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Positive Solutions for Abstract Hammerstein Equations and Applications

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Abstract

The authors use fixed point index properties to prove existence of positive solutions to the abstract Hammerstein equation $u = LFu$ where $L : E \to E$ is a compact linear operator, $F : K \to K$ is a continuous and bounded mapping, E is a Banach space, and K is a cone in E. The results obtained are used to prove existence results for positive solutions to two point boundary value problems associated with differential equations.

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1 Introduction

Existence and multiplicity of solutions to boundary value problems (BVPs) associated with ordinary differential equations (ODEs) is a subject that has been widely investigated in the last several decades; see, for example, [3, 4, 5, 13, 16, 17, 18, 19, 20, 21, 15], and the references therein. Often those BVPs are formulated as a fixed point problem in a Banach space E having the form $u = LFu$, where $L \in L(E)$ is compact and $F : E \to E$ is continuous and bounded (maps bounded sets into bounded sets). This equation is known as the abstract Hammerstein equation (see [23, Chapter 7]).

In many of the papers cited above, existence and multiplicity results are obtained under the condition that the nonlinearity varies between 0 and $+\infty$ or between $-\infty$ and $+\infty$. For instance, in [4], the author obtain existence and multiplicity of positive solutions to the boundary value problem

$$\begin{cases} -u''(x) = f(x,u(x)), & x \in (0,1), \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0, \end{cases} \tag{1.1}$$

where a, b, c, and d are nonnegative real numbers such that $ac + ad + cb > 0$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, where $\mathbb{R}^+ = [0,+\infty)$. It is known (see [4, Proposition 3.2]) that problem (1.1) has no positive solutions if either

$$\frac{f(t,x)}{x} > \lambda_1 \quad \text{for all} \quad (t,x) \in [0,1] \times (0,+\infty)$$

or

$$\frac{f(t,x)}{x} < \lambda_1 \quad \text{for all} \quad (t,x) \in [0,1] \times (0,+\infty),$$

where λ_1 is the smallest positive eigenvalue of the linear boundary value problem

$$\begin{cases} -u''(x) = \lambda u(x), & x \in (0,1), \\ au(0) - bu'(0) = 0, \\ cu(1) + du'(1) = 0. \end{cases}$$

This result means that a necessary condition for the existence of a positive solution to problem (1.1) is that the nonlinearity f must cross the linear function $\lambda_1 u$ at least once. An existence result is obtained under the hypothesis

$$f(t,u) \geq \alpha u \text{ for all } (t,u) \in [0,1] \times [p,q] \quad \text{and} \quad f(t,u) \leq \beta u \text{ for all } (t,u) \in [0,1] \times [r,s]$$

with $\alpha > \lambda_1 > \beta$ and other suitable conditions. Moreover, this result holds if the intervals $[p,q]$ and $[r,s]$ are neighborhoods of 0 and $+\infty$ (see [4, Corollary 3.7]).

Results similar to [4, Corollary 3.7] are often seen in the literature; in the case of second order BVPs see, for example, [3, 6, 15, 16, 17], and in the singular case, see [5]; for fourth order BVPs, see [22].

It is clear from the above discussion that the eigenvalues of L play some role in the existence of solutions to the abstract Hammerstein equation. Thus, in this paper, we focus our attention on existence of positive solutions (solutions belonging to a cone) to the equation

$u = LFu$. Roughly speaking, we will prove that there exists two nonnegative real numbers $\lambda^+ \leq \lambda^-$ such that the Hammerstein equation has no positive solutions if the nonlinearity F lies above the linear function $(\lambda^+)^{-1} u$ or below $(\lambda^-)^{-1} u$, and we obtain existence otherwise (see Theorems 3.7 and 3.10 in Section 3 below). We may also ask when do λ^+ and λ^- coincide with a positive eigenvalue of L? We answer this question in Theorems 3.13 and 3.15.

In order to illustrate the importance of our results, we conclude this paper with two applications. Throughout, we let $A^* := A \setminus \{0\}$ where A is any subset of a Banach space.

2 Preliminaries

In all that follows, E denotes a real Banach space, $L(E)$ is the set of all continuous linear maps from E into E, and $Q(E)$ is the subset of $L(E)$ consisting of compact maps. For $L \in L(E)$, $r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}}$ denotes the *spectral radius* of L.

Definition 2.1. Let K be a nonempty closed convex subset of E. Then K is said to be a cone if $K \cap (-K) = \{0\}$ and $(tK) \subset K$ for all $t \geq 0$.

It is well known that a cone induces a partial ordering in the Banach space E. We write for all $x, y \in E$, $x \leq y$ if $y - x \in K$; $x < y$ if $y - x \in K$ and $y \neq x$; $x \nleq y$ if $y - x \notin K$; and $x \ll y$ if $\mathrm{int}K \neq \emptyset$ and $y - x \in \mathrm{int}K$. The notations $\geq$, $>$, $\ngeq$, and $\gg$ are defined similarly.

Definition 2.2. Let K be a cone in E. Then:

(i) K is reproducing if $E = K - K$;

(ii) K is total if $E = \overline{K - K}$;

(iii) K is normal if there exists a positive constant N such that for all $u, v \in K$, $u \leq v$ implies $\|u\| \leq N\|v\|$.

Remark 2.3. A cone with nonempty interior is a typical example of a reproducing cone.

Definition 2.4. Let K be a cone in E and $L \in L(E)$. Then:

(i) L is said to be increasing if $L(K) \subset K$;

(ii) An increasing operator $L \in L(E)$ is K-normal if there exists a positive constant N such that for all $u, v \in K$, $u \leq v$ implies $\|Lu\| \leq N\|Lv\|$.

We will make extensive use of fixed point index theory. For the sake of completeness, we recall some basic facts related to this; see, for example, [7, 14, 15].

Let K be a nonempty closed subset of E. Then K is called a *retract* of E if there exists a continuous mapping $r : E \to K$ such that $r(x) = x$ for all $x \in K$. Such a mapping is called a *retraction*. From a theorem by Dugundji, every nonempty closed convex subset of E is a retract of E. In particular, every cone in E is a retract of E.

Let K be a retract of E and U be a bounded open subset of K such that $U \subset B(0,R)$, where $B(0,R)$ is the ball centered at 0 of radius R. For any completely continuous mapping $f : \overline{U} \to K$ with $f(x) \neq x$ for all $x \in \partial U$, the integer given by

$$i(f,U,K) = \deg\Big(I - f \circ r, B(0,R) \cap r^{-1}(U), 0\Big),$$

where deg is the Leray-Schauder degree, is well defined and is called the *fixed point index*. *Properties of the fixed point index*

Normality: $i(f,U,K) = 1$ if $f(x) = x_0 \in \overline{U}$ for all $x \in \overline{U}$.

