \cup \{\infty\}$. When f and g share $(1,m)$, $m \geq 1$ then $N_E^{1)}(r,1;f) = N(r,1;f \mid= 1)$.

Definition 1.6. [12, 13] Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g. Clearly $\overline{N}_*(r,a;f,g) = \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

2 Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H the following function.

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right).$$

Let f and g be two non-constant meromorphic function and

$$F = R(f), \qquad G = R(g), \tag{2.1}$$

where $R(w)$ is given by (1.2). From (1.2) and (2.1) it is clear that

$$T(r,f) = \frac{1}{n}T(r,F) + S(r,f), \quad T(r,g) = \frac{1}{n}T(r,G) + S(r,g). \tag{2.2}$$

Lemma 2.1. *[2] Let F, G be given by (2.1) and $H \not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k). Then*

$$\begin{aligned} N_E^{1)}(r,1;F) \leq\ & \overline{N}_L(r,1;F) + \overline{N}_L(r,1;G) + \overline{N}(r,0;f) + \overline{N}(r,b;f) + \overline{N}_*(r,\infty;f,g) \\ & + \overline{N}(r,0;g) + \overline{N}(r.b;g) + \overline{N}_0(r,0;f') + \overline{N}_0(r,0;g'), \end{aligned}$$

where $\overline{N}_0(r,0;f')$ denotes the reduced counting function corresponding to the zeros of f' which are not the zeros of $f(f-b)$ and $F-1$, $\overline{N}_0(r,0;g')$ is defined similarly.

Lemma 2.2. *Let f and g be two non-constant meromorphic functions sharing $(1,m)$, where $0 \leq m < \infty$. Then*

$$\overline{N}(r,1;f) + \overline{N}(r,1;g) - N_E^{1)}(r,1;f) + \left(m - \frac{1}{2}\right)\overline{N}_*(r,1;f,g) \leq \frac{1}{2}\left[N(r,1;f) + N(r,1;g)\right].$$

Proof. Let z_0 be a 1- point of f of multiplicity p and a 1-point of g of multiplicity q. Since f, g share $(1,m)$, we note that the 1-points of f and g up to multiplicity m are same. When $p=q=1$, z_0 is counted once, both in left and right hand side of the above inequality but when $2\leq p=q\leq m$, z_0 is counted 2 times in the left hand side of the above inequality whereas it is counted p times in the right hand side of the same. If $p=m+1$ then the possible values of q are as follows. (i) $q=m+1$, (ii) $q\geq m+2$. When $p=m+2$ then q can take the following possible values (i) $q=m+1$, (ii) $q=m+2$, (iii) $q\geq m+3$. Similar explanations hold if we interchange p and q. Clearly when $p=q\geq m+1$, z_0 is counted 2 times in the left hand side and $p\geq m+1$ times in the right hand side of the above inequality. When $p>q\geq m+1$, in view of *Definition 1.6* we know z_0 is counted $m+\frac{3}{2}$ times in the left hand side and $\frac{p+q}{2}\geq m+\frac{3}{2}$ times in the right hand side of the above inequality. When $q>p$ we can explain similarly. Hence the lemma follows. □

Lemma 2.3. *[17] Let f be a non-constant meromorphic function and $P(f)=a_0+a_1f+a_2f^2+\ldots+a_nf^n$, where $a_0,a_1,a_2\ldots,a_n$ are constants and $a_n\neq 0$. Then $T(r,P(f))=nT(r,f)+O(1)$.*

Lemma 2.4. *Let F, G be given by (2.1) where $n\geq 6$ is an integer and $H\not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k), where $0\leq m<\infty$. Then*

$$\begin{aligned}
&\left\{\frac{n}{2}+1\right\}\{T(r,f)+T(r,g)\}\\
\leq\;& 2\left[\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,b;f)+N(r,b;g)\right]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\
&+\overline{N}_*(r,\infty;f,g)-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. By the second fundamental theorem we get

$$\begin{aligned}
&(n+1)T(r,f)+(n+1)T(r,g) \qquad (2.3)\\
\leq\;& \overline{N}(r,1;F)+\overline{N}(r,0;f)+\overline{N}(r,b;f)+\overline{N}(r,\infty;f)+\overline{N}(r,1;G)+\overline{N}(r,0;g)\\
&+\overline{N}(r,b;g)+\overline{N}(r,\infty;g)-N_0(r,0;f^{'})-N_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using *Lemmas 2.1*, *2.2* and *2.3* we see that

