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LARGE POPULATION STOCHASTIC DYNAMIC GAMES: CLOSED-LOOP MCKEAN-VLASOV SYSTEMS AND THE NASH CERTAINTY EQUIVALENCE PRINCIPLE*

MINYI HUANG†, ROLAND P. MALHAMÉ‡, AND PETER E. CAINES§

Abstract. We consider stochastic dynamic games in large population conditions where multi-class agents are weakly coupled via their individual dynamics and costs. We approach this large population game problem by the so-called Nash Certainty Equivalence (NCE) Principle which leads to a decentralized control synthesis. The McKean-Vlasov NCE method presented in this paper has a close connection with the statistical physics of large particle systems: both identify a consistency relationship between the individual agent (or particle) at the microscopic level and the mass of individuals (or particles) at the macroscopic level. The overall game is decomposed into (i) an optimal control problem whose Hamilton-Jacobi-Bellman (HJB) equation determines the optimal control for each individual and which involves a measure corresponding to the mass effect, and (ii) a family of McKean-Vlasov (M-V) equations which also depend upon this measure. We designate the NCE Principle as the property that the resulting scheme is consistent (or soluble), i.e. the prescribed control laws produce sample paths which produce the mass effect measure. By construction, the overall closed-loop behaviour is such that each agent's behaviour is optimal with respect to all other agents in the game theoretic Nash sense.

Key words: Stochastic dynamic games, large populations, multi-class agents, interacting particle systems, statistical physics, decentralized control, Hamilton-Jacobi-Bellman equation, McKean-Vlasov equation, Nash equilibria

1. Introduction.

The modelling and analysis of dynamic systems with many interacting agents has gained research attention from a wide range of disciplines. In this paper, the investigation of large weakly coupled systems has its motivation coming from many complex phenomena arising in engineering and socioeconomic settings, among others, for instance, dynamic economic models involving competing agents [18, 14, 35, 25], biological models on animal competition and conflicts [36, 38], wireless power control [23, 24], road traffic engineering [46, 22], and shared data buffer modelling [3]. Also, large-scale weakly coupled stochastic dynamic systems appear in

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†Minyi Huang is with the Department of Information Engineering, Research School of Information Sciences and Engineering, The Australian National University, Canberra, ACT 0200, Australia. E-mail: minyi.huang@rsise.anu.edu.au. Please address all correspondence to this author.

‡Roland P. Malhamé is with the Department of Electrical Engineering, École Polytechnique de Montréal, Montreal, QC H3C 3A7, Canada, and also affiliated with GERAD, Montreal. E-mail: roland.malhame@polymtl.ca.

§Peter E. Caines is with the Department of Electrical and Computer Engineering, McGill University, Montreal, QC H3A 2A7, Canada. E-mail: peterc@cim.mcgill.ca.

mathematical biology and have received justification by field observations [41]. Usually, a key common feature to these systems is that while each agent only receives a negligible influence from any other given individual, the effect of the overall population is significant to each agent.

In this paper, we study the optimization of large-scale nonlinear stochastic systems wherein many agents are each coupled with others via the individual dynamics and costs. This kind of weak coupling in both dynamics and costs is used to model the mutual impact of agents during competitive decision-making. In particular, the dynamic coupling specifies the impact of the environment on an individual's decision-making, and the underlying model takes the form of weakly coupled diffusions subject to individual controls. We aim to develop a methodology for multi-agent optimization with local information. In contrast to the extensive literature on dynamic games (see, e.g., [6, 42, 16, 31, 45]), we are particularly interested in large populations. We note that games with a large or infinite population have long been a major research area in game theory [33, 1, 2, 40, 43, 19], but traditionally most work has been based upon static models. And moreover, in our modelling each agent only has local information about its own state and dynamics in addition to the statistical property about the population parameter variation. This leads to a situation of decision with incomplete information [20, 21, 4, 39]. In general, for analysis of such games, Harsanyi's approach via Bayesian players [20] is important, though it involves great complexity in specifying the types of different players. However, in large population conditions, it is possible to develop efficient decision-making without paying excessive attention to the fine details of the system structure since each individual agent's effect is extremely weak with respect to the overall population's behaviour which becomes more relevant to a given agent's optimization as the population increases. Such an intuitive fact will provide the main motivation in developing our methodology which has a close connection with large particle systems and statistical physics [5, 44, 34, 32, 11].

Based upon the interaction between the individual and mass, we develop state aggregation for the underlying dynamic models which extracts the overall effect a given agent receives from the population. Our general methodology based upon state aggregation and upon individual-mass interaction was first developed for a population with linear individual dynamics or mild nonlinearity [23, 25, 28, 29, 27, 26, 30], combined with quadratic individual costs, which usually led to explicit calculations for the individual strategies. The aim of this paper is to extend this general approach to fully nonlinear models. Our analysis is facilitated by considering a controlled McKean-Vlasov (M-V) equation associated with the large population limit. The key step is to construct a mutually consistent pair of (i) the mass effect and (ii) the individual strategies such that the latter not only each constitute an optimal response to the mass effect but also collectively produce that mass effect. By the NCE Methodology we mean the overall game decomposition into a (non-standard) optimal control problem whose HJB equation involves a mass effect measure and a closed-loop McKean-Vlasov equation; these two parts are related to each other by the optimal control law derived from the former. In this setting, we designate the NCE Principle as the property that the resulting scheme is consistent in the sense that the prescribed control laws produce sample paths which produce the mass effect with respect to which the optimal control is derived via the HJB equation. It is a property of this overall closed-loop behaviour that each agent's optimal behaviour with respect to all other agents holds in the game theoretic Nash sense.

So it turns out that the application of the NCE methodology amounts to determining a certain mass effect such that in the first step we solve a stochastic optimal control problem which generates the individual strategies, and such that in the second step, the closed-loop M-V equation will generate the same mass effect which has been used in the first place. In carrying out these two steps, we introduce the so-called best response map and measure-flow inducing map. The solution to the overall problem, and hence the demonstration of the NCE Principle, or Property, relies on finding a fixed point to the composite action of the two maps in the appropriate metrized space of measures. The fixed point analysis is facilitated by introducing the so-called Vasershtein metric for probability measures on the space of continuous functions [13, 44].

The rest of the paper is organized as follows. In Section 2 we introduce the stochastic dynamic game model involving a set of interacting diffusions, and the well-known McKean-Vlasov equation for interacting particles. In Section 3 we describe the NCE principle which yields the HJB equation together with individual strategies, and the closed-loop M-V equation. Section 4 introduces some important assumptions for the system dynamics. Sections 5 and 6 analyze the decoupled HJB equation and M-V equation, respectively. In Section 7 we present an existence result for the consistent pair under suitable regularity and gain conditions. Section 8 is devoted to an asymptotic equilibrium analysis.

For the sake of exposition, we make a few conventions about notation. For a system involving many scalar state components (e.g., z_i, y_i, etc.), we use the integer-valued subscript as the label for a certain agent. When the system involves only a single agent or particle (e.g., x_t, w_t), we may use the real-valued subscript to indicate time. Throughout the paper, $|\cdot|$ denotes the Euclidean norm of a vector. The integer n is reserved to denote the population size of the game. We use C, C_1, C_2, etc. to denote a generic constant independent of the population size, and they may vary from place to place.

2. The Weakly Coupled Systems. We first formulate the large population stochastic dynamic game, which is then followed by the introduction of the interacting particle (IP) system. The motivation for such an organization is that, while the former will be the focus of our analysis in this paper, the modelling and optimization approach in the nonlinear stochastic dynamic game problem has a close connection with the modelling approach in the latter, especially with respect to the individual-mass interaction aspect and the microscopic analysis based upon single agents or particles.

2.1. The Stochastic Dynamic Game. We consider an n dimensional nonlinear stochastic system where the evolution of each state component is described by

$$dz_i = (1/n)\sum_{j=1}^{n} f_{a_i}(z_i, u_i, z_j)dt + \sigma dw_i, \quad 1 \le i \le n, \quad t \ge 0, \tag{1}$$

where $\{w_i, 1 \le i \le n\}$ denotes n independent standard scalar Wiener processes. The initial states $z_i(0)$ are mutually independent, and also independent of $\{w_i, 1 \le i \le n\}$. In addition, $E|z_i(0)|^2 < \infty$. Each state component shall be referred to as the state of the corresponding agent (also to be called player). The control input u_i takes its values in a compact set $U = [\alpha, \beta]$. The function f_{a_i} is from $\mathbb{R} \times U \times \mathbb{R}$ to $\mathbb{R}$. For simplicity of analysis we take the diffusion coefficient to be the same constant $\sigma > 0$ for the n agents. Unless otherwise stated, throughout the paper z_i is described by the dynamics (1).

The nonlinear functions $f_{(\cdot)}$ are indexed by $a \in \mathcal{A}$ where a is called the dynamic parameter and $\mathcal{A}$ is an index set. Note that we indicate no explicit dependence of $f_{(\cdot)}$ on a, and for different values of a, f_a is allowed to take different forms. For the ith agent, its dynamic parameter takes the specific value $a_i \in \mathcal{A}$. The dynamic parameter may vary from agent to agent; this property is used to describe the heterogeneity of the population. In the special case when $\mathcal{A}$ is a singleton, we get a population of uniform agents.

In the analysis below we assume a finite set $\mathcal{A} = \{\theta_1, \cdots, \theta_K\}$ for modelling multi-class agents, where K is the number of agent classes. We further assume that the distribution for the initial state depends upon the class of the agent. Hence we may denote $\mu_0^{a_i}$ for the initial distribution of z_i, and there are a total of K classes of initial distributions, listed by $\mu_0^{\theta_1}, \cdots, \mu_0^{\theta_K}$.

The cost function for the ith agent is given in the form:

$$J_i(u_i) \triangleq E\int_0^T \Big[(1/n)\sum_{j=1}^{n} L(z_i, u_i, z_j)\Big]dt, \tag{2}$$

where $T \in (0, \infty)$ is the terminal time, L is a nonlinear function from $\mathbb{R} \times U \times \mathbb{R}$ to $\mathbb{R}_+ = [0, \infty)$.

To emphasize the control objective in relation to its own state and control processes, it is possible to consider a more general form for the cost function of the ith agent:

$$J_i'(u_i) \triangleq E \int_0^T \Theta\Big(z_i, u_i, (1/n) \sum_{j=1}^{n} L(z_i, u_i, z_j)\Big) dt, \tag{3}$$

where Θ is a function from $\mathbb{R} \times U \times \mathbb{R}$ to $\mathbb{R}_+$. However, in this paper we will only consider the optimization based upon the cost (2), and under suitable conditions, our analysis may be easily adapted to deal with the cost structure (3).

For the system configuration $z = (z_1, \cdots, z_n)$, define the empirical distribution $\varepsilon_z = (1/n) \sum_{i=1}^{n} \delta_{z_i}$ where $\delta_\bullet$ is the Dirac measure. Then the coupling terms in the individual dynamics and costs are functionals of ε_z which is insensitive to the ordering of the entries in z. This feature is important and will be exploited in developing the aggregation technique for control synthesis.

For the above system, the objective is to seek individual control strategies and appropriately characterize their optimality, and a standard approach is to analyze Nash (equilibrium) strategies. However, this approach requires that each agent has full information on the states and dynamic parameters of all agents, which results in very high control complexity under large-population conditions. This motivates us to search for lower complexity control strategies.