Homotopy invariance: Let $H : [0,1]\times\overline{U} \to K$ be a completely continuous mapping such that $H(t,x) \neq x$ for all $(t,x) \in [0,1]\times\partial U$. The integer $i(H(t,\cdot),U,K)$ is independent of t.

Additivity: $i(f,U,K) = i(f,U_1,K)+i(f,U_2,K)$ whenever U_1 and U_2 are two disjoint open subsets of U such that f has no fixed point in $\overline{U}\setminus(U_1\cup U_2)$.

Permanence: If K' is a retract of K with $f(\overline{U}) \subset K'$, then $i(f,U,K) = i(f,U\cap K',K')$.

Solution property: If $i(f,U,K) \neq 0$, then f admits a fixed point in U.

Now we assume that K is a cone in E and for all $R > 0$, we let $K_R = B(0,R)\cap K$. We will need the following lemmas related to the computation of the index $i(f,K_R,K)$.

Lemma 2.5. *If $f(x) \neq \lambda x$ for all $x \in \partial K_R = \partial B(0,R)\cap K$ and $\lambda \geq 1$, then*

$$i(f,K_R,K) = 1.$$

Lemma 2.6. *If $f(x) \neq \lambda x$ for all $x \in \partial K_R = \partial B(0,R)\cap K$ and $\lambda \in (0,1]$, and if* $\inf\{\|f(x)\| : x \in \partial K_R\} > 0$*, then*

$$i(f,K_R,K) = 0.$$

Lemma 2.7. *If $f(x) \ngeq x$ for all $x \in \partial K_R = \partial B(0,R)\cap K$, then*

$$i(f,K_R,K) = 1.$$

Lemma 2.8. *If $f(x) \nleq x$ for all $x \in \partial K_R = \partial B(0,R)\cap K$, then*

$$i(f,K_R,K) = 0.$$

For additional details and proofs of these lemmas, we refer the reader to [14].

3 Main results

Let K be a cone in E, $L \in L(E)$ be increasing, and $F : K \to K$ be a continuous bounded mapping. We focus our attention in this section on the existence of positive solutions to the abstract equation

$$u = LFu. \tag{3.1}$$

By a *positive solution* to (3.1), we mean a vector $u \in K^*$ satisfying $u = LFu$. We recall that $\lambda \geq 0$ is a *positive eigenvalue* of L if there exists $u \in K^*$ such that $Lu = \lambda u$, and it is an *interior eigenvalue* if there exists $u \in \text{int}K$ such that $Lu = \lambda u$. For any subset P of K with $P^* \neq \emptyset$, let

$$\Lambda_P^+(L) = \{\lambda \geq 0 : \text{ there exists } u \in P^* \text{ such that } Lu \leq \lambda u\}$$

and

$$\Lambda_P^-(L) = \{\lambda \geq 0 : \text{ there exists } u \in P^* \text{ such that } Lu \geq \lambda u\}.$$

When these quantities exist, we set

$$\lambda_P^+ = \inf \Lambda_P^+(L),\ \lambda_P^- = \sup \Lambda_P^-(L),\ \lambda^+ = \inf \Lambda_K^+(L), \text{ and } \lambda^- = \sup \Lambda_K^-(L).$$

Remark 3.1. (i) Note that $0 \in \Lambda_P^-(L)$, and if $\lambda \in \Lambda_P^-(L)$, then $[0,\lambda] \subset \Lambda_P^-(L)$. (ii) If $\lambda \in \Lambda_P^+(L)$, then $[\lambda,+\infty) \subset \Lambda_P^+(L)$. (iii) We have $\Lambda_P^+(L) \subset \Lambda_K^+(L)$ and $\Lambda_P^-(L) \subset \Lambda_K^-(L)$.

The following lemmas provide sufficient conditions for the existence of λ_P^+ and λ_P^-. Let $\mathbb{N}_0 = \{0,1,2,\dots\}$.

Lemma 3.2. *If P is a cone and $L(K) \subset P$, then $\Lambda_P^+(L) \neq \emptyset$.*

Proof. For $\lambda > r(L)$, $\left(I - \frac{L}{\lambda}\right)^{-1} = \sum_{n\in\mathbb{N}_0} \frac{L^n}{\lambda^n}$, and since for all integers n, $L^n(K) \subset P$, we obtain $\left(I - \frac{L}{\lambda}\right)^{-1}(K) \subset P$. Thus, for any $u \in K^*$, $v = \left(I - \frac{L}{\lambda}\right)^{-1}(u) \in P^*$. In other words, $\lambda v > Lv$, and so $\lambda \in \Lambda_P^+(L)$. □

Lemma 3.3. *If* $\mathrm{int}K \neq \emptyset$*, then* $\Lambda_{\mathrm{int}K}^+(L) \neq \emptyset$.

Proof. For $\lambda > r(L)$, $\left(I - \frac{L}{\lambda}\right)^{-1} = \sum_{n\in\mathbb{N}_0} \frac{L^n}{\lambda^n}$ is a homeomorphism of E, so $\left(I - \frac{L}{\lambda}\right)^{-1}(\mathrm{int}K)$ is an open set contained in K. Therefore, $\left(I - \frac{L}{\lambda}\right)^{-1}(\mathrm{int}K) \subset \mathrm{int}K$. Thus, for any $u \in \mathrm{int}K$, $v = \left(I - \frac{L}{\lambda}\right)^{-1}(u) \in \mathrm{int}K$, i.e., $\lambda v > Lv$, so $\lambda \in \Lambda_{\mathrm{int}K}^+(L)$. □

Lemma 3.4. *Assume that K is normal. Then for any nonempty subset $P \subset K$, $\Lambda_P^-(L)$ is bounded from above by $r(L)$.*

Proof. If $\lambda > 0$ and $u \in P^*$ with $\|u\| = 1$ are such that $Lu \geq \lambda u$, then

$$u \leq T^n u \text{ for all } n \in \mathbb{N}^*,$$

where $T = \frac{L}{\lambda}$. Hence, the normality of K implies that

$$1 \leq N^{\frac{1}{n}} \left\|T^n u\right\|^{\frac{1}{n}} = N^{\frac{1}{n}} \frac{\|L^n u\|^{\frac{1}{n}}}{\lambda} \leq N^{\frac{1}{n}} \frac{\|L^n\|^{\frac{1}{n}}}{\lambda},$$

where N is the constant of normality of K. Letting $n \to \infty$, we have

$$\lambda \leq \lim_n N^{\frac{1}{n}} \left\|L^n\right\|^{\frac{1}{n}} = r(L),$$

which proves the lemma. □

Lemma 3.5. *Assume that L is K-normal. Then, for any cone $P \subset K$ with $L(K) \subset P$, $\Lambda_P^-(L)$ is bounded from above by $r(L)$.*