$$\begin{aligned}
&\overline{N}(r,1;F)+\overline{N}(r,1;G) \qquad (2.4)\\
\leq\;& \frac{1}{2}\left[N(r,1;F)+N(r,1;G)\right]+N_E^{1)}(r,1;F)-\left(m-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\
\leq\;& \frac{n}{2}\{T(r,f)+T(r,g)\}+\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)+\overline{N}_*(r,\infty;f,g)\\
&-\left(m-\frac{3}{2}\right)\overline{N}_*(r,1;F,G)+\overline{N}_0(r,0;f^{'})+\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g).
\end{aligned}$$

Using (2.4) in (2.3) the lemma follows. □

Lemma 2.5. *Let F, G be given by (2.1) and $H\not\equiv 0$. If F, G share $(1,m)$ and f, g share (∞,k), where $0\leq m<\infty$, $0\leq k<\infty$, then*

$$\begin{aligned}
&[(n-2)k+n-3)]\,\overline{N}(r,\infty;f\mid\geq k+1)=[(n-2)k+n-3)]\overline{N}(r,\infty;g\mid\geq k+1)\\
\leq\;& \overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Proof. The proof of the lemma can be found in *Lemma 2.16* [2]. □

Lemma 2.6. *Let f, g be two non-constant meromorphic functions sharing $(\infty,0)$ and suppose α_1 and α_2 are two distinct roots of the equation $n(n-1)w^2-2n(n-2)bw+(n-1)(n-2)b^2=0$. Then*

$$\frac{f^n}{(f-\alpha_1)(f-\alpha_2)}\ \frac{g^n}{(g-\alpha_1)(g-\alpha_2)} \not\equiv \frac{n^2(n-1)^2}{a^2},$$

where n (≥ 3) is an integer.

Proof. We omit the proof since the proof can be found out in the proof of *Theorem 3* [8]. □

Lemma 2.7. *Let F, G be given by (2.1), where $n \geq 6$ is an integer. If $F \equiv G$, then $f \equiv g$.*

Proof. We omit the proof since the proof can be found out in [8]. □

Lemma 2.8. *Let F, G be given by (2.1). Also let S be given as in* Theorem 1.3*, where $n \geq 3$ is an integer. If $E_f(S,0) = E_g(S,0)$ then $S(r,f) = S(r,g)$.*

Proof. Since $E_f(S,0) = E_g(S,0)$, it follows that F and G share $(1,0)$. We denote the distinct elements of S by w_j, $j = 1,2,\dots n$. Since F, G share $(1,0)$ from the second fundamental theorem we have

$$(n-2)T(r,g) \leq \sum_{j=1}^{n} \overline{N}(r,w_j;g) + S(r,g) = \sum_{j=1}^{n} \overline{N}(r,w_j;f) + S(r,g) \leq nT(r,f) + S(r,g).$$

Similarly we can deduce $(n-2)T(r,f) \leq nT(r,g) + S(r,f)$. The last inequalities imply $T(r,f) = O(T(r,g))$ and $T(r,g) = O(T(r,f))$ and so we have $S(r,f) = S(r,g)$. □

3 Proof of the main theorem

Proof of Theorem 1.3. Let F, G be given by (2.1). Since $E_f(S,2) = E_g(S,2)$ it follows that F, G share $(1,2)$. Also since $E_f(\{\infty\},\infty) = E_g(\{\infty\},\infty)$ we see that $\overline{N}_*(r,\infty;f,g) \equiv 0$. If possible let us suppose that $H \not\equiv 0$. Since $n \geq 7$ using *Lemma 2.4* for $m = 2$ and $k = \infty$, *Lemma 2.5* for $k = 0$ we obtain for $\varepsilon(>0)$