In specifying the structure of the individual dynamics and costs, we assume that neither f_{a_i} nor L explicitly depend upon the control of other agents. However, each z_i is under an indirect influence of u_j, $j \neq i$ via the coupling state variable z_j appearing in $f_{a_i}(z_i, u_i, z_j)$. For illustrating this class of models, we consider a concrete example as follows.

EXAMPLE 1. *We take* $f_{a_i}(z_i, u_i, z_j) = f_{a_i}^0(z_i, u_i) + f_{a_i}^1(z_j)$ *and* $L(z_i, u_i, z_j) = L^0(z_i, u_i) + L^1(z_j)$. *This gives the drift term* $(1/n) \sum_{j=1}^{n} f_{a_i}(z_i, u_i, z_j) = f_{a_i}^0(z_i, u_i) + (1/n) \sum_{j=1}^{n} f_{a_i}^1(z_j)$ *and cost integrand* $(1/n) \sum_{j=1}^{n} L(z_i, u_i, z_j) = L^0(z_i, u_i) + (1/n) \sum_{j=1}^{n} L^1(z_j)$, *corresponding to (1) and (2), respectively.*

The modelling in Example 1 yields a typical situation where each agent's dynamics and cost are closely related to its own state and control selection while receiving an average impact from the population. This reflects an important feature in many practical situations for decision-making.

2.2. The Interacting Particle System. In an interacting particle (IP) system, the state evolution of an individual particle is affected by an empirical average of coupling terms with all other particles. Mathematically, this leads to a set of weakly coupled diffusions, each describing the motion of a single particle, where an averaging across the population produces the coupling term in the individual dynamics. We introduce the following dynamics [12] in the form of N coupled stochastic differential equations (SDE):

$$dx_i = (1/N)\sum_{k=1}^{N} b(x_i, x_k)dt + \sigma dw_i, \quad 1 \le i \le N, \quad t \ge 0, \tag{4}$$

where $b(\cdot,\cdot)$ is a function from $\mathbb{R}^2$ to $\mathbb{R}$, N is the number of particles and all x_i's are assumed to have i.i.d. initial conditions at $t = 0$. Here we assume x_i is a scalar although the modelling is also applicable to the case of vector particle states. The noises $\{w_i, 1 \le i \le N\}$ are N independent Wiener processes independent of the initial conditions $x_i(0)$, $1 \le i \le N$. Let $\varepsilon_x = (1/N)\sum_{i=1}^{N} \delta_{x_i}$ denote the empirical measure of the particle configuration $(x_1, x_2, \cdots, x_N)$. Then the drift term in (4) may be expressed as a function of x_i and ε_x.

For this class of particle models, one can achieve a remarkable degree of economy in the description of population dynamics, by expressing the aggregate coupling term in terms of an expectation over a typical individual's probability distribution function which evolves with time. This is based upon the following intuition: as the number of particles grows to infinity, the particles become essentially indistinguishable while each individual being negligible, and furthermore, there is a decoupling effect such that a single particle's statistical properties can effectively approximate the empirical distribution produced by all particles [10]. More specifically, as N tends to infinity, the individual dynamics may be written in the limiting form:

$$dx_t = b[x_t, \mu_t]dt + \sigma dw_t, \quad t \ge 0, \tag{5}$$

which is the celebrated McKean-Vlasov (M-V) equation. Here $b[x, \mu_t] = \int b(x, y)\mu_t$ (dy) for some probability distribution μ_t on $\mathbb{R}$. This equation, as well as its variants, has been extensively studied in physics, stochastic analysis, and partial differential equations [12, 44, 37, 7, 8]. The noise w_t may be determined in different ways. For instance, if one intends to approximate x_1 in (4) by x_t, one may set $w_t = w_1$ as the driving Brownian motion in (5) and $x_0 = x_1(0)$. Note that by introducing the density function $p_t(x)$, associated with μ_t for x_t, one may recast (5) in the form of a Fokker-Planck equation whose coefficients depend upon the density $p_t(x)$ itself.

Definition 2. *A pair (x_t, μ_t), $t \ge 0$, is said to be a consistent pair if x_t is a solution to the SDE (5) and μ_t is its distribution at time t, i.e., $P(x_t \le \alpha) = \int_{-\infty}^{\alpha} \mu_t(dy)$ for all $\alpha \in \mathbb{R}$ and $t \ge 0$.* □

It is obvious that μ_0 in Definition 2 is determined as the distribution of x_0. For a detailed analysis on the existence and uniqueness of a solution to (5) and asymptotic properties, see [7, 8]. For a weak convergence relationship between solutions of (4) and (5), and large deviation analysis for μ_t, see [44, 12] and references therein.

3. The Nash Certainty Equivalence Principle via Population Limit. For the above n-agent dynamic game, as $n \to \infty$, we will use the M-V equation (5), as well as the notion of a consistent state-distribution pair, as a motivating scheme to develop reduction methods for the mass coupling term in the individual dynamics (1). Specifically, we attempt to use a probability distribution μ_t^i on $\mathbb{R}$, to approximate the empirical distribution of the sub-configuration $(z_{k_1}, \cdots z_{k_l})$ of $(z_1, \cdots, z_n)$ at time t, which corresponds to the l states sharing a dynamic parameter $a = \theta_i$. Roughly speaking, the parameter a is used as a classifier for a family of empirical distributions each being induced by the same class of agents. The rationale for using the K distributions μ_t^i, $1 \le i \le K$, to approximate the overall population effect is that as $n \to \infty$, if the population is well randomized so as to give a sufficient number of agents in each class appearing in the game, then it is possible to approximate the mass effect by the superposition of these K distributions, provided that all agents have sufficiently weak coupling and generate their effect on any given agent additively in a certain manner. However, it should be clear that this is only a heuristic argument, and its justification via the exact characterization of the individual-mass interaction needs to be based upon rigorous mathematical analysis, which will be the main focus of what follows.

For the sequence $\{a_i, i \ge 1\}$ where $a_i \in \mathcal{A}$, we define the empirical distribution associated with the first n agents

$$F_n(\{\theta_k\}) = (1/n) \sum_{i=1}^{n} 1_{(a_i = \theta_k)},$$

which gives a discrete distribution function on $\mathcal{A}$ for each given n. We introduce the assumption below as a characterization of the population statistics.

(H0) There exists a distribution function F on $\mathcal{A} = \{\theta_1, \cdots, \theta_K\}$, denoted as $\pi = (\pi_1, \cdots, \pi_K)$ (i.e., $F(\{\theta_k\}) = \pi_k$), such that $\lim_{n\to\infty} F_n(\{\theta_k\}) = F(\{\theta_k\})$ for each $\theta_k \in \mathcal{A}$. □

Before tackling the large population game, we first consider its limiting form via specifying the behaviour of a single agent, as in the M-V equation. To facilitate the exposition, we call a K-tuple $(\mu_t^1, \cdots, \mu_t^K)$ of K probability measures on $\mathbb{R}$, defined for all $t \in [0, T]$, a probability measure flow on $[0, T]$. Also, we may simply call it a measure flow.

We write the dynamics of a representative agent with the scalar state variable x_t:

$$dx_t = f_a[x_t, u_t, \mu_t^1, \cdots, \mu_t^K]dt + \sigma dw_t, \tag{6}$$

where the distribution of x_0 will be specified, depending upon the value of a, as the common initial distribution of that class of agents, and the drift coefficient is defined as

$$f_a[x, u, \mu_t^1, \cdots, \mu_t^K] = \sum_{i=1}^K \pi_i \int_{\mathbb{R}} f_a(x, u, y)\mu_t^i(dy), \tag{7}$$

which results from the superposition of the average effects, each determined by integrating with respect to a certain μ_t^i, of multi-class agents. We note that the state variable x_t, control variable u_t and noise w_t are associated with a specific value of a; however, for notational brevity, we do not include a matching index a for x_t, u_t and w_t. In fact, if we write equation (6) for K agents in distinct classes, we will get a system of K equations which have their own independent initial conditions and Brownian motions and are coupled by μ_t^i, $1 \le i \le n$. These equations are mutually independent if each individual control is adapted to the σ-algebra generated by the Brownian motion in the same equation.

And corresponding to (2), we define the cost function:

$$J(u, \mu^1, \cdots, \mu^K) \triangleq E\int_0^T L[x_t, u_t, \mu_t^1, \cdots, \mu_t^K]dt, \tag{8}$$

where

$$L[x, u, \mu_t^1, \cdots, \mu_t^K] = \sum_{i=1}^K \pi_i \int_{\mathbb{R}} L(x, u, y)\mu_t^i(dy).$$

Within the context of the limiting game problem, we give the interpretation for the controlled system dynamics (6) as follows. The control u_t should be sought such that (i) it is optimal for the minimization of $J(u, \mu^1, \cdots, \mu^K)$ when the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, is treated as an *exogenous* signal, and (ii) the distribution of x_t in the closed-loop system coincides with μ_t^i, i.e., $P(x_t \le \alpha) = \int_{-\infty}^{\alpha} \mu_t^i(dy)$ for all $\alpha \in \mathbb{R}$ and $t \ge 0$, when the dynamic parameter is set as $a = \theta_i$. This forms the basis for our subsequent control synthesis and this control design scheme is called the Nash Certainty Equivalence (NCE) Methodology. Equation (6) may be looked at as a controlled McKean-Vlasov equation for multi-class agents with the control performance measured by (8). Notice that in step (i), u_t is formally regarded as not affecting $(\mu_t^1, \cdots, \mu_t^K)$ during the strategy selection. The reason for so doing is that in relation to the game with a large but finite population, μ_t^i is used to model the collective effect of the ith class of agents whose states are under their own controls and as such, it is expected to become asymptotically insensitive to the control of the individual agent in question. This consequently leads to a decoupled stochastic control problem involving an isolated agent, which is indicated by its dynamic parameter to be in one of the K classes of agents. Subsequently, in the closed-loop equation we get μ_t^i as the distribution of x_t associated with the parameter $a = \theta_i$.

3.1. The NCE Methodology and the NCE Equation. Since the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, is treated as an exogenous signal in the NCE methodology, the control for the agent described by (6) and (8) is determined by a standard optimal control problem. We formally proceed to write the following equations for which the existence analysis will be developed subsequently. Assuming the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, is given first, we use $V_a(t,x)$, $(t,x) \in [0,T] \times \mathbb{R}$ to denote the value function of agents with parameter a, i.e., $V_a(t,x) = \inf_{u(\cdot)\in\mathcal{U}} \int_t^T L[x_s, u_s, \mu_s^1, \cdots, \mu_s^K] ds$ subject to the dynamics (6). The admissible control set $\mathcal{U}$ consists of all controls such that $u_t \in U$ and is adapted to the σ-algebra $\sigma(x_0, w_s, s \le t)$ with the associated Brownian w_t in this agent's dynamics. We write the HJB equation

$$-\frac{\partial V_a}{\partial t} = \inf_{u\in U} \left\{ f_a[x, u, \mu_t^1, \cdots, \mu_t^K] \frac{\partial V_a}{\partial x} + L[x, u, \mu_t^1, \cdots, \mu_t^K] \right\} + \frac{\sigma^2}{2} \frac{\partial^2 V_a}{\partial x^2}, \tag{9}$$

where the initial time-state pair is $(t,x) \in [0,T) \times \mathbb{R}$ and the terminal condition is $V_a(T,x) = 0$.

We denote the resulting optimal control law by

$$u_t = \varphi_a(t, x | \mu_\cdot^1, \cdots, \mu_\cdot^K), \tag{10}$$

where $(t,x) \in [0,T] \times \mathbb{R}$. It should be noted that the notation in (10) indicates that the value of φ_a at time t depends upon the measure flow $(\mu_t^1, \cdots, \mu_t^K)$ on the whole interval $[0,T]$.