Proof. If $\lambda > 0$ and $u \in P^*$ with $\|Lu\| = 1$ are such that $Lu \geq \lambda u$, then

$$Lu \leq T^n Lu \text{ for all } n \in \mathbb{N}^*,$$

where $T = \dfrac{L}{\lambda}$. Hence, the $K-$normality of L implies that

$$1 \leq N^{\frac{1}{n}} \left\|T^n Lu\right\|^{\frac{1}{n}} = N^{\frac{1}{n}} \frac{\|L^n Lu\|^{\frac{1}{n}}}{\lambda} \leq N^{\frac{1}{n}} \frac{\|L^n\|^{\frac{1}{n}}}{\lambda},$$

where N is the constant of the $K-$normality of L. Letting $n \to \infty$, we obtain

$$\lambda \leq \lim_n N^{\frac{1}{n}} \left\|L^n\right\|^{\frac{1}{n}} = r(L),$$

which completes the proof. □

Before presenting existence results for equation (3.1), we need to draw attention to the following fact. If L admits a positive eigenvalue λ, then $\lambda^+ \leq \lambda^-$ and $\lambda \in [\lambda^+, \lambda^-]$. In what follows, we will prove that for any cone P, with $L(K) \subset P \subset K$, if L is completely continuous, no matter if L has a positive eigenvalue or not, we always have $\lambda_P^+ \leq \lambda_P^-$. To prove this we need following results.

Proposition 3.6. *Let either*

$$Fu \leq \alpha u \text{ for all } u \in P^* \text{ with } \alpha\lambda_P^- < 1 \tag{3.2}$$

or

$$Fu \geq \beta u \text{ for all } u \in P^* \text{ with } \beta\lambda_P^+ > 1 \tag{3.3}$$

hold, where $P \subset K$ is nonempty with $L(K) \subset P$. Then equation (3.1) *has no positive solutions.*

Proof. We present the proof in the case where (3.2) holds; the proof in the other case is similar. Assume there exists $u \in K^*$ such that $LFu = u$. Then $u \in P^*$, and since $Fu \leq \alpha u$, it follows that $Lu \geq \dfrac{1}{\alpha} u$ and $\dfrac{1}{\alpha} \leq \lambda_P^-$, which contradicts $\alpha\lambda_P^- < 1$. This completes the proof. □

From [14, Theorem 2.3.3] we can obtain the following existence result.

Theorem 3.7. *Assume that $L \in Q(E)$, $P \subset K$ is a cone with $L(K) \subset P$, and there exist real numbers α, β, R_1, and R_2 with $\alpha\lambda_P^- < 1$, $\beta\lambda_P^+ > 1$, and $0 < R_1 < R_2$. If either*

$$Fu \leq \alpha u \text{ for all } u \in P \cap \partial B(0,R_1) \text{ and } Fu \geq \beta u \text{ for all } u \in P \cap \partial B(0,R_2), \tag{3.4}$$

or

$$Fu \geq \beta u \text{ for all } u \in P \cap \partial B(0,R_1) \text{ and } Fu \leq \alpha u \text{ for all } u \in P \cap \partial B(0,R_2), \tag{3.5}$$

then equation (3.1) *admits a positive solution u with $R_1 < \|u\| < R_2$.*

We also have the following comparison result.

Theorem 3.8. *Assume that $L \in Q(E)$. Then for any cone $P \subset K$ with $L(K) \subset P$, we have $\lambda_P^+ \le \lambda_P^-$.*

Proof. The case $\lambda_P^+ = 0$ is obvious, so assume that $\lambda_P^+ > \lambda_P^- \ge 0$ and consider the function $G : K \to K$ defined by

$$Gu = \frac{\beta u + \alpha \|u\| u}{1 + \|u\|}$$

with $0 < \beta < \alpha$ and $\beta\lambda_P^+ > 1 > \alpha\lambda_P^-$. On one hand, we have

$$Gu - \alpha u = \frac{(\beta - \alpha) u}{1 + \|u\|} < 0 \text{ for all } u \in K^*,$$

so by Proposition 3.6, the equation $u = LGu$ admits no positive solution. On the other hand, for any $0 < R_1 < R_2$, we have

$$Gu \le \alpha u \text{ for all } u \in K \cap \partial B(0, R_1) \text{ with } \alpha\lambda_P^- < 1$$

and

$$Gu - \beta u = \frac{(\alpha - \beta) u \|u\|}{1 + \|u\|} > 0 \text{ for all } u \in K \cap \partial B(0, R_2) \text{ with } \beta\lambda_P^+ > 1.$$

Condition (3.4) is satisfied, so by Theorem 3.7, the equation $u = LGu$ has a positive solution. This contradiction implies $\lambda_P^+ \le \lambda_P^-$. □

Remark 3.9. From Lemmas 2.7 and 2.8 we see that if $L \in Q(E)$, then for any cone $P \subset K$ with $L(K) \subset P$ and any $R > 0$, we have

1. $i(\alpha L, B(0,R) \cap P, P) = 1$ if $\alpha\lambda_P^- < 1$, and
2. $i(\beta L, B(0,R) \cap P, P) = 0$ if $\beta\lambda_P^+ > 1$.

Next we present an existence result for positive solutions to the Hammerstein equation (3.1) in case the cone K is normal. This result includes those covered by [4, Corollary 3.7].

Theorem 3.10. *Assume that $L \in Q(E)$, K is normal, $P \subset K$ is a cone with $L(K) \subset P$, and there exist nonnegative real numbers α, β, and γ, and continuous functions $G_i : K \to K$, $i = 1,2,3$, with*

$$\alpha\lambda_P^- < 1 \ \textit{and} \ \beta\lambda_P^+ > 1,$$

$$Fu \le \alpha u + G_1 u \ \textit{for all} \ u \in P^* \cap B(0,\delta) \ \textit{for some} \ \delta > 0,$$

and

$$\beta u - G_2 u \le Fu \le \gamma u + G_3 u \ \textit{for all} \ u \in P^*.$$

If either

$$G_1 u = o(\|u\|) \ \textit{as} \ u \to 0 \ \textit{and} \ G_i u = o(\|u\|) \ \textit{as} \ u \to \infty \ \textit{for} \ i = 2,3, \tag{3.6}$$

or

$$G_1 u = o(\|u\|) \ \textit{as} \ u \to \infty \ \textit{and} \ G_i u = o(\|u\|) \ \textit{as} \ u \to 0 \ \textit{for} \ i = 2,3, \tag{3.7}$$

then equation (3.1) *has a positive solution.*

Proof. We give the proof in case (3.6) holds; the proof if (3.7) holds is similar. All we need to do is to show the existence of $0 < r < R$ such that

$$i(LF, B(0,r)\cap P, P) = 1 \text{ and } i(LF, B(0,R)\cap P, P) = 0.$$

Then the additivity and the solution properties of the fixed point index will imply that

$$i(LF, \left(B(0,R)\setminus \overline{B}(0,r)\right)\cap P, P) = i(LF, B(0,R)\cap P, P) - i(LF, B(0,r)\cap P, P) = -1$$

and equation (3.1) has a positive solution u with $r < \|u\| < R$.