$$
\begin{aligned}
& \left(\frac{n}{2}+1\right)\ \{T(r,f)+T(r,g)\} \\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)\right\}+\overline{N}(r,\infty;f) \\
& +\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)-\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\
\leq\ & 2\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}(r,b;f)+\overline{N}(r,b;g)\right\}+\frac{1}{2}\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\} \\
& +\frac{1}{n-3}\left\{\overline{N}(r,0;f)+\overline{N}(r,0;g)\right\}+S(r,f)+S(r,g) \\
\leq\ & \left(\frac{9}{2}-2\Theta(0;f)-2\Theta(b;f)-\frac{1}{2}\Theta(\infty;f)+\frac{1}{n-3}+\varepsilon\right)T(r,f) \\
& +\left(\frac{9}{2}-2\Theta(0;g)-2\Theta(b;g)-\frac{1}{2}\Theta(\infty;f)-\frac{1}{n-3}+\varepsilon\right)\ T(r,g) \\
& +S(r,f)+S(r,g).
\end{aligned}
$$

That is

$$
\begin{aligned}
& \left(\frac{n}{2}-\frac{7}{2}-\frac{1}{n-3}+2\Theta(0;f)+2\Theta(b;f)+\frac{1}{2}\Theta(\infty;f)-\varepsilon\right)\ T(r,f) \\
& +\left(\frac{n}{2}-\frac{7}{2}-\frac{1}{n-3}+2\Theta(0;g)+2\Theta(b;g)+\frac{1}{2}\Theta(\infty;g)-\varepsilon\right)\ T(r,g) \\
& \leq S(r,f)+S(r,g).
\end{aligned} \tag{3.1}
$$

Without the loss of generality, we may suppose that there exists a set I with infinite linear measure such that

$$T(r,g) \leq T(r,f), \qquad r \in I.$$

From (3.1) and *Lemma 2.8* we have

$$\left[\frac{1}{2}(\Theta_f+\Theta_g)-7-\frac{2}{n-3}+n-2\varepsilon\right]\ T(r,g)\ \leq S(r,g), \qquad r \in I\backslash E,$$

which leads to a contradiction for $\varepsilon > 0$. Hence $H \equiv 0$. Then

$$F \equiv \frac{AG+B}{CG+D}, \tag{3.2}$$

where A, B, C, D are constants such that $AD-BC \neq 0$. Also

$$T(r,F) = T(r,G)+O(1),$$

and hence from *Lemma 2.3* we have

$$T(r,f) = T(r,g)+O(1). \tag{3.3}$$

From (1.4) we note that $\overline{N}(r,c;F) \leq \overline{N}(r,b;f)+(n-3)T(r,f) \leq (n-2)T(r,f)+S(r,f)$. Similarly $\overline{N}(r,c;G) \leq (n-2)T(r,g)+S(r,g)$. From (3.2) and the condition f and g share

$(\infty,0)$ it follows that ∞ is Picard exceptional value of f and g. So in view of (1.2) and (2.1) we observe that $\overline{N}(r,\infty;F)=\overline{N}(r,\alpha_1;f)+\overline{N}(r,\alpha_2;f)$ and $\overline{N}(r,\infty;G)=\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)$. We now consider the following cases.

Case I. Let $AC\neq 0$. Suppose $B\neq 0$. From (3.2) we get

$$\overline{N}\left(r,-\frac{B}{A};G\right)=\overline{N}(r,0;F). \tag{3.4}$$

In view of (3.3), (3.4), *Lemma 2.3* and the second fundamental theorem we get

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,-\frac{B}{A};G\right)+S(r,G)\\ &\leq \overline{N}(r,0;g)+\overline{N}(r,\alpha_1;g)+\overline{N}(r,\alpha_2;g)+\overline{N}(r,0;f)+S(r,g)\\ &\leq 3T(r,g)+T(r,f)+S(r,g)\leq 4T(r,g)+S(r,g), \end{aligned}$$

which is a contradiction for $n\geq 7$.

So we must have $B=0$ and in this case (3.2) changes to

$$F\equiv\frac{\frac{A}{C}\,G}{G+\frac{D}{C}}. \tag{3.5}$$

From (3.5) we see that

$$\overline{N}(r,\infty;F)=\overline{N}\left(r,-\frac{D}{C};G\right). \tag{3.6}$$

Now in view of (3.6), *Lemma 2.3* and the second fundamental theorem we obtain

$$\begin{aligned} nT(r,g) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}\left(r,-\frac{D}{C};G\right)+S(r,G)\\ &\leq \overline{N}(r,0;g)+2T(r,g)+2T(r,f)+S(r,g)\leq 5T(r,g)+S(r,g), \end{aligned}$$

which implies a contradiction for $n\geq 7$.