Substituting (10) into (6), we write the closed-loop dynamics

$$dx_t = f_a[x_t, \varphi_a(t, x | \mu_\cdot^1, \cdots, \mu_\cdot^K), \mu_t^1, \cdots, \mu_t^K] dt + \sigma dw_t, \quad 0 \le t \le T. \tag{11}$$

Now, the NCE methodology amounts to finding a solution (x_t, u_t) for each of the K classes of agents and a measure flow $(\mu_t^1, \ldots, \mu_t^K)$ such that (9)-(11) hold, where the distribution of x_t in equation (11) is equal to μ_t^i for $t \in [0,T]$ when $a = \theta_i \in \mathcal{A}$.

Within the framework of the NCE methodology it is required that φ_a is derived from the HJB equation, and in this setting equation (11) may be regarded as a generalized McKean-Vlasov equation where the right hand side has a functional dependence on the distributions of multi-class particles.

For a better appreciation of the interaction between the individual and the mass, we generate K copies of (11) by taking K distinct values of a combined with different initial conditions and driving Brownian motions. This leads to the following coupled M-V equation system:

$$\begin{cases} dx_1 &= f_1[x_1, \varphi_1(t, x_1 | \mu_\cdot^1, \cdots, \mu_\cdot^K), \mu_t^1, \cdots, \mu_t^K] dt + \sigma dw_1, \\ & \vdots \\ dx_K &= f_K[x_K, \varphi_K(t, x_K | \mu_\cdot^1, \cdots, \mu_\cdot^K), \mu_t^1, \cdots, \mu_t^K] dt + \sigma dw_K, \end{cases} \tag{12}$$

where we write f_a (φ_a, resp.) as f_i (φ_i, resp.) when $a = \theta_i$. The K components have independent initial conditions and $\mu_t^k|_{t=0} = \mu_0^{\theta_k}$, $1 \le k \le K$, where $(\mu_0^{\theta_1}, \cdots, \mu_0^{\theta_K})$ has been specified in Section 2.1. The independent Brownian motions here may be different from those in (1). The subsequent analysis is concerned with the existence of a consistent pair $(x_1, \cdots, x_K)$ and $(\mu^1, \cdots, \mu^K)$ solving the equation system (12) coupled by the measure flow where φ_i is derived from the associated HJB equation with a corresponding parameter $a = \theta_i$.

4. Main Assumptions and Restriction on Measure Flows. Our strategy to approach the limiting game problem is to first detach the HJB equation (9) from the closed-loop M-V equation (11). In the analysis below, the measure flows such as $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, *do not* necessarily satisfy (9)-(11) simultaneously.

We introduce the following assumptions:

(H1) U is a compact interval.

(H2) For each $a \in \mathcal{A}$, $f_a(x, u, y)$ and $L(x, u, y)$ are continuous and bounded on $\mathbb{R} \times U \times \mathbb{R}$, and Lipschitz continuous in (x, y), i.e., there exist constants $B_i > 0$ such that

$$|f_a(x, u, y) - f_a(x', u, y')| \le B_1|x - x'| + B_2|y - y'|,$$
$$|L(x, u, y) - L(x', u, y')| \le B_3|x - x'| + B_4|y - y'|,$$

for all $u \in U$ and $x, x', y, y' \in \mathbb{R}$. In addition, there exists a modulus of continuity $m(\cdot) : (0, \infty) \to \mathbb{R}_+$ (satisfying $m(0+) = 0$) such that $\sup_{x,y,u\neq u'} |\psi(x, u, y) - \psi(x, u', y)| \le m(|u - u'|)$, where ψ stands for f_a or L.

(H3) For both $f_a(x, u, y)$ and $L(x, u, y)$, their first and second order derivatives (w.r.t. x) are all uniformly continuous and bounded on $\mathbb{R} \times U \times \mathbb{R}$, and Lipschitz continuous in y.

(H4) For each $a \in \mathcal{A}$, $f_a(x, u, y)$ is Lipschitz continuous in $u \in U$, i.e., there exists $B_5 > 0$ such that $|f_a(x, u, y) - f_a(x, u', y)| \le K_5|u - u'|$ for any $u, u' \in U$.

(H5) For any $q \in \mathbb{R}$, $a \in \mathcal{A}$ and any probability distribution $\nu(dy)$ on $\mathbb{R}$, the set

$$S_a(x, q) = \arg\min_{u \in U} \Big\{ \int_{y\in\mathbb{R}} [q f_a(x, u, y) + L(x, u, y)]\nu(dy) \Big\}$$

is a singleton and the resulting u, as a function of (x, q), is Lipschitz continuous in $(x, q) \in \mathbb{R}^2$, uniformly with respect to the choice of $\nu(dy)$.

The assumptions (H1)-(H2) are mostly standard conditions used in the stochastic control literature. In (H3), we only need the differentiability condition with respect to x rather than (x, y) since in the limiting game the y component will be averaged out to lead to time-varying dynamics involving (t, x).

We need (H4) and (H5) to ensure some regularity of the closed-loop McKean-Vlasov equation in order to analyze its solvability. It should be noted that when (H4) holds, the modulus of continuity condition for f_a in (H2) is automatically satisfied. The first part of (H5) may be satisfied under suitable convexity conditions with respect to u. We may give a simple example where L contains a quadratic term u^2 multiplied by a positive term $g(x)$ and where f_a is affine in u. For illustration, consider the model $f_a = f_a^0(x,y) + f_a^1(x,y)u$ and $L = L^0(x,y) + g(x,y)u^2$, where $\inf_{x,y} f_a^1(x,y) > 0$ and $\inf_{x,y} g(x,y) > 0$; by applying (H5) to such a pair (f_a, L) for given (x,q) and $\nu(dy)$, it leads to the minimization of a quadratic function in terms of the argument u restricted to a compaction interval. Obviously this procedure yields a unique minimizer.

If the probability measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, is fixed, $f_a[x, u, \mu_t^1, \cdots, \mu_t^K]$ and $L[x, u, \mu_t^1, \cdots, \mu_t^K]$ each become a function of (t, x, u), and accordingly, we denote

$$f_a^*(t,x,u) \triangleq f_a[x,u,\mu_t^1,\cdots,\mu_t^K], \qquad L^*(t,x,u) \triangleq L[x,u,\mu_t^1,\cdots,\mu_t^K]. \tag{13}$$

In order to analyze the HJB equation involving f^* and L^*, we need to impose some restrictions on the measure flow, which will be useful for examining the individual equations in (9)-(11). We introduce the following class $\mathcal{M}_{[0,T]}$ of measure flows.

DEFINITION 3. *A probability measure flow $(\mu_t^1, \cdots, \mu_t^K)$ on $[0,T]$ is in $\mathcal{M}_{[0,T]}$, if there exists $\beta \in (0,1]$ such that for any bounded and Lipschitz continuous function ψ on $\mathbb{R}$,*

$$\sup_{1 \le j \le K} \Big| \int_{\mathbb{R}} \psi(y)\mu_{t'}^j(dy) - \int_{\mathbb{R}} \psi(y)\mu_{t''}^j(dy) \Big| \le B_6 |t' - t''|^\beta \tag{14}$$

for all $t', t'' \in [0,T]$, where for the given $(\mu_t^1, \cdots, \mu_t^K)$, the constant B_6 may be selected to depend only upon the Lipschitz constant of ψ. The constant β, to be called the Hölder exponent, depends upon the specific $(\mu_t^1, \cdots, \mu_t^K)$. □

The set $\mathcal{M}_{[0,T]}$ is nonempty since we may take all μ_t^j, $1 \le j \le K$ and $0 \le t \le T$, to be the Dirac measure at any constant y_0. We give some explanation on (14) by relating it to weak convergence of measures. Let t' be fixed and take $t'' \to t'$. If $u_{t''}$ is only known to weakly converge to $u_{t'}$, the left hand side is a vanishing term for any bounded and continuous function ψ, but this in general leads to no explicit vanishing rate. Thus in defining $\mathcal{M}_{[0,T]}$, the convergence rate is strengthened.

PROPOSITION 4. *Let f_a^*, $a \in \mathcal{A}$, and L^* be defined by (13) for which $(\mu_t^1, \cdots, \mu_t^K) \in \mathcal{M}_{[0,T]}$ is fixed with Hölder exponent β in (14), and in the following we assume (H1) always holds. We have:*

(i) Under (H2), f_a^* and L^* are continuous and bounded on $[0,T] \times \mathbb{R} \times U$, and in addition $f_a^*(t,x,u)$ and $L^*(t,x,u)$ are Hölder continuous in t with exponent β, i.e.,

$$\sup_{u \in U, x \in \mathbb{R}} \sup_{0 \le s < t \le T} \frac{|\psi(t,x,u) - \psi(s,x,u)|}{|t-s|^\beta} \le c, \tag{15}$$

where $\psi = f_a^*, L^*$, *and* c *is a finite constant.*

(ii) Under (H3), for $\psi = f_a^*, L^*$, *the partial derivatives* ψ_x *and* ψ_{xx} *are continuous and bounded on* $[0,T]\times\mathbb{R}\times U$.

(iii) Under (H4), there exists $c>0$ *such that* $\sup_{(t,x)\in[0,T]\times\mathbb{R}}|f_a^*(t,x,u)-f_a^*(t,x,u')|\le c|u-u'|$, *for each* $a\in\mathcal{A}$, *i.e., each* f_a^* *is Lipschitz continuous in* $u\in U$.

(iv) Under (H5), for any $q\in\mathbb{R}$, *the set of minimizers* $\arg\min_{u\in U}[f_a^*(t,x,u)q+L^*(t,x,u)]$ *is a singleton.*

Proof. (i) We analyze f_a^* only, and the case for L^* is similar. We first prove (15) for f_a^*. Since $(\mu_t^1,\cdots,\mu_t^K)\in\mathcal{M}_{[0,T]}$ with Hölder exponent β, and since $|f_a(x,u,y)-f_a(x,u,y')|\le B_2|y-y'|$ with the Lipschitz constant B_2 independent of (x,u), we may select a finite constant $c>0$ such that (15) holds.

The boundedness of f_a^* is obvious. We take (t,x,u) and (t',x',u'), both from the set $[0,T]\times\mathbb{R}\times U$. We have

$$\begin{aligned} &|f_a^*(t,x,u)-f_a^*(t',x',u')| \\ &= |f_a^*(t,x,u)-f_a^*(t,x',u')|+|f_a^*(t,x',u')-f_a^*(t',x',u')|. \end{aligned} \tag{16}$$

By (H2), it is easy to show that $|f_a(x,u,y)-f_a(x',u',y)|\to 0$ uniformly, as $|x-x'|+|u-u'|\to 0$. By the definition of f_a^*, this implies that $|f_a^*(t,x,u)-f_a^*(t,x',u')|\to 0$ uniformly as $|x-x'|+|u-u'|\to 0$. By combining (16) with (15), it follows that $|f_a^*(t,x,u)-f_a^*(t',x',u')|\to 0$ uniformly, as $|t-t'|+|x-x'|+|u-u'|\to 0$.

(ii) Under (H3), the partial derivatives ψ_x, ψ_{xx} exist, and for $i=1,2$,

$$\frac{\partial[f_a^*(t,x,u)]^i}{\partial x^i}=\sum_{k=1}^K \pi_k\int\frac{\partial[f_a(x,u,y)]^i}{\partial x^i}\mu_t^k(dy),$$

$$\frac{\partial[L^*(t,x,u)]^i}{\partial x^i}=\sum_{k=1}^K \pi_k\int\frac{\partial[L(x,u,y)]^i}{\partial x^i}\mu_t^k(dy),$$

where the integration and differentiation are interchangeable due to the boundedness of the derivatives of f_a and L in x. The continuity of the derivatives may be proved by following similar steps as in (i).