Consider the function $H_1 : [0,1]\times K \to K$ defined by $H_1(t,u) = tLFu + (1-t)\beta Lu$. We want to show the existence of $R > 0$ large enough so that for all $t \in [0,1]$, the equation $H_1(t,u) = u$ has no solution in $\partial B(0,R)\cap P$. To the contrary, suppose that for all integers $n \geq 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial B(0,n)\cap P$ such that

$$u_n = t_n LFu_n + (1-t_n)\beta Lu_n.$$

Note that $v_n = \dfrac{u_n}{\|u_n\|} \in \partial B(0,1)\cap P$ and satisfies

$$v_n = t_n L\left(\frac{Fu_n}{\|u_n\|}\right) + (1-t_n)\beta Lv_n. \tag{3.8}$$

Thus, the normality of the cone K combined with the inequalities

$$\beta v_n - \frac{G_2u_n}{\|u_n\|} \leq \frac{Fu_n}{\|u_n\|} \leq \gamma v_n + \frac{G_3u_n}{\|u_n\|} \tag{3.9}$$

and the fact that $G_i(u_n) = o(\|u_n\|)$ at ∞ for $i = 2,3$, implies that $\dfrac{Fu_n}{\|u_n\|}$ is bounded. From the compactness of L, we obtain the existence of a subsequence of (v_n), also denoted by (v_n), that converges to $v \in \partial B(0,1)\cap P$. Taking limits as $n \to \infty$ in (3.8) and (3.9) shows $v \geq \beta Lv$. That is, $\dfrac{1}{\beta} \geq \lambda_P^+$, which contradicts $\beta\lambda_P^+ > 1$.

For such an $R > 0$, from the homotopy property of the fixed point index and Remark 3.9, we have

$$\begin{aligned} i(LF, B(0,R)\cap P, P) &= i(H_1(1,\cdot), B(0,R)\cap P, P) \\ &= i(H_1(0,\cdot), B(0,R)\cap P, P) = i(\beta L, B(0,R)\cap P, P) = 0. \end{aligned}$$

In a similar way, we consider the function $H_2 : [0,1]\times K \to K$ defined by $H_2(t,u) = tLFu + (1-t)\alpha Lu$ and we prove the existence of $r > 0$ small enough so that for all $t \in [0,1]$, the equation $H_2(t,u) = u$ has no solution in $\partial B(0,r)\cap P$. To the contrary, suppose that for every integer $n \geq 1$ with $1/n < \delta$, there exists $t_n \in [0,1]$ and $u_n \in \partial B(0,1/n)\cap P$ such that

$$u_n = t_n LFu_n + (1-t_n)\alpha Lu_n.$$

Now $v_n = \dfrac{u_n}{\|u_n\|} \in \partial B(0,1)\cap P$ and satisfies

$$v_n = t_n L\left(\frac{Fu_n}{\|u_n\|}\right) + (1-t_n)\alpha Lv_n.$$

The normality of the cone K combined with the inequality

$$\frac{Fu_n}{\|u_n\|} \le \alpha v_n + \frac{G_1 u_n}{\|u_n\|}$$

and the fact that $G_1(u_n) = o(\|u_n\|)$ at 0 shows that $\frac{Fu_n}{\|u_n\|}$ is bounded. Then, from the compactness of L, we conclude the existence of a subsequence of (v_n), also denoted by (v_n), that converges to $v \in \partial B(0,1) \cap P$. Again taking limits shows that $v \le \alpha L v$. Thus, $\frac{1}{\alpha} \le \lambda_P^-$, which contradicts $\alpha\lambda_P^- < 1$.

For such an $r > 0$ the homotopy property of the fixed point index and Remark 3.9 show that

$$\begin{aligned} i(LF, B(0,r) \cap P, P) &= i(H_2(1,\cdot), B(0,r) \cap P, P) \\ &= i(H_2(0,\cdot), B(0,r) \cap P, P) = i(\alpha L, B(0,r) \cap P, P) = 1. \end{aligned}$$

This completes the proof of the theorem. □

Remark 3.11. Note that if $\ker L \cap K^* = \emptyset$, then for every subset $P \subset K$ with $L(K) \subset P$,

$$\Lambda_P^+(L) = \Lambda_K^+(L), \quad \Lambda_P^-(L) = \Lambda_K^-(L), \quad \lambda_P^+ = \lambda^+, \quad \text{and} \quad \lambda_P^- = \lambda^-.$$

In fact, if $\lambda > 0$ and $u \in K^*$ are such that $Lu \le \lambda u$ (resp. $Lu \ge \lambda u$), then $U = Lu \in P^*$ and $LU \le \lambda U$ (resp. $LU \ge \lambda U$).

Let P be a cone such that $L(K) \subset P \subset K$. In our previous results, we saw the role played by the constants λ_P^+ and λ_P^- in the existence of positive solutions for the Hammerstein equation (3.1). Now we will present two results in which λ_P^+ and λ_P^- coincide with the unique positive eigenvalue of L. To do this, we need the following definition.

Definition 3.12. Let $\chi : E \times E \to \mathbb{R}$ be a bilinear form. We say that χ is positive if for all u, $v \in K$, $\chi(u,v) \ge 0$, and we say that χ is increasing if for all u_1, u_2, v_1, $v_2 \in K$,

$$u_1 \le u_2 \text{ implies } \chi(u_1,v_1) \le \chi(u_2,v_1) \text{ and } v_1 \le v_2 \text{ implies } \chi(u_1,v_1) \le \chi(u_1,v_2).$$

Theorem 3.13. *Assume that $L \in Q(E)$, $\lambda^+ > 0$, and there exists a positive increasing bilinear form $\chi : E \times E \to \mathbb{R}$ such that*

$$0 < \chi(Lu, v) = \chi(u, Lv) \text{ for all } u, v \in K^*.$$

Then for every subset P of K with $L(K) \subset P$, we have $\lambda_P^+ = \lambda_P^- = \lambda^+ = \lambda^-$, and $\lambda_1 = \lambda^+ = \lambda^-$ is the unique positive eigenvalue of L.