Case II. Let $A\neq 0$ and $C=0$. Then $F=\alpha G+\beta$, where $\alpha=\frac{A}{D}$ and $\beta=\frac{B}{D}$.

If F has no 1-point, by the second fundamental theorem and *Lemma 2.3* we get

$$nT(r,f)\leq\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+S(r,f)\leq 3T(r,f)+S(r,f),$$

which implies a contradiction for $n\geq 7$.

If F and G have some 1-points then $\alpha+\beta=1$ and so

$$F\equiv\alpha G+1-\alpha. \tag{3.7}$$

Suppose $\alpha\neq 1$. If $1-\alpha\neq c$ then in view of (3.3), *Lemma 2.3* and the second fundamental theorem we get

$$\begin{aligned} 2nT(r,f) &\leq \overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,1-\alpha;F)+\overline{N}(r,\infty;F)+S(r,F)\\ &\leq (n+1)T(r,f)+\overline{N}(r,0;G)+S(r,f)\leq(n+2)T(r,f)+S(r,f), \end{aligned}$$

which implies a contradiction for $n \geq 7$. If $1-\alpha = c$, then we have from (3.7)

$$F \equiv (1-c)G + c.$$

Since $c \neq 1$, by the second fundamental theorem we can obtain using (3.3) and *Lemma 2.3* that

$$\begin{aligned} 2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}\left(r, \frac{c}{c-1};G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq (n+1)T(r,g) + \overline{N}(r,0;F) + S(r,g) \leq (n+2)T(r,g) + S(r,g), \end{aligned}$$

which implies a contradiction since $n \geq 7$.

So $\alpha = 1$ and hence $F \equiv G$. So by *Lemma 2.7* we get $f \equiv g$.

Case III. Let $A = 0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma G + \delta}$, where $\gamma = \frac{C}{B}$ and $\delta = \frac{D}{B}$.

If F has no 1-point then as in *Case II* we can deduce a contradiction.

If F and G have some 1-points then $\gamma + \delta = 1$ and so

$$F \equiv \frac{1}{\gamma G + 1 - \gamma}. \tag{3.8}$$

Suppose $\gamma \neq 1$ If $\frac{1}{1-\gamma} \neq c$, then by the second fundamental theorem and *Lemma 2.3* we get

$$\begin{aligned} 2nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r, \frac{1}{1-\gamma};F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,f) \\ &\leq (n+3)T(r,f) + \overline{N}(r,0;G) + S(r,f) \leq (n+4)T(r,f) + S(r,f), \end{aligned}$$

which gives a contradiction for $n \geq 7$. If $\frac{1}{1-\gamma} = c$, from (3.8) we have

$$F \equiv \frac{c}{(c-1)\,G + 1}. \tag{3.9}$$

If $c \neq \frac{1}{1-c}$ the second fundamental theorem with the help of (3.3), (3.9) and *Lemma 2.3* yields

$$\begin{aligned} 2nT(r,g) &\leq \overline{N}(r,0;G) + \overline{N}(r,c;G) + \overline{N}\left(r, \frac{1}{1-c};G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq (n+1)T(r,g) + \overline{N}(r,\infty;F) + S(r,g) \leq (n+3)T(r,g) + S(r,g), \end{aligned}$$

which implies a contradiction since $n \geq 7$. On the other hand if $c = \frac{1}{1-c}$ then from (3.9) we have

$$G \equiv \frac{c(F-c)}{F}.$$

So from the second fundamental theorem it follows that

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,0;F) + \overline{N}(r,c;F) + \overline{N}(r,\infty;F) + S(r,F) \\ &\leq 3T(r,f) + \overline{N}(r,0;G) + S(r,f) \leq 4T(r,f) + S(r,f), \end{aligned}$$

which implies a contradiction since $n \geq 7$. So we must have $\gamma = 1$ then $FG \equiv 1$, which is impossible by *Lemma 2.6*. This completes the proof of the theorem. □

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