The proof of (iii) is obvious. Since both f_a^* and L^* are defined using the measure $\sum_{i=1}^K\pi_i\mu_t^i(dy)$, (iv) follows. □

5. HJB Equation for Optimal Control of a Single Agent. The NCE methodology translates into the analysis of the three coupled equations specifying (i) the HJB equation for the optimal control problem based upon a single agent, (ii) the optimal strategy for each type of agents as classified by the dynamic parameter a, and (iii) the closed-loop M-V equation. Instead of directly analyzing the coupled equation system, we shall begin by dealing with the decoupled individual equations. This will provide insight into the structure of the underlying game problem.

By using the notation in (13), we write the HJB equation

$$-\frac{\partial V_a}{\partial t} = \inf_{u\in U}\left\{ f_a^*(t,x,u)\frac{\partial V_a}{\partial x} + L^*(t,x,u)\right\} + \frac{\sigma^2}{2}\frac{\partial^2 V_a}{\partial x^2}, \tag{17}$$

where the terminal condition is $V_a(T,x) = 0$. As in Section 4, the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, which is involved in f_a^* and L^*, is not restricted to satisfy (9)-(11) simultaneously. The first step is to identify suitable conditions, so that the HJB equation (17) gives a unique classical solution where the measure flow is only generally specified.

Let $Q_T = (0,T)\times\mathbb{R}$ and $\overline{Q}_T = [0,T]\times\mathbb{R}$. We denote by $C^{1,2}(\overline{Q}_T)$ (resp., $C^{1,2}(Q_T)$) the set of continuous functions $v(t,x)$ with continuous derivatives v_t, v_{xx} on $\overline{Q}_T$ (resp., Q_T). Let $C_b^{1,2}(\overline{Q}_T)$ be the set consisting of all bounded functions in $C^{1,2}(\overline{Q}_T)$.

THEOREM 5. *Suppose (H1)-(H4) hold, and the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$ is in the class $\mathcal{M}_{[0,T]}$ with Hölder exponent $\beta \in (0,1]$. Then equation (17) has a unique solution $V_a \in C_b^{1,2}(\overline{Q}_T)$.*

Proof. We give the proof by standard methods in optimal control of non-degenerate diffusion processes. First, we can follow exactly the argument for proving Theorem 6.2 in Appendix of [15] to show that there exists a solution (actually determined as a continuous function on $\overline{Q}_T$) $V_a \in C^{1,2}(Q_T)$ by only assuming that both f_a^* and L^* satisfy Hölder continuity in t rather than have a continuous derivative in t [15]. In constructing this particular solution V_a by the approximation procedure, we may obtain an *a priori* constant upper bound for V_a by use of the boundedness of f_a^* and L^*.

For any $R > 0$, let $B_R = (-R,R)$ and $\overline{B}_R = [-R,R]$. Subsequently, we can show by standard Hölder estimate [15] (pp. 207-208) that for a small positive constant $\delta \in (0,1)$, $V_a \in C^{1+\delta/2,2+\delta}((0,T)\times B_R)$ by restricting V_a to the domain $(0,T)\times B_R$. In other words, under the parabolic distance $d((t,x),(s,y)) = (|t-s| + |x-y|^2)^{1/2}$, the functions V_a, $(V_a)_t$ and $(V_a)_{xx}$ are all Hölder continuous on $(0,T)\times B_R$ with exponent δ. This further implies the first and second order derivatives of V_a appearing in the HJB equation may be extended to $[0,T]\times\overline{B}_R$ in an obvious way and $V_a \in C^{1+\delta/2,2+\delta}([0,T]\times\overline{B}_R)$. Since R is arbitrary, we have $V_a \in C^{1,2}(\overline{Q}_T)$.

The uniqueness follows from the standard verification theorem by interpreting V_a as the value function of an associated stochastic optimal control problem. This completes the proof. □

As in [15], by the verification theorem we may obtain uniqueness in the wider class $C_p^{1,2}(Q_T)\cap C(\overline{Q}_T)$ (i.e., when $v \in C(\overline{Q}_T)$ is restricted to Q_T, it is also in $C_p^{1,2}(Q_T)$) where $C_p^{1,2}(Q_T) \subset C^{1,2}(Q_T)$ consists of functions satisfying a polynomial growth in the spatial variable x.

Given a measure flow in $\mathcal{M}_{[0,T]}$, by Theorem 5 we obtain a smooth solution for V_a and subsequently, corresponding to each $(t,x) \in [0,T] \times \mathbb{R}$, under (H5) we get $u(t,x)$ as a well defined function minimizing the right hand side of (9). Hence we write the optimal control law in a feedback form

$$u = \varphi_a(t, x|\mu_\cdot^1, \cdots, \mu_\cdot^K), \tag{18}$$

where $(t,x) \in [0,T] \times \mathbb{R}$, $(\mu_t^1, \cdots, \mu_t^K) \in \mathcal{M}_{[0,T]}$, and a is the dynamic parameter for the associated agent.

We introduce an additional assumption for the control law (18).

(H6) For each $a \in \mathcal{A}$ and $(\mu_t^1, \cdots, \mu_t^K) \in \mathcal{M}_{[0,T]}$, the function $\varphi_a(t, x|\mu_\cdot^1, \cdots, \mu_\cdot^K)$ is continuous in $(t,x) \in [0,T] \times \mathbb{R}$, and Lipschitz continuous in $x \in \mathbb{R}$.

Note that to explicitly verify the Lipschitz continuity in x for the feedback control law, we usually need more concrete assumptions, such as affine linearity in u_i for the dynamics of the ith agent, combined with smoothness and convexity of the cost integrand with respect to the control. In the literature, Lipschitz continuity of the feedback has been a well studied topic; see, e.g., [15].

Denote $\mu_t^o = (\mu_t^1, \cdots, \mu_t^K) \in \mathcal{M}_{[0,T]}$ where $0 \le t \le T$. By use of (18) we may write a vector of feedback control laws with K distinct values of a. As in (12), for $a = \theta_i$ we will simply write φ_a as φ_i, and consequently we define the following map from $\mathcal{M}_{[0,T]}$ to the K-fold product set $[C(\overline{Q}_T)]^K$

$$\Gamma(\mu_\cdot^o) = (\varphi_1(t, x|\mu_\cdot^o), \cdots, \varphi_K(t, x|\mu_\cdot^o)) \tag{19}$$

where $\mu_\cdot^o = (\mu_\cdot^1, \cdots, \mu_\cdot^K)$ and we use the same argument x inside each component function. When these individual control laws are used by the corresponding agents, the variable x is substituted by its own state variable (see (12)).

The nonlinear map Γ gives an important characterization of the individual-mass interaction, and it turns out to have a close relation to the well known best response map in noncooperative game theory. In a static n-person noncooperative game [17], once the actions for the other $n-1$ players are assumed, the best response map of the given player will determine its optimal choice of one or more actions which is optimal conditioned on other agents' actions assumed in the first place. In our large population multi-class agent game model, we may view the K-tuple $\mu_\cdot^o = (\mu_\cdot^1, \cdots, \mu_\cdot^K)$ as the effect of a virtual player, and then each φ_i may be regarded as the local optimal strategy in response to that given $\mu_\cdot^o$. For this reason, we just extend the conventional name by calling Γ the best response map.

6. The M-V Equation with Decentralized Lipschitz Feedback . We recall that in (11) the feedback control has a functional dependence on the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$. That causes the M-V equation (11) to be coupled with the

HJB equation (9). In this section and in parallel to the treatment in Section 5, we proceed by assuming a known feedback control law $\phi_a(t,x)$ (which may be written as $\phi_k(t,x)$ when $a=\theta_k$) for the associated agents with dynamic parameter a, which leads to the following auxiliary equation

$$\text{(20)}\qquad \begin{cases} dx_1 &= f_1[x_1,\phi_1(t,x_1),\mu_t^1,\cdots,\mu_t^K]dt+\sigma dw_1,\\ &\vdots\\ dx_K &= f_K[x_K,\phi_K(t,x_K),\mu_t^1,\cdots,\mu_t^K]dt+\sigma dw_K,\end{cases}$$

where the function $\phi_k(t,x)$ is the class $C_{Lip(x)}(\overline{Q}_T,U)$ consisting of continuous functions from $\overline{Q}_T=[0,T]\times\mathbb{R}$ to U, which are Lipschitz continuous in x. The initial conditions and the Brownian motions in (20) are specified in the same form as in (12). The key question is whether there exists a well defined vector process $(x_1,\cdots,x_K)$ satisfying equation (20) in the McKean-Vlasov sense.

Before establishing existence results for (20), we introduce some preliminary material about the metric on a space of probability measures. For related treatment, the reader is referred to [44] (pp.172-174). Let $C([0,T],\mathbb{R}^K)$ be the space of continuous functions on $[0,T]$ and we write it in the abbreviated form $C^{(K)}$. For $x,y\in C([0,T],\mathbb{R}^K)$, define the norm $\|x-y\|=\sup_{t\in[0,T]}|x(t)-y(t)|$. Then $(C^{(K)},\|\cdot\|)$ constitutes a Banach space. Also, we introduce the metric $\rho(x,y)=\sup_{t\in[0,T]}|x(t)-y(t)|\wedge 1$, and it is easy to verify that $(C^{(K)},\rho)$ forms a complete metric space, i.e., each Cauchy sequence has a limit in $(C^{(K)},\rho)$. It is well known in functional analysis that $(C^{(K)},\|\cdot\|)$ is a separable space. On the other hand, the convergence of a sequence in $(C^{(K)},\|\cdot\|)$ is equivalent to its convergence in the metric ρ. Hence $(C^{(K)},\rho)$ is a separable and complete metric space.

On $C([0,T],\mathbb{R}^K)$, we specify the σ-algebra $\mathcal{F}^{(K)}$ induced by all cylindrical sets of the form $\{x(\cdot)\in C^{(K)}: x_{t_i}\in B_i,\ t_i\in[0,T], i=1,2,\cdots,l\}$ where each B_i is a Borel set in $\mathbb{R}^K$ and l is any positive integer. Let $\mathcal{M}(C^{(K)})$ be the space of all probability measures m on $(C^{(K)},\mathcal{F}^{(K)})$. Thus each $(C^{(K)},\mathcal{F}^{(K)},m)$ is a probability space. For the product space $C^{(K)}\times C^{(K)}$, the space of probability measures is defined in an obvious manner, and denoted as $\mathcal{M}(C^{(K)}\times C^{(K)})$. We introduce the canonical process X to be a random process with the sample space $C^{(K)}$, i.e., $X_t(\omega)=\omega_t$ for $\omega\in C^{(K)}$.

By the same procedure but replacing the dimension number K by one, we may define the space of probability measures on $C([0,T],\mathbb{R})$, and we denote it by $\mathcal{M}(C^{(1)})$.