Proof. Note that $\lambda^+ > 0$ implies $\ker L \cap K^* = \emptyset$, and for every subset P of K with $L(K) \subset P \subset K$, we have $\lambda_P^+ = \lambda^+$ and $\lambda_P^- = \lambda^-$. We claim that L has a positive eigenvalue. By Remark 3.9, for any $R > 0$, $i(\alpha L, B(0,R) \cap K, K) = 0$ with $\alpha\lambda^+ > 1$. Hence, we see from Lemma 2.5 that there exist $\theta \ge 1$ and $u \in K \cap \partial B(0,R)$ such that $\alpha L u = \theta u$. That is, $\frac{\theta}{\alpha}$ is a positive eigenvalue of L.

Let λ_1 be a positive eigenvalue of L and let ϕ be the associated eigenvector. On one hand, we have

$$0 < \lambda^+ \leq \lambda_1 \leq \lambda^- \leq +\infty. \tag{3.10}$$

At the same time, if u, $v \in K^*$ and λ, μ, are such that $Lu \leq \lambda u$ and $Lv \geq \mu v$, then the properties of χ lead to

$$0 < \lambda_1 \chi(\phi, u) = \chi(L\phi, u) = \chi(\phi, Lu) \leq \lambda \chi(\phi, u)$$

and

$$\lambda_1 \chi(\phi, v) = \chi(L\phi, v) = \chi(\phi, Lv) \geq \mu \chi(\phi, v),$$

which imply

$$\mu \leq \lambda_1 \leq \lambda,$$

that is,

$$\lambda^- \leq \lambda_1 \leq \lambda^+. \tag{3.11}$$

Combining (3.10) and (3.11) gives $\lambda^- = \lambda^+ = \lambda_1$ is the unique positive eigenvalue of L. □

Remark 3.14. If we add to Theorem 3.13 the condition that K is a total cone, then it follows from [7, Theorem 19.2] (or [23, Proposition 7.26]) that $\lambda^- = \lambda^+ = \lambda_1$ is the principal and unique positive eigenvalue of L.

Theorem 3.15. *Assume that $L \in Q(E)$, $\operatorname{int}K \neq \emptyset$, and either K is normal or L is K-normal. Then*

$$\lambda^- \leq \lambda^+_{\operatorname{int}K}.$$

Moreover, if $\lambda^+ > 0$ and K is a total cone, then $\lambda^- = r(L) > 0$ is the principal eigenvalue of L.

Proof. Assume that $\lambda^+_{\operatorname{int}K} < \lambda^-$ and $\lambda \in \left(\lambda^+_{\operatorname{int}K}, \lambda^-\right)$. For such a λ, there exists $u \in \operatorname{int}K$ and $v \in K^*$ such that $Lu \leq \lambda u$ and $Lv \geq \lambda v$. Now $u \in \operatorname{int}K$ implies the existence of $t > 0$ such that $u > v_t = tv$.

If K is normal, then the operator $T = \dfrac{L}{\lambda}$ maps the closed bounded convex interval $[v_t, u]$ into itself. So Schauder's fixed point theorem guarantees the existence of a fixed point w of T such that $v_t \leq w \leq u$ and λ is an eigenvalue of L.

If L is K-normal, then the operator $T = \dfrac{L}{\lambda}$ maps the closed bounded convex set $\overline{L([v_t, u])}$ into itself. Schauder's fixed point theorem then guarantees the existence of a fixed point $w \in \overline{L([v_t, u])}$ of T and λ is an eigenvalue of L.

This shows that in the two cases, $\left(\lambda^+_{\operatorname{int}K}, \lambda^-\right) \subset \operatorname{sp}(L)$, where $\operatorname{sp}(L)$ is the spectrum of L, and this contradicts L being compact.

Now if $0 < \lambda^+$, then $r(L) > 0$, and since K is total, [7, Theorem 19.2] (or [23, Proposition 7.26]) ensures that $r(L)$ is a positive eigenvalue of L and $r(L) = \lambda^-$. □

Remark 3.16. Note that Theorem 3.15 guarantees that L has at most one interior eigenvalue. In fact, if λ_1 is an interior eigenvalue, then

$$\lambda^+ \leq \lambda_1 \leq \lambda^- \leq \lambda^+_{\operatorname{int}K} \leq \lambda_1,$$

which implies

$$\lambda_1 = \lambda^- = \lambda^+_{\text{int}K}.$$

Moreover, if 0 is an interior eigenvalue, then λ_1 is the unique positive eigenvalue of L. If this is the case, then $\lambda^+_{\text{int}K} = \lambda^+ = \lambda^-$.

Theorem 3.17. *Assume that* $L \in Q(E)$, $\text{int}K \neq \emptyset$, $L(\partial K \setminus \{0\}) \subset \text{int}K$, *and either K is normal or L is K-normal. Then,* $\lambda^- = \lambda^+ = r(L)$ *is the principal and unique positive eigenvalue of L.*

Proof. We need to prove that $\lambda^+ = \lambda^+_{\text{int}K}$. To this end, we show that $\Lambda^+_K(L) \subset \Lambda^+_{\text{int}K}(L)$. If $\lambda \in \Lambda^+_K(L)$, then there exists $u \in K^*$ such that $Lu \leq \lambda u$ and there are two possibilities.

First, we could have $u \in \text{int}K$. Then $\lambda \in \Lambda^+_{\text{int}K}(L)$. Second, we could have $u \in \partial K$. In this case, $U = Lu \in \text{int}K$ and $LU \leq \lambda U$. This again implies $\lambda \in \Lambda^+_{\text{int}K}(L)$.

Let $u \in \partial K \setminus \{0\}$; then $Lu \in \text{int}K$, so there exists $t > 0$ such that $Lu \geq tu$. This implies that $\lambda^- > 0$, and by Lemmas 3.4 and 3.5, $r(L) \geq \lambda^- > 0$. Thus, it follows from the Krein-Rutman Theorem (see [7, Theorem 19.3] or [23, Theorem 7.C]) that $r(L) = \lambda^-$ is the principal and positive eigenvalue of L. Finally, we see that the condition $L(\partial K \setminus \{0\}) \subset \text{int}K$ implies that L has only interior eigenvalues, so uniqueness follows from Remark 3.16. □

Combining the Krein-Rutman Theorem with Theorem 3.17, we obtain the following result.