Based upon the metric ρ in $(C^{(K)},\rho)$, we now introduce the so-called Vasershtein metric on $\mathcal{M}(C^{(K)})$. For $m_1,m_2\in\mathcal{M}(C^{(K)})$, define

$$\text{(21)}\qquad D_T(m_1,m_2)=\inf_m\int_{C^{(K)}\times C^{(k)}}\Big(\sup_{s\le T}|X_s(\omega_1)-X_s(\omega_2)|\wedge 1\Big)dm(\omega_1,\omega_2),$$

where $m\in\mathcal{M}(C^{(K)}\times C^{(K)})$, $p_1\circ m=m_1$ and $p_2\circ m=m_2$ denoting the two marginal distributions of m, and m is called a coupling of m_1 and m_2. This gives a complete metric on $\mathcal{M}(C^{(K)})$, which implies each fundamental sequence $\{m_i, i \geq 1\}$ under the metric D_T has a limit in $\mathcal{M}(C^{(K)})$ [44]. We note that for both $C^{(K)}$ and $\mathcal{M}(C^{(K)})$, we may define the corresponding space on a smaller interval $[0,t] \subset [0,T]$. Similarly, $D_t(m_1, m_2)$ may be defined for $t < T$ by considering the restriction of $m_i \in \mathcal{M}(C^{(K)})$ to $C([0,t], \mathbb{R}^K)$, which is embedded in a natural way as a subspace of $C([0,T], \mathbb{R}^K)$.

Remark. If $(\cdot \wedge 1)$ in (21) is replaced by $(\cdot \wedge d)$ for $d > 0$, we may denote the associated metric by $D_T^d(m_1, m_2)$, and all analysis in this section still holds. □

Let $m \in \mathcal{M}(C^{(K)})$ and define the random process $y = (y_1, \cdots, y_K)$ on $[0,T]$ as follows.

$$(22)\quad y_i(t) = x_i(0) + \sigma w_i(t) + \int_0^t \int_{C^{(K)}} \sum_{j=1}^{K} \pi_j f_i(y_i(s), \phi_i(s, y_i(s)), \omega_s^j) dm(\omega) ds,$$

where $0 \leq t \leq T$ and the canonical process is denoted as $\omega_\cdot = (\omega_\cdot^1, \cdots, \omega_\cdot^K)$. We denote the law of y on $[0,T]$ by $\Phi(m)$ which clearly belongs to $\mathcal{M}(C^{(K)})$. The existence proof below is based upon a fixed point method to show that there exists a unique m such that $\Phi(m) = m$, and we finish the proof by determining m as a product form.

THEOREM 6. *Under (H1)-(H6), there is a unique consistent solution pair $(x_1, \cdots, x_K)$ and $(\mu^1, \cdots, \mu^K)$ to (20).*

Proof. We break the proof into two steps where the first step is similar to the proof of Theorem 1.1 in [44] which deals with a conventional M-V equation for a single class of particles. For the reader's convenience, we give the details of the fixed point analysis.

Step 1. Take $m, \hat{m} \in \mathcal{M}(C^{(K)})$, and let y be defined by (22). Similarly, $\hat{y}$ is defined by (22) after replacing m by $\hat{m}$. Both y_i and $\hat{y}_i$ have the same initial condition $x_i(0)$, $1 \leq i \leq K$. Denote

$$\zeta(s, y_i(s), \omega_s) = \sum_{j=1}^{K} \pi_j f_i(y_i(s), \phi_i(s, y_i(s)), \omega_s^j),$$
$$\hat{\zeta}(s, \hat{y}_i(s), \omega_s) = \sum_{j=1}^{K} \pi_j f_i(\hat{y}_i(s), \phi_i(s, \hat{y}_i(s)), \omega_s^j),$$

where $0 \leq s \leq T$. It follows that

(23)
$$\sup_{0 \leq s \leq t} |y_i(s) - \hat{y}_i(s)| \leq \int_0^t \Big| \int_{C^{(K)}} \zeta(s, y_i(s), \omega_s) dm(\omega) - \int_{C^{(K)}} \hat{\zeta}(s, \hat{y}_i(s), \omega_s) d\hat{m}(\omega) \Big| ds.$$

For any $\overline{m} \in \mathcal{M}(C^{(K)} \times C^{(K)})$ such that $p_1 \circ \overline{m} = m$ and $p_2 \circ \overline{m} = \hat{m}$, we have

$$\begin{aligned}\xi_s &\triangleq \Big| \int_{C^{(K)}} \zeta(s, y_i(s), \omega_s) dm(\omega) - \int_{C^{(K)}} \hat{\zeta}(s, \hat{y}_i(s), \omega_s) d\hat{m}(\omega) \Big| \\ &= \Big| \int_{C^{(K)} \times C^{(K)}} \zeta(s, y_i(s), \omega_s) d\overline{m}(\omega, \hat{\omega}) - \int_{C^{(K)} \times C^{(K)}} \hat{\zeta}(s, \hat{y}_i(s), \hat{\omega}_s) d\overline{m}(\omega, \hat{\omega}) \Big| \\ &\le C_1(|y_i(s) - \hat{y}_i(s)| \wedge 1) + \sum_j \int_{C^{(K)} \times C^{(K)}} C_2(|\omega_s^j - \hat{\omega}_s^j| \wedge 1) d\overline{m}(\omega, \hat{\omega}), \end{aligned} \tag{24}$$

where C_1 and C_2 are two constants obtained from the boundedness and Lipschitz continuity of both f_i and ϕ_i. Clearly, for all trajectories of the canonical process, we have

$$\sum_j (|\omega_s^j - \hat{\omega}_s^j| \wedge 1) \le K(|\omega_s - \hat{\omega}_s| \wedge 1). \tag{25}$$

Hence it follows from (24) and (25) that

$$\xi_s \le C_1(|y_i(s) - \hat{y}_i(s)| \wedge 1) + \int_{C^{(K)} \times C^{(K)}} C_2(|\omega_s - \hat{\omega}_s| \wedge 1) d\overline{m}(\omega, \hat{\omega}),$$

which implies

$$\xi_s \le C_1(|y_i(s) - \hat{y}_i(s)| \wedge 1) + C_2 D_s(m, \hat{m}) \tag{26}$$

since $\overline{m}$ is any coupling of m and $\hat{m}$. Consequently, it follows from (23) and (26) that

$$\sup_{0 \le s \le t} |y_i(s) - \hat{y}_i(s)| \le \int_0^t \Big[C_1(|y_i(s) - \hat{y}_i(s)| \wedge 1) + C_2 D_s(m, \hat{m}) \Big] ds. \tag{27}$$

Then (27) combined with Gronwall's lemma gives

$$\sup_{0 \le s \le t} |y_i(s) - \hat{y}_i(s)| \wedge 1 \le C_T \int_0^t D_s(m, \hat{m}) ds \tag{28}$$

where the constant C_T depends upon the terminal time T. Subsequently,

$$\sup_{0 \le s \le t} |y(s) - \hat{y}(s)| \wedge 1 \le K C_T \int_0^t D_s(m, \hat{m}) ds. \tag{29}$$

We see that y and $\hat{y}$ induce two probability distributions, denoted as $\Phi(m)$ and $\Phi(\hat{m})$, respectively, on $C^{(K)}$, and furthermore, the joint distribution of $(y, \hat{y})$ gives a measure $\overline{m}_\Phi$ on $C^{(K)} \times C^{(K)}$. By taking expectation on the left hand side of (29) and expressing the calculation in terms of the canonical process and $\overline{m}_\Phi \in \mathcal{M}(C^{(K)} \times C^{(K)})$, we get an upper bound for $D_t(\Phi(m), \Phi(\hat{m}))$. This leads to the inequality

$$D_t(\Phi(m_1), \Phi(m_2)) \le K C_T \int_0^t D_s(m_1, m_2) ds. \tag{30}$$

Now, following a similar argument as in [44], we can show that $\{\Phi^k(m), k \ge 1\}$ forms a fundamental sequence, and there exists a unique $m \in \mathcal{M}(C^{(K)})$ satisfying $\Phi(m) = m$. This completes the first step.

Step 2. Next we examine the structure of the fixed point probability measure m arising in the existence proof for a solution of (20). Let $x = (x_1, \cdots, x_K)$ denote the solution of (22) on $[0, T]$ when $m = \Phi(m)$. Since all x_i have independent initial conditions and Brownian motions, they form n independent processes. We denote product $B_i = B_i^1 \times \cdots \times B_i^K$, where $i = 1, 2, \cdots$, and each B_i^k is a Borel set in $\mathbb{R}$. Since m is the law of the associated solution x on $[0, T]$ which has K independent components, we have

$$
\begin{aligned}
m(\omega_{t_i} \in B_i, i = 1, 2, \cdots, l) &= P(x_1(t_i) \in B_i^1, \cdots, x_K(t_i) \in B_i^K, i = 1, 2 \cdots, l) \\
&= \prod_{k=1}^{K} P(x_k(t_i) \in B_i^k, i = 1, 2, \cdots, l) \\
&= \prod_{k=1}^{K} m^i(\omega_{t_i}^k \in B_i^k, i = 1, 2, \cdots, l) \qquad (31)
\end{aligned}
$$

where m^k is the law of x_k, $1 \le k \le K$. We get the second equality by independence in terms of the underlying probability space $(\Omega, \mathcal{F}, P)$. Since (31) holds for all product Borel sets in $\mathbb{R}^K$, we see that $m = m^1 \times \cdots \times m^K$ on $\mathcal{F}^{(K)}$.

Now for $m = m^1 \times \cdots \times m^K$, equation (22) for x_i reduces to

$$
\begin{aligned}
x_i(t) &= x_i(0) + \sigma w_i(t) + \int_0^t \int_{C^{(K)}} \sum_j \pi_j f_i(x_i(s), \phi_i(s, x_i(s)), \omega_s^j) dm(\omega) ds \\
&= x_i(0) + \sigma w_i(t) + \int_0^t \sum_j \pi_j \int_{C([0,T],\mathbb{R})} f_i(x_i(s), \phi_i(s, x_i(s)), \omega_s^j) dm^j(\omega^j) ds \\
&= x_i(0) + \sigma w_i(t) + \int_0^t \int_{\mathbb{R}} \sum_j \pi_j f_i(x_i(s), \phi_i(s, x_i(s)), y) \mu_s^j(dy) ds
\end{aligned}
$$

where μ_s^j is the marginal distribution of m^j at time s. We may further express the right hand side of (22) in terms of the product measure $\mu_t^1 \times \cdots \times \mu_t^K$ which is the marginal distribution at time t for the law of x, reducing (22) to the form (20); uniqueness of the consistent pair follows from Step 1. This completes the proof. □

For a set of functions $(\phi_1, \cdots, \phi_K)$ such that $\phi_k \in C_{Lip(x)}(\overline{Q}_T, U)$, for $1 \le k \le K$, we implement it as the set of control laws in (20), which leads to a well defined solution $x = (x_1, \cdots, x_K)$ on $[0, T]$ by Theorem 6. Let the law of the resulting solution x be denoted by m, and define the map from $(C_{Lip(x)}(\overline{Q}_T, U))^K$ to $\mathcal{M}(C^{(K)})$:

$$
m = \widehat{\Gamma}(\phi_1, \cdots, \phi_K). \qquad (32)
$$

By taking marginal distributions for the solution process or equivalently for m in (32), we get a measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$. We introduce the map:

$$(\mu_\cdot^1, \cdots, \mu_\cdot^K) = \overline{\Gamma}(\phi_1, \cdots, \phi_K). \tag{33}$$

Each $\mu_\cdot^k$ on the left hand side corresponds to all marginal distributions on $[0,T]$ of the component x_t^k, $0 \le t \le T$. It is easy to see that there is a well defined map relating $(\mu_\cdot^1, \cdots, \mu_\cdot^K)$ to m:

$$(\mu_\cdot^1, \cdots, \mu_\cdot^K) = p_\mu(m), \tag{34}$$

where $m \in \widehat{\Gamma}([C_{Lip(x)}(\overline{Q}_T, U)]^K)$. We will call p_μ the projection map.

LEMMA 7. *Under the assumptions in Theorem 6, for any $(\phi_1, \cdots, \phi_K)$ such that $\phi_k \in C_{Lip(x)}(\overline{Q}_T, U)$, for $1 \le k \le K$, the measure flow $(\mu_t^1, \cdots, \mu_t^K)$, $0 \le t \le T$, obtained from (33) is in the class $\mathcal{M}_{[0,T]}$. Hence $\overline{\Gamma}$ is a map from $(C_{Lip(x)}(\overline{Q}_T, U))^K$ to $\mathcal{M}_{[0,T]}$.*

Proof. By Theorem 6, there exists a unique solution $x_t = (x_t^1, \cdots, x_t^K)$, $0 \le t \le T$, to (20) when the set of control laws $(\phi_1, \cdots, \phi_K)$ is used. Take any $0 \le t' < t'' \le T$. For any bounded and Lipschitz continuous function $\psi(y)$ with a Lipschitz constant $\mathrm{Lip}(\psi)$, we have

$$\begin{aligned} \Big| \int \psi(y)\mu_{t'}^j(dy) - \int \psi(y)\mu_{t''}^j(dy) \Big| &= |E\psi(x_{t'}^j) - E\psi(x_{t''}^j)| \\ &\le \mathrm{Lip}(\psi) E|x_{t'}^j - x_{t''}^j|. \end{aligned}$$

On the other hand, we have

$$x_{t''}^j = x_{t'}^j + \int_{t'}^{t''} f_j[x_s^j, \phi_j(s, x_s^j), \mu_t^1, \cdots, \mu_t^K]ds + \sigma(w_j(t'') - w_j(t')),$$

and it follows that

$$E|x^j(t'') - x^j(t')|^2 \le 2C_1^2|t'' - t'|^2 + 2\sigma^2|t'' - t'|,$$

where C_1 is an upper bound for $f_j(x, u, y)$. Hence

$$\begin{aligned} \Big| \int \psi(y)\mu_{t'}^j(dy) - \int \psi(y)\mu_{t''}^j(dy) \Big| &\le \sqrt{2}\mathrm{Lip}(\psi)(C_1|t' - t''| + |\sigma||t' - t''|^{1/2}) \\ &\le \sqrt{2}\mathrm{Lip}(\psi)(C_1\sqrt{T} + |\sigma|)|t' - t''|^{1/2}, \end{aligned}$$

for all $t', t'' \in [0, T]$. By the arbitrariness of ψ and j, the lemma follows. □

The implication of this lemma is that, after all agents apply Lipschitz control laws determined by their own parameter type, the resulting measure flow maintains a certain continuity. Consequently, one can obtain a well defined new strategy by solving an HJB equation involving that measure flow. Such a procedure makes it feasible to develop strategy or policy iteration by use of the population limit. In other words, one may repeatedly apply the two operators Γ and $\overline{\Gamma}$ alternatively.

7. Feedback Regularity and Strategy Revision. We have the following proposition about the composite map formed by $\overline{\Gamma}$ and Γ.

PROPOSITION 8. *Assume (H1)-(H6) and let* $\mu_t^o = (\mu_t^1, \cdots, \mu_t^K) \in M_{[0,T]}$. *We have* $\overline{\Gamma} \circ \Gamma(\mu_\cdot^o) \in M_{[0,T]}$, *i.e.,* $\Gamma_{\mathcal{M}} \triangleq \overline{\Gamma} \circ \Gamma$ *is a map from* $\mathcal{M}_{[0,T]}$ *to* $\mathcal{M}_{[0,T]}$.

Proof. Given $\mu_t^o \in \mathcal{M}_{[0,T]}$, we obtain a set of well defined feedback control laws $(\phi_1, \cdots, \phi_K)$ by the results in Section 5. Subsequently, the Lipschitz feedback assumption combined with Lemma 7 implies this proposition. □

Now it is clear that we obtain a solution to the equation system (9)-(11) for the NCE methodology if we can find $\mu_t^o \in \mathcal{M}_{[0,T]}$ to satisfy the fixed point equation

$$\overline{\Gamma} \circ \Gamma(\mu_\cdot^o) = \mu_\cdot^o. \tag{35}$$

Here we need to restrict the solution to the set $\mathcal{M}_{[0,T]}$ so that the machinery of the decoupled HJB equation and McKean-Vlasov equation approach may be employed. This is not an essential restriction since once there indeed exists a solution μ_t^o, we can derive from (11) that $\mu_t^o \in \mathcal{M}_{[0,T]}$ only by use of the boundedness of $f_i(x, u, y)$, $1 \le i \le K$.

However, a drawback of directly analyzing (35) is that we only know μ_t^o is a "flow" of measures with limited structural information. This makes it difficult to develop fixed point analysis. For this reason, we will use an embedding strategy by finding a certain measure $m \in \mathcal{M}(C^{(K)})$, which is associated with $\mu_\cdot^o$ via the projection map (34).

Under (H1)-(H6), for any $\mu_\cdot^o \in \mathcal{M}_{[0,T]}$, we obtain a set of control laws $\Gamma(\mu_\cdot^o) = (\phi_{\theta_1}, \cdots, \phi_{\theta_K})$ by solving the HJB equation (17). Subsequently we get $m = \widehat{\Gamma} \circ \Gamma(\mu_\cdot^o) \in \mathcal{M}(C^{(K)})$, which further induces $p_\mu \circ \widehat{\Gamma} \circ \Gamma(\mu_\cdot^o) \in \mathcal{M}_{[0,T]}$. Now we introduce an auxiliary fixed point equation

$$\widehat{\Gamma} \circ \Gamma \circ p_\mu(m) = m, \tag{36}$$

where m belongs to the image $\widehat{\Gamma} \circ \Gamma(\mathcal{M}_{[0,T]}) \subset \mathcal{M}(C^{(K)})$ of the map $\widehat{\Gamma} \circ \Gamma$ on $\mathcal{M}_{[0,T]}$.

For analyzing equation (36), we introduce the regularity condition as follows. Take two measures $m, \tilde{m} \in \widehat{\Gamma} \circ \Gamma(\mathcal{M}_{[0,T]})$. Denote the two measure flows $\mu_\cdot^o = p_\mu(m)$ and $\tilde{\mu}_\cdot^o = p_\mu(\tilde{m})$ on $[0,T]$. We write the associated drift vector at time t as $f_a[x_t, u_t, \mu_t^o]$ and $f_a[x_t, u_t, \tilde{\mu}_t^o]$. The cost integrands are given as $L[x_t, u_t, \mu_t^o]$ and $L[x_t, u_t, \tilde{\mu}_t^o]$. Accordingly, the associated HJB equations (17) are solved to give two sets of feedback control laws $\phi_a(t,x)$ and $\tilde{\phi}_a(t,x)$, $a \in \mathcal{A}$, with each individual control law being in $C_{Lip(x)}(\overline{Q}_T, U)$ under (H6). For instance, we have $(\phi_{\theta_1}, \cdots, \phi_{\theta_K}) = \Gamma \circ p_\mu(m)$. The feedback regularity (FR) condition is given as: There exists a constant $c_1 > 0$ such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |\phi_a(t,x) - \tilde{\phi}_a(t,x)| \le c_1 D_T(m, \tilde{m}), \qquad \forall\ a \in \mathcal{A}. \tag{37}$$

The FR condition (37) characterizes a sensitivity property for the control law when a perturbation of the pre-image of the measure flow, associated with the map p_μ, is involved. This, in turn, is related to certain continuous dependence of the solution V_a (as well as its derivatives) to the HJB equation (17) on the functions f_a^* and L^* appearing there.

LEMMA 9. *Under (H1)-(H6), there exists a constant c_2 such that*

$$D_T(m,\tilde{m}) \le c_2 \sup_{a\in\mathcal{A},(t,x)\in\overline{Q}_T} |\phi_a(t,x) - \tilde{\phi}_a(t,x)| \tag{38}$$

where $\overline{Q}_T = [0,T]\times\mathbb{R}$, and $m,\tilde{m}\in\mathcal{M}(C^{(K)})$ are induced by (32) using the two sets of control laws ϕ_a and $\tilde{\phi}_a$, $a\in\mathcal{A}$, respectively.

Proof. Recall that for $a_j=\theta_k$, we write f_{a_j} (resp., ϕ_{a_j}) as f_k (resp., ϕ_k). For the two solutions, denoted by x_t and $\tilde{x}_t$, we write their equations by components,

$$x_i(t) = x_i(0) + \sigma w_i(t) + \int_0^t \int_{C^K} \sum_{j=1}^K \pi_j f_i(x_i(s),\phi_i(s,x_i(s)),\omega_s^j)dm(\omega)ds,$$

$$\tilde{x}_i(t) = x_i(0) + \sigma w_i(t) + \int_0^t \int_{C^K} \sum_{j=1}^K \pi_j f_i(\tilde{x}_i(s),\tilde{\phi}_i(s,\tilde{x}_i(s)),\omega_s^j)d\tilde{m}(\omega)ds,$$

where the initial condition is $x_i(0)$, $1\le i\le K$, and the Brownian motion is w_i. By use of the Lipschitz continuity of both f_i and the feedback control laws, we get

$$\begin{aligned}
&|f_i(x_i(s),\phi_i(s,x_i(s)),\omega_s^j) - f_i(\tilde{x}_i(s),\tilde{\phi}_i(s,\tilde{x}_i(s)),\tilde{\omega}_s^j)| \\
&\le |f_i(x_i(s),\phi_i(s,x_i(s)),\omega_s^j) - f_i(\tilde{x}_i(s),\phi_i(s,\tilde{x}_i(s)),\tilde{\omega}_s^j)| \\
&\quad + |f_i(\tilde{x}_i(s),\phi_i(s,\tilde{x}_i(s)),\tilde{\omega}_s^j) - f_i(\tilde{x}_i(s),\tilde{\phi}_i(s,\tilde{x}_i(s)),\tilde{\omega}_s^j)| \\
&\le C_1(|x_i(s)-\tilde{x}_i(s)|\wedge 1) + C_2 \sup_{(s,x)\in\overline{Q}_T} |\phi_i(s,x)-\tilde{\phi}_i(s,x)| + C_3(|\omega_s^j - \tilde{\omega}_s^j|\wedge 1). \qquad (39)
\end{aligned}$$

Now, similar to the derivation of (27), we use (39) to obtain

$$\begin{aligned}
|x(t)-\tilde{x}(t)| \le{}& C_1\int_0^t (|x(s)-\tilde{x}(s)|\wedge 1)ds + C_2 t \sup_{a\in\mathcal{A},(s,x)\in\overline{Q}_T} |\phi_a - \tilde{\phi}_a| \\
&+ C_3\int_0^t D_s(m,\tilde{m})ds
\end{aligned}$$

which together with Gronwall's lemma gives

$$\sup_{0\le s\le t} |x(s)-\tilde{x}(s)|\wedge 1 \le C_2 t \sup_{a\in\mathcal{A},(t,x)\in\overline{Q}_T} |\phi_a-\tilde{\phi}_a| + C_3\int_0^t D_s(m,\tilde{m})ds.$$

Subsequently,

$$D_t(m,\tilde{m}) \le C_2 t \sup_{a\in\mathcal{A},(t,x)\in\overline{Q}_T} |\phi_a-\tilde{\phi}_a| + C_3\int_0^t D_s(m,\tilde{m})ds.$$

By using Gronwall's lemma again, we complete the proof. ◻

THEOREM 10. *Assume (H1)-(H6) hold. If the constants c_1 for (37) and c_2 for (38) can be selected to satisfy the composite gain (CG) condition $c_1c_2 < 1$, then there exists a unique solution for (35) and hence a unique solution for the NCE equation system (9)-(11).*

Proof. It follows easily from a fixed point argument that there is a unique $m \in \mathcal{M}(C^{(K)})$ satisfying equation (36). Consequently, we can construct $\mu^o_\cdot = p_\mu(m)$ to satisfy (35). Assume we have two measure flows $\mu^o_\cdot, \tilde{\mu}^o_\cdot \in \mathcal{M}_{[0,T]}$ satisfying (35), and then by the construction of the mappings $\Gamma, \overline{\Gamma}, \widehat{\Gamma}$, it is easy to derive the associated fixed points m and $\tilde{m}$ for (36) such that $\mu^o_\cdot = p_\mu(m)$ and $\tilde{\mu}^o_\cdot = p_\mu(\tilde{m})$. Since we necessarily have $m = \tilde{m}$, it follows that $\mu^o_\cdot = \tilde{\mu}^o_\cdot$. This completes the proof. ◻

7.1. Discussion on Feedback Regularity. For illustrating the inequality in the FR condition (37), we examine a highly simplified situation by considering a linear quadratic model with nonlinear coupling, which is a variant of (2). In addition, there is no bound constraints on the control. We note that for general nonlinear models, it is much more difficult to obtain the corresponding estimates explicitly.