Corollary 3.18. *Assume that* $L \in Q(E)$, $\text{int}K \neq \emptyset$, $L(K \setminus \{0\}) \subset \text{int}\, K$, *and either K is normal or L is K-normal. Then* $\lambda^- = \lambda^+ = r(L)$ *is the principal and unique positive eigenvalue of L.*

Remark 3.19. Common to both Theorems 3.13 and 3.17 is that 0 cannot be an eigenvalue of L and so for every subset $P \subset K$ with $L(K) \subset P$, we have $\lambda^+_P = \lambda^-_P = \lambda^+ = \lambda^-$.

4 Applications

In this section we apply our results to some specific boundary value problems.

4.1 Third order boundary value problem

Consider the third order boundary value problem

$$\begin{cases} -u'''(x) = a(x) f(u(x)), & x \in (0,1), \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{4.1}$$

where $a \in C([0,1], \mathbb{R}^+)$ does not vanish identically on any subinterval of $[0,1]$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. We also consider the associated linear eigenvalue problem

$$\begin{cases} -u'''(x) = \mu a(x) u(x), & x \in (0,1), \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{4.2}$$

Theorem 4.1. *The linear eigenvalue problem* (4.2) *has a unique positive eigenvalue* $\mu_1 > 0$. *Moreover, problem* (4.1) *has no positive solution if either*

$$\inf\{f(t,u)/u,\ t\in[0,1]\ \ u>0\} > \mu_1$$

or

$$\sup\{f(t,u)/u,\ t\in[0,1]\ \ u>0\} < \mu_1.$$

Proof. Let $X = \left\{u \in C^2([0,1]) \ :\ u(0) = u'(0) = u'(1) = 0\right\}$ be equipped with the norm defined for $u \in X$ by $\|u\| = \sup\{|u''(t)|, t \in [0,1]\}$ and consider the operator $L : X \to X$ given by

$$Lu(x) = \int_0^x \left(\int_0^1 G(s,t)a(t)u(t)dt\right), \tag{4.3}$$

where

$$G(s,t) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

is the Green function associated with the differential operator $-\dfrac{d^2}{dx^2}$ and Dirichlet boundary conditions. It is clear that $\mu > 0$ is a positive eigenvalue of (4.2) if and only if μ^{-1} is a positive eigenvalue of L. Let Q be the natural cone in X, i.e., $Q = \{u \in X \ :\ u \ge 0 \text{ in } [0,1]\}$. In view of Corollary 3.18, let us prove that $L(Q^*) \subset \operatorname{int}Q$. To this end, consider the set

$$S = \{u \in X : u' > 0 \text{ in } (0,1),\ u''(0) > 0, \text{ and } u''(1) < 0\}.$$

We have $S \subset Q$ and S is an open set; in fact, $X \setminus S = F_1 \cup F_2 \cup F_3$ where

$$\begin{aligned} F_1 &= \{u \in X : \text{there exists } x \in (0,1) \text{ with } u'(x) \le 0\}, \\ F_2 &= \{u \in X : u''(0) \le 0\}, \text{ and} \\ F_3 &= \{u \in X : u''(1) \ge 0\}. \end{aligned}$$

It is clear that F_2 and F_3 are closed sets in X so let $(u_n) \subset F_1$ tending to u in X and $(x_n) \subset (0,1)$ tending to $\overline{x} \in [0,1]$ with $u_n'(x_n) \le 0$. Now if $\overline{x} \in (0,1)$, then $u'(\overline{x}) = \lim u_n'(x_n) \le 0$, and so $u \in F_1$. If $\overline{x} = 0$, we have $u''(0) = \lim_{n\to\infty} \frac{u_n'(x_n)}{x_n} \le 0$, which implies $u \in F_2$. Finally, if $\overline{x} = 1$, then $u''(1) = \lim_{n\to\infty} \frac{u_n'(x_n)}{x_n - 1} \ge 0$, so $u \in F_3$.

Now let $u \in Q^*$ and $v = Lu$; we have

$$v'(x) = \int_0^1 G(x,t)a(t)u(t)dt > 0 \ \text{ for any } \ x \in (0,1),$$

$$v''(0) = \int_0^1 (1-t)a(t)u(t)dt > 0, \text{ and } v''(1) = -\int_0^1 ta(t)u(t)dt < 0,$$

that is, $L(Q^*) \subset S \subset \operatorname{int}Q$. Since Q is not a normal cone in X, to complete our proof we need to show that L is a Q-normal operator. Let $u_1, u_2 \in Q$ with $u_1 \le u_2$, $v_1 = Lu_1$, and $v_2 = Lu_2$. For $i = 1,2$, v_i' are concave functions on $[0,1]$ and $\|v_i\| = \max\left\{v_i''(0), -v_i''(1)\right\}$. We have

$$v_1''(0) = \int_0^1 (1-t)a(t)u_1(t)dt \le \int_0^1 (1-t)a(t)u_2(t)dt = v_2''(0)$$

and

$$-v_1''(1) = \int_0^1 ta(t)u_1(t)dt \le \int_0^1 ta(t)u_2(t)dt = -v_2''(0).$$

That is, $\|v_1\| \le \|v_2\|$ and so L is Q-normal. The conclusion of the theorem then follows from Corollary 3.18 and Proposition 3.6. □

In order to present an existence result, we introduce the following notations:

$$f^0 = \limsup_{u\to 0}\left(\max_{t\in[0,1]} \frac{f(t,u)}{u}\right), \quad f^\infty = \limsup_{u\to +\infty}\left(\max_{t\in[0,1]} \frac{f(t,u)}{u}\right),$$

$$f_0 = \liminf_{u\to 0}\left(\min_{t\in[0,1]} \frac{f(t,u)}{u}\right), \quad f_\infty = \liminf_{u\to +\infty}\left(\min_{t\in[0,1]} \frac{f(t,u)}{u}\right).$$

Theorem 4.2. *If either*

$$f^0 < \mu_1 < f_\infty \le f^\infty < \infty$$

or

$$f^\infty < \mu_1 < f_0 \le f^0 < \infty$$

holds, then problem (4.1) *has a positive solution.*

Proof. Let $E = C([0,1])$ equipped with its sup-norm, $L : E \to E$ be the operator defined by (4.3), and $F : C \to C$ be the Nemytskii operator defined for $u \in C$ by $Fu(t) = f(t,u(t))$, where C is a positive cone in E. It is clear that continuity of f implies that F is continuous and maps bounded sets of C into bounded sets of C. Also, L is an increasing and compact operator and u is a positive solution of problem (4.1) if and only if u is a positive fixed point of LF. Let λ_C^+ and λ_C^- be defined by

$$\lambda_C^+ = \inf\{\lambda \ge 0 : \; Lu \le \lambda u \text{ for some } u \in C^*\}$$

and

$$\lambda_C^- = \sup\{\lambda \ge 0 : \; Lu \ge \lambda u \text{ for some } u \in C^*\}.$$