We consider a system of uniform agents, i.e., $\mathcal{A}$ degenerates to a singleton. Let the dynamics be given by $dz_i = az_i dt + bu_i dt + (\alpha/n)\sum_{j=1}^n \phi(z_j)dt + \sigma dw_i$, and the cost is $J_i = E\int_0^T [(z_i - (\beta/n)\sum_{j=1}^n \psi(z_j))^2 + ru_i^2]dt$, $1 \le i \le n$. First, assume two measure flows $\mu_t \triangleq \mu_t^{(1)}$ and $\tilde{\mu}_t \triangleq \mu_t^{(2)}$ on $[0,T]$, which are the marginal distributions on $\mathbb{R}$, respectively, of two probability measures m_1 and m_2, both in $\mathcal{M}(C^{(1)})$. We determine two sets of functions $f^{(k)}(t) = \int_{\mathbb{R}} \phi(y)\mu_t^{(k)}(dy)$, $z^{*(k)}(t) = \int_{\mathbb{R}} \psi(y)\mu_t^{(k)}(dy)$, where $k = 1, 2$, $t \in [0,T]$ and ϕ, ψ are bounded and Lipschitz continuous functions on $\mathbb{R}$. Accordingly, we obtain two sets of control laws

$$u_t^{(k)} = \varphi^{(k)}(t, z_i) = -\frac{b}{r}(\Pi_t z_i + s_t^{(k)}), \qquad k = 1, 2,$$

which minimizes $J_i = E\int_0^T [(z_i - \beta z^{*(k)})^2 + ru_i^2]dt$ subject to $dz_i = az_i dt + bu_i dt + \alpha f^{(k)}dt + \sigma dw_i$. In the above, we have

$$\frac{d\Pi_t}{dt} + 2a\Pi_t - \frac{b^2}{r}\Pi_t^2 + 1 = 0, \tag{40}$$

$$\frac{ds_t^{(k)}}{dt} + (a - \frac{b^2}{r}\Pi_t)s_t^{(k)} + \alpha\Pi_t f^{(k)} - \beta z^{*(k)} = 0, \tag{41}$$

where $0 \le t \le T$, and $\Pi_T = 0$, $s_T^{(k)} = 0$. Let $C_\Pi = \sup_{0\le t\le T}\Pi_t$. For a proof of the optimality of the control law, see e.g. [9]. By combining Gronwall's lemma with (41), we see that there exist constants c_1 and c_2 depending upon C_Π such that

$$\sup_{0\le t\le T}|s_t^{(2)} - s_t^{(1)}| \le c_1|\alpha| \sup_{0\le\tau\le T}|f^{(2)}(\tau) - f^{(1)}(\tau)| + c_2|\beta| \sup_{0\le\tau\le T}|z^{*(2)}(\tau) - z^{*(1)}(\tau)|.$$

Letting g stand for ϕ and ψ, we have

$$\int g(x)\mu_t(dx) - \int g(x)\tilde{\mu}_t(dx) = \int g(X_t(\omega_1))dm_1(\omega_1) - \int g(X_t(\omega_2))dm_2(\omega_2)$$
$$= \int g(X_t(\omega_1))dm(\omega_1,\omega_2) - \int g(X_t(\omega_2))dm(\omega_1,\omega_2)$$

where X is the canonical process and m is a coupling of m_1 and m_2. Hence

$$\Big|\int g(x)\mu_t(dx) - \int g(x)\tilde{\mu}_t(dx)\Big|$$
$$\le c_3 \int_{C([0,T],\mathbb{R}^2)} \sup_{0\le t\le T} (|X_t(\omega_1) - X_t(\omega_2)| \wedge 1) dm(\omega_1,\omega_2)$$

for a constant c_3 depending upon ϕ and ψ. Noticing that $|\varphi^{(2)}(t,z) - \varphi^{(1)}(t,z)| = \frac{|b|}{r}|s_t^{(2)} - s_t^{(1)}|$, we get

$$|\varphi^{(2)}(t,z) - \varphi^{(1)}(t,z)| \le \frac{|b|}{r} c_3(c_1|\alpha| + c_2|\beta|) D_T(m_1,m_2).$$

It is seen that the coefficient $\frac{|b|}{r}c_3(c_1|\alpha| + c_2|\beta|)$ can be made sufficiently small by choosing suitably small α and β.

8. Asymptotic Equilibrium Analysis. Within the context of a population of n agents with dynamics (1), for any $1 \le k \le n$, the kth agent's admissible control set $\mathcal{U}_k$ consists of all Lipschitz feedback controls u_k adapted to the σ-algebra $\mathcal{F}(z_i(\tau), \tau \le t, 1 \le i \le n)$ (i.e., $u_k(t)$ is a continuous function of $(t, z_1(t), \cdots, z_n(t))$ and Lipschitz continuous in $(z_1(t), \cdots, z_n(t))$), which ensures a unique strong solution to the closed-loop system of the n agents exists on $[0,\infty)$. In parallel, we define $\mathcal{U}_k^d$, as a subset of $\mathcal{U}_k$, such that the Lipschitz feedback control $u_k(t)$ depends upon $(t, z_k(t))$. We denote $\mathcal{U}^n = \mathcal{U}_1 \times \cdots \times \mathcal{U}_n$ and $\mathcal{U}^{d,n} = \mathcal{U}_1^d \times \cdots \times \mathcal{U}_n^d$.

Note that $\mathcal{U}_k$ is not restricted to be decentralized, and this will give a stronger characterization of the ε-Nash strategies introduced below. For each i, we denote by u_{-i} the vector obtained by deleting u_i in $(u_1, \cdots, u_i, \cdots, u_n)$.

For the dynamic game problem specified by (1)-(2), we rewrite the individual cost (2) for the ith agent in the form

$$J_i(u_i, u_{-i}) = E\int_0^T \Big[(1/n)\sum_{j=1}^n L(z_i, u_i, z_j)\Big]dt, \qquad 1 \le i \le n, \tag{42}$$

which indicates the effect of the control laws of other agents due to the coupling in dynamics and individual costs.

DEFINITION 11. *For the n players, let the costs be given by $J_k, 1 \le k \le n$. A set of controls $u_k \in \mathcal{U}_k$ (resp., $u_k \in \mathcal{U}_k^d$) each given as a Lipschitz feedback φ_k, $1 \le k \le n$, is called an ε-Nash equilibrium with respect to the strategy space $\mathcal{U}^n$ (resp., $\mathcal{U}^{d,n}$), if* *there exists* $\varepsilon \geq 0$ *such that for any fixed* $1 \leq i \leq n$*, we have*

$$J_i(u_i, u_{-i}) \leq J_i(u_i', u_{-i}) + \varepsilon,$$

when any alternative $u_i' \in \mathcal{U}_i$ *(resp.,* $u_i' \in \mathcal{U}_i^d$*,), determined as another Lipschitz feedback* $\tilde{\varphi}_i$*, is applied by the* i*th player.* □

In relation to (u_i, u_{-i}) in Definition 11, we may also call u_i an ε-Nash strategy. This is with an underlying assumption that other players use strategies u_{-i}.

8.1. Decoupling Rate and Crossing Perturbation. Let $\phi_{\theta_k}(t,x) \in C_{Lip(x)}(\overline{Q}_T, U)$, $1 \leq k \leq K$, be a set of K functions associated with different values θ_k, not necessarily satisfying the NCE principle. These functions are used by the n agents for their individual control laws. We write the closed-loop equation as follows:

$$dz_i = (1/n)\sum_{j=1}^{n} f_{a_i}(z_i, \phi_{a_i}(t, z_i), z_j)dt + \sigma dw_i, \qquad 1 \leq i \leq n, \tag{43}$$

where ϕ_{a_i} reduces to ϕ_{θ_k} if $a_i = \theta_k$. By the Lipschitz condition of the feedback, we see that there exists a unique strong solution $(z_1, \cdots, z_n)$ on $[0, T]$. For the agent with dynamic parameter a_i, we write the associated M-V equation system as follows

$$d\hat{z}_i = f_{a_i}[\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), \mu_t^1, \cdots, \mu_t^K]dt + \sigma dw_i, \qquad 1 \leq i \leq n, \tag{44}$$

where the initial condition is $\hat{z}_i(0) = z_i(0)$. In contrast to the coupled processes in (43), equation (44) gives n independent processes. Denote

$$\varepsilon_n = \sup_{1\leq k\leq K} |F_n(\{\theta_k\}) - \pi_k| \tag{45}$$

which measures the gap between the empirical distribution of the dynamic parameter and its limit. Under (H0), we have $\lim_{n\to\infty} \varepsilon_n = 0$.

We have the following decoupling result which shows that each process z_i may be approximated by the corresponding process $\hat{z}_i$ as $n \to \infty$.