Since 0 is not an eigenvalue of L and $L(C) \subset Q$, where Q is the cone defined in the proof of Theorem 4.1, it follows from Remark 3.19 that

$$(\mu_1)^{-1} = \lambda_C^- = \lambda_C^+.$$

Moreover, $f^0 < \mu_1 < f_\infty \le f^\infty < \infty$ (the other case is similar) implies there exists $\varepsilon > 0$ and positive constants C_1, C_2 such that

$$F(u) \le (\mu_1 - \varepsilon)u + G(u) \;\text{ for all }\; u \in Q^* \cap B(0,\delta)$$

and

$$(\mu_1 + \varepsilon)u - C_1 \le F(u) \le (f^\infty + \varepsilon)u + C_2 \text{ for all } u \in Q^*,$$

where $Gu(t) = \max\left\{f(t,u(t)) - f^0 u(t), 0\right\}$. The conclusion of the theorem follows from Theorem 3.10. □

4.2 Positive solution for the generalized Fisher like equation posed on the positive half line

Consider the boundary value problem

$$\begin{cases} -u''(x) + cu'(x) + \lambda u(x) = a(x)f(x,u(x)), \ x \in (0,+\infty), \\ u(0) = u(+\infty) = 0, \end{cases} \tag{4.4}$$

where c, λ are positive constants, $a \in C(\mathbb{R}^+,\mathbb{R}^+)$ does not vanish identically on any subinterval of $[0,+\infty)$, and $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function. Also consider the associated linear eigenvalue problem

$$\begin{cases} -u''(x) + cu'(x) + \lambda u(x) = \mu a(x)u(x), \ x \in (0,+\infty), \\ u(0) = u(+\infty) = 0. \end{cases} \tag{4.5}$$

Let G be the Green function associated with (4.4) given by

$$G(x,t) = \frac{1}{k}\begin{cases} e^{r_1(x-t)}(1-e^{(r_1-r_2)x}), & x \geq t, \\ e^{r_2(x-t)}(1-e^{(r_1-r_2)t}), & x \leq t, \end{cases}$$

where $r_1 < 0 < r_2$ are the two roots of $-X^2 + cX + \lambda = 0$ and $k = r_2 - r_1$. For the mathematical origin and physical significance of this equation we refer the reader to [9]–[12].

Denote by E the Banach space of all continuous functions defined on $[0,+\infty)$ that vanish at 0 and $+\infty$ equipped with its sup-norm. Let $L : E \to E$ be the linear operator defined by

$$Lu(x) = \int_0^{+\infty} G(x,t)a(t)u(t)dt$$

and $F : U \to U$ be the Nemytskii operator defined by

$$Fu(t) = f(t,u(t)),$$

where U is the normal positive cone of E. It is clear that F is continuous and maps bounded sets into bounded sets and $u \in E$ is a positive solution of (4.4) if and only if u is a positive fixed point of LF. Moreover, $\mu > 0$ is a positive eigenvalue of (4.5) if and only if μ^{-1} is a positive eigenvalue of L.

In order to prove the compactness of the operator L, we will make use of the following lemmas.

Lemma 4.3. ([1], [2]) *A subset $A \subset E$ is relatively compact if and only if the following conditions are satisfied*

(i) A is uniformly bounded;

(ii) A is equicontinuous on every compact interval of $\mathbb{R}^+$;

(iii) A is equiconvergent.

By *equiconvergence* in Lemma 4.3 we mean that for every $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that, for all $u \in A$ and $t > T_\varepsilon$, we have $|u(t)| < \varepsilon$.

Lemma 4.4. *If $a \in E$, then $L \in Q(E)$.*

Proof. First note that $\int_0^{+\infty} G(0,t)a(t)dt = 0$, and by Hôpital's rule, we conclude from the fact that $a \in E$ that $\lim_{x\to+\infty} \int_0^{+\infty} G(x,t)a(t)dt = 0$. Let $[a,b] \subset [0,+\infty)$ and $a \le x < y \le b$. Straightforward computations show that

$$\left|\int_0^{+\infty} G(x,t)a(t)dt - \int_0^{+\infty} G(y,t)a(t)dt\right| \le 2\theta^* \int_x^y e^{-r_1 t}a(t)dt + 2\gamma^* \int_x^y e^{-r_2 t}a(t)dt\,,$$

where $\theta^* = \sup\left\{e^{-r_1 x}\left(1 - e^{(r_1-r_2)x}\right) : x \in [a,b]\right\}$ and $\gamma^* = \sup\{e^{r_2 x} : x \in [a,b]\}$.

Since the functions $z \to \int_0^z e^{-r_1 t}a(t)dt$ and $z \to \int_0^z e^{-r_2 t}a(t)dt$ are uniformly continuous on $[a,b]$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for all x, $y \in [a,b]$ with $|x-y| < \delta$, we have

$$\left|\int_0^{+\infty} G(x,t)a(t)dt - \int_0^{+\infty} G(y,t)a(t)dt\right| < \varepsilon. \tag{4.6}$$

Therefore, the function $x \to \int_0^{+\infty} G(x,t)a(t)dt$ is continuous and $\sup_{x\ge0}\{\int_0^{+\infty} G(x,t)a(t)dt\} < \infty$. Thus, for all $u \in E$, $Lu(0) = 0$, $\lim_{x\to+\infty} Lu(x) = 0$, and Lu is continuous on $[0,+\infty)$, that is, $Lu \in E$. In addition,

$$|Lu(x)| \le \left(\sup_{x\ge0}\left\{\int_0^{+\infty} G(x,t)a(t)dt\right\}\right)\|u\|,$$

so $L \in L(E)$.

To show the compactness of L, let B be a subset of E bounded by $M > 0$. Then $L(B)$ is bounded by $\left(\sup_{x\ge0}\{\int_0^{+\infty} G(x,t)a(t)dt\}\right)M$, and for all $u \in B$ and x, $y \in [a,b] \subset [0,+\infty)$ with $0 < y - x < \delta$, (4.6) implies

$$|Lu(x) - Lu(y)| \le M\varepsilon,$$

that is, $L(B)$ is equicontinuous on any compact subinterval of $[0,+\infty)$.

Now, for any $u \in B$, we have

$$|Lu(x)| \le M\int_0^{+\infty} G(x,t)a(t)dt,$$

so from the fact that $\lim_{x\to+\infty} \int_0^{+\infty} G(x,t)a(t)dt = 0$, for any $\varepsilon > 0$ there exists $T_\varepsilon > 0$ such that

$$|Lu(x)| \le M\int_0^{+\infty} G(x,t)a(t)dt < \varepsilon \text{ for any } x > T_\varepsilon.$$

Hence, $L(B)$ is equiconvergent. Thus, Lemma 4.3 guarantees $L \in Q(E)$. □

Now consider the functional $\alpha : U \to \mathbb{R}^+$ defined by

$$\alpha(u) = \min\{u(x), x \in [\gamma,\delta]\},$$

where $[\gamma,\delta] \subset (0,+\infty)$ is a given interval. It is easy to see that α has the following properties:

$\alpha(\lambda u) = \lambda\alpha(u)$ for any $u \in U$ and $\lambda \ge 0$;

$u \le v$ implies $\alpha(u) \le \alpha(v)$ where $u, v \in U$;

$\alpha(Lu) = 0$ implies $u = 0$;

and for all $u \in U$,

$$\alpha(Lu) \ge C_0 \alpha(u), \tag{4.7}$$

where $C_0 = \min\left\{\int_\gamma^\delta G(x,t)a(t)dt : x \in [\gamma,\delta]\right\} > 0$.

Thus, if $\lambda \ge 0$ and $u \in U^*$ are such that $Lu \le \lambda u$, then

$$0 < \alpha(Lu) \le \lambda \alpha(u),$$

and by (4.7),

$$0 < C_0 \alpha(u) \le \alpha(Lu) \le \lambda \alpha(u),$$

that is, $\lambda \ge C_0$, and so $\lambda^+ \ge C_0 > 0$.

Consider the bilinear form $\chi : E \times E \to \mathbb{R}$ defined for $u, v \in E$ by

$$\chi(u,v) = \int_0^{+\infty} e^{-cx} a(x)u(x)v(x)dx.$$

It is clear that χ is positive, increasing, and for all $u, v \in U^*$, $\chi(Lu,v) > 0$. Let $u, v \in U$, $W_1 = Lu$, and $W_2 = Lv$. We need to prove that $e^{-cx}W_1'(x)$ and $e^{-cx}W_2'(x)$ are bounded functions. Let $x_0 \ge 0$ be such that $W_1'(x_0) = 0$. Then,

$$\left|e^{-cx}W_1'(x)\right| = \left|\int_{x_0}^x e^{-cx}(mu - \lambda W_1)\right| \le \left(\int_0^{+\infty} e^{-cx}dx\right)(\|m\|\,\|u\| + \lambda\,\|W_1\|) < \infty,$$

and similarly

$$\left|e^{-cx}W_2'(x)\right| \le \left(\int_0^{+\infty} e^{-cx}dx\right)(\|m\|\,\|v\| + \lambda\,\|W_2\|) < \infty.$$

Two integrations by parts then lead to

$$\chi(Lu,v) = \int_0^{+\infty} e^{-cx}a(x)W_1(x)v(x)dx$$
$$= \int_0^{+\infty} e^{-cx}a(x)W_1(x)\left(-W_2''(x) + cW_2'(x) + \lambda W_2(x)\right)dx$$

and

$$\int_0^{+\infty} aW_1\left(-\left(e^{-cx}W_2'\right)' + \lambda e^{-cx}W_2\right)dx$$
$$= \int_0^{+\infty} aW_2\left(-\left(e^{-cx}W_1'\right)' + \lambda e^{-cx}W_1\right)dx = \chi(u,Lv).$$

The hypotheses of Theorem 3.13 are satisfied, so we have the following result.

Theorem 4.5. *The linear eigenvalue problem* (4.5) *has a unique positive eigenvalue* $\mu_1 > 0$. *Moreover, problem* (4.4) *has no positive solutions if either*

$$\inf\{f(t,u)/u,\ t \in [0,1]\ u > 0\} > \mu_1$$

or

$$\sup\{f(t,u)/u,\ t \in [0,1]\ u > 0\} < \mu_1.$$

To prove our existence result we need the following notation:

$$f^0 = \limsup_{u\to 0}\left(\max_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right), \quad f^\infty = \limsup_{u\to+\infty}\left(\max_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right)$$

$$f_0 = \liminf_{u\to 0}\left(\min_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right), \quad f_\infty = \liminf_{u\to+\infty}\left(\min_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right).$$

Theorem 4.6. *If either*

$$f^0 < \mu_1 < f_\infty \le f^\infty < \infty$$

or

$$f^\infty < \mu_1 < f_0 \le f^0 < \infty,$$

then problem (4.4) *has a positive solution.*

Proof. The condition $f^0 < \mu_1 < f_\infty \le f^\infty < \infty$ (the other case is similar) implies that there exists $\varepsilon > 0$ and positive constants C_1, C_2 such that

$$F(u) \le (\mu_1 - \varepsilon)\,u + G(u) \text{ for all } \in U^* \cap B(0,\delta)$$

and

$$(\mu_1 + \varepsilon)\,u - C_1 \le F(u) \le (f^\infty + \varepsilon)\,u + C_2 \text{ for all } u \in U^*,$$

where $Gu(t) = \max\left\{f(t,u(t)) - f^0 u(t), 0\right\}$. The conclusion then follows from Theorem 3.10. □

Remark 4.7. The generalized Fisher equation posed on the real line has been studied in [8] and [9]. Arguing as in Sub-section 4.2, we can prove the existence of $0 < \mu^+ \le \mu^-$ such that, if

$$f^0 < \mu^+ \le \mu^- < f_\infty \le f^\infty < \infty$$

or

$$f^\infty < \mu^+ \le \mu^- < f_0 \le f^0 < \infty$$

holds, then the boundary value problem

$$\begin{cases} -u''(x) + cu'(x) + \lambda u(x) = a(x) f(x,u(x)),\ x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{cases}$$

has a positive solution in the case where $a \in C(\mathbb{R}, \mathbb{R}^+)$ does not vanish identically on any subinterval of $\mathbb{R}$, and vanishes at $\pm\infty$, $f \in C(\mathbb{R}\times\mathbb{R}^+, \mathbb{R}^+)$, and

$$f^0 = \limsup_{u\to 0}\left(\max_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right), \quad f^\infty = \limsup_{u\to+\infty}\left(\max_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right),$$

$$f_0 = \liminf_{u\to 0}\left(\min_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right), \quad f_\infty = \liminf_{u\to +\infty}\left(\min_{t\in[0,+\infty)} \frac{f(t,u)}{u}\right).$$

Moreover, the eigenvalue problem

$$\begin{cases} -u''(x) + cu(x) + \lambda u(x) = \mu a(x)u(x), \ x \in \mathbb{R}, \\ u(-\infty) = u(+\infty) = 0, \end{cases}$$

admits at least one positive eigenvalue.

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