THEOREM 12. *Assume (H0)-(H3) and let* z_i *and* $\hat{z}_i$*,* $1 \leq i \leq n$*, be given by (43)-(44). We have*

$$\sup_{1\leq i\leq n} E \sup_{0\leq t\leq T} |z_i(t) - \hat{z}_i(t)|^\kappa = O(n^{-1/2} + \varepsilon_n),$$

where $\kappa = 1, 2$*,* ε_n *is given by (45), and the right hand side may depend upon the terminal time* T*.*

Proof. We begin by considering the case $\kappa = 1$. Denote the relation $j \in S(\theta_k)$ if the dynamic parameter associated with z_j is $a_j = \theta_k$. It follows that

$$\begin{aligned} z_i(s) - \hat{z}_i(s) &= \int_0^s (1/n) \sum_{j=1}^n f_{a_i}(z_i, \phi_{a_i}(t, z_i), z_j) dt \\ &\quad - \int_0^s f_{a_i}[\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), \mu_t^1, \cdots, \mu_t^K] dt \\ &= \int_0^s \sum_{k=1}^K \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(z_i, \phi_{a_i}(t, z_i), z_j) dt \\ &\quad - \int_0^s \sum_{k=1}^K \pi_k \int_{y \in \mathbb{R}} f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), y) \mu_t^k(dy) dt. \end{aligned} \tag{46}$$

Let

$$D_k^i(t) \triangleq \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(z_i, \phi_{a_i}(t, z_i), z_j) - \pi_k \int_{\mathbb{R}} f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), y) \mu_t^k(dy).$$

We have

$$\begin{aligned} &D_k^i(t) \\ &= \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(z_i, \phi_{a_i}(t, z_i), z_j) - \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), z_j) \quad (\triangleq D_{k,1}^i) \\ &\quad + \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), z_j) - \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), \hat{z}_j) \quad (\triangleq D_{k,2}^i) \\ &\quad + \sum_{j \in S(\theta_k)} (1/n) f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), \hat{z}_j) - \pi_k \int f_{a_i}(\hat{z}_i, \phi_{a_i}(t, \hat{z}_i), y) \mu_t^k(dy). \quad (\triangleq D_{k,3}^i) \end{aligned}$$

By the Lipschitz continuity condition of ϕ_{a_i} and f_{a_i}, there exists a constant $C > 0$, independent of n, such that

$$|D_{k,1}^i| + |D_{k,2}^i| \le C \sum_{j \in S(\theta_k)} (1/n)[|z_i - \hat{z}_i| + |z_j - \hat{z}_j|]. \tag{47}$$

It follows from (46)-(47) that

$$\begin{aligned} \sup_{0 \le s \le t} |z_i(s) - \hat{z}_i(s)| &\le C \int_0^t |z_i(s) - \hat{z}_i(s)| ds + C \int_0^t (1/n) \sum_{j=1}^n |z_j(s) - \hat{z}_j(s)| ds \\ &\quad + \int_0^t \sum_{k=1}^K |D_{k,3}^i(s)| ds, \end{aligned} \tag{48}$$

which gives

$$\begin{aligned} \sum_{i=1}^n \sup_{0 \le s \le t} |z_i(s) - \hat{z}_i(s)| &\le 2C \sum_{i=1}^n \int_0^t |z_i(s) - \hat{z}_i(s)| ds + \int_0^t \sum_{i=1}^n \sum_{k=1}^K |D_{k,3}^i(s)| ds \\ &\le 2C \sum_{i=1}^n \int_0^t \sup_{0 \le \tau \le s} |z_i(\tau) - \hat{z}_i(\tau)| ds + \int_0^t \sum_{i=1}^n \sum_{k=1}^K |D_{k,3}^i(s)| ds. \end{aligned} \tag{49}$$

Now we show

$$\sup_{0\le t\le T}\sum_{k=1}^{K} E|D^i_{k,3}(t)| = O(n^{-1/2}+\varepsilon_n). \tag{50}$$

In fact, we have

$$\begin{aligned} &E|D^i_{k,3}(t)|^2\\ &\le 2E\bigg|\sum_{j\in S(\theta_k)}\Big[(1/n)f_{a_i}(\hat z_i,\phi_{a_i}(t,\hat z_i),\hat z_j)-\int(1/n)f_{a_i}(\hat z_i,\phi_{a_i}(t,\hat z_i),y)\mu^k_t(dy)\Big]\bigg|^2\\ &\quad+2E\bigg|\Big[(1/n)\sum_{j=1}^{n}1_{\{j\in S(\theta_k)\}}-\pi_k\Big]\int f_{a_i}(\hat z_i,\phi_{a_i}(t,\hat z_i),y)\mu^k_t(dy)\bigg|^2\\ &\triangleq 2S_1+2S_2. \end{aligned}$$

We clearly have $S_2=O(\varepsilon_n^2)$ by (H0). By independence of the components $\hat z_j$, $1\le j\le n$, the crossing terms in S_1 generated by taking $j'\ne j''$ (for j) is zero after taking expectations, which implies $S_1=O(1/n)$. Thus it follows that

$$E|D^i_{k,3}(t)|^2=O(1/n+\varepsilon_n^2),$$

where the right hand side does not depend upon t, and therefore, $E(\sum_{k=1}^K|D^i_{k,3}(t)|)^2 = O(1/n+\varepsilon_n^2)$ by the elementary inequality $(y_1+\cdots+y_K)^2\le K(y_1^2+\cdots+y_K^2)$ for real numbers. Hence $E\sum_{k=1}^K|D^i_{k,3}(s)|=O(n^{-1/2}+\varepsilon_n)$, and (50) follows.

Now by (50) and (49), it follows from Gronwall's lemma that

$$\sum_{i=1}^{n} E\sup_{0\le s\le T}|z_i(s)-\hat z_i(s)| = O(n^{1/2}+n\varepsilon_n),$$

which, combined again with Gronwall's lemma and (48), yields $E\sup_{0\le s\le T}|z_i(s)-\hat z_i(s)|=O(n^{-1/2}+\varepsilon_n)$. This completes the case for $\kappa=1$.

By combining the finished part with boundedness of $\sup_{1\le i\le n}\sup_{0\le t\le T}|z_i(t)-\hat z_i(t)|$, the case with $\kappa=2$ follows. This completes the proof. □

The following performance analysis involves extensive crossing perturbation estimate. Specifically, when a given agent changes its control, it will result in state process variations for other agents. In turn, these variations, together with the initial control change, will affect the dynamics of that agent.

THEOREM 13. *Assume (H0)-(H6) hold and there exists a set of Lipschitz control laws* $(u_1,\cdots,u_n)=(\varphi_{a_1},\cdots,\varphi_{a_n})$ *satisfying the NCE principle, i.e., the set of control laws* $(\varphi_{\theta_1},\cdots,\varphi_{\theta_K})$ *is derived from the HJB equation (9) together with the M-V equation (11), and satisfies the Lipschitz condition (H6). When the* i*th agent changes its control from* u_i *to* $u_i'\in\mathcal{U}_i^d$ *and the control laws of all other agents remain the same, the cost* $J_i(u_i',u_{-i})$ *can be decreased by at most* $O(n^{-1/2}+\varepsilon_n)$*, i.e.,* $J_i(u_i',u_{-i})\ge J_i(u_i,u_{-i})-O(n^{-1/2}+\varepsilon_n)$*, where* ε_n *is given by (45).*

Proof. To simplify the notation, we consider a strategy change for the first agent. We write

$$dz_i = (1/n)\sum_{j=1}^{n} f_{a_1}(z_i, \varphi_{a_i}(t, z_i), z_j)dt + \sigma dw_i, \qquad 1 \le i \le n, \tag{51}$$

and

$$\begin{cases} dz_1' &= (1/n)\sum_{j=1}^n f_{a_1}(z_1', \tilde{\varphi}_{a_1}(t, z_1'), z_j')dt + \sigma dw_1 \\ dz_2' &= (1/n)\sum_{j=1}^n f_{a_2}(z_2', \varphi_{a_2}(t, z_2'), z_j')dt + \sigma dw_2 \\ &\vdots \\ dz_n' &= (1/n)\sum_{j=1}^n f_{a_n}(z_n', \varphi_{a_n}(t, z_n'), z_j')dt + \sigma dw_n, \end{cases} \tag{52}$$

where the equation system (52) gives the closed-loop dynamics after the first agent takes the control $\tilde{\varphi}_{a_1}$ instead of φ_{a_1}. The two equation systems (51) and (52) have the same initial condition $(z_1(0), \cdots, z_n(0))$. We further introduce the auxiliary M-V equation system

$$dz_i'' = f_{a_i}[z_i'', \varphi_{a_i}(t, z_i''), \mu_t^1, \cdots, \mu_t^K]dt + \sigma dw_i, \tag{53}$$

where the initial condition is still $(z_1(0), \cdots, z_n(0))$.

We compare $(z_2, \cdots, z_n)$ and $(z_2', \cdots, z_n')$ in (51) and (52) by treating z_1 and z_1' as an additional quantity, and show by Gronwall's lemma that

$$\sup_{2\le j\le n}\sup_{0\le s\le T}|z_j(s) - z_j'(s)| \le O(1/n), \tag{54}$$

where the right hand side holds uniformly with respect to the choices of $\tilde{\varphi}_{a_1}$.

By the same argument as in proving Theorem 12, we get

$$\sup_{2\le j\le n} E \sup_{0\le t\le T}|z_j''(t) - z_j'(t)|^\kappa + \sup_{1\le j\le n} E \sup_{0\le t\le T}|z_j''(t) - z_j(t)|^\kappa = O(n^{-1/2} + \varepsilon_n),$$

for $\kappa = 1, 2$.

We introduce the new equation

$$d\hat{z}_1' = (1/n)\sum_{j=1}^{n} f_{a_1}(\hat{z}_1', \tilde{\varphi}_{a_1}(t, \hat{z}_1'), z_j'')dt + \sigma dw_1, \tag{55}$$

where $\hat{z}_1'(0) = z_1(0)$ and $(z_1'', \cdots, z_n'')$ is given by (53). By (52) and (55), once $\tilde{\varphi}_{a_1}$ is selected, we can show that

$$E\sup_{0\le t\le T}|\hat{z}_1'(t) - z_1'(t)|^\kappa = O(n^{-1/2} + \varepsilon_n), \qquad \kappa = 1, 2.$$

Next, we construct

$$d\tilde{z}_1' = f_{a_1}[\tilde{z}_1', \tilde{\varphi}_{a_1}(t, \tilde{z}_1'), \mu_t^1, \cdots, \mu_t^K]dt + \sigma dw_1, \tag{56}$$

where $\tilde{z}_1'(0) = z_1(0)$. Combining (55)-(56) and applying a similar argument as in proving the decoupling results in Theorem 12, we can show that

$$E \sup_{0\le t\le T} |\hat{z}_1' - \tilde{z}_1'| = O(n^{-1/2} + \varepsilon_n).$$

Subsequently, we can show that

$$\begin{aligned}
&E\int_0^T (1/n)\sum_{j=1}^n L(z_1', \tilde{\varphi}_{a_1}(t, z_1'), z_j')dt \\
&\ge E\int_0^T (1/n)\sum_{j=1}^n L(z_1', \tilde{\varphi}_{a_1}(t, z_1'), z_j'')dt - O(n^{-1/2} + \varepsilon_n) \\
&\ge E\int_0^T (1/n)\sum_{j=1}^n L(\tilde{z}_1', \tilde{\varphi}_{a_1}(t, \tilde{z}_1'), z_j'')dt - O(n^{-1/2} + \varepsilon_n) \\
&\ge E\int_0^T L[\tilde{z}_1', \tilde{\varphi}_{a_1}(t, \tilde{z}_1'), \mu_t^1, \cdots, \mu_t^K]dt - O(n^{-1/2} + \varepsilon_n) \\
&\ge E\int_0^T L[z_1'', \varphi_{a_1}(t, z_1''), \mu_t^1, \cdots, \mu_t^K]dt - O(n^{-1/2} + \varepsilon_n),
\end{aligned}$$

where the last inequality follows from the optimality interpretation of φ_{a_1} for (53). By further comparing $E\int_0^T L[z_1'', \varphi_{a_1}(t, z_1''), \mu_t^1, \cdots, \mu_t^K]dt$ with the cost $E\int_0^T (1/n) \sum_{j=1}^n L(z_1, \varphi_{a_1}(t, z_1), z_j)dt$ under the control laws $(\varphi_{a_1}, \cdots, \varphi_{a_n})$ in (51) for the n agents, we can show the theorem holds when u_1 changes within $\mathcal{U}_1^d$. This completes the proof. □

In Theorem 13, the equilibrium analysis is based upon the general nonlinear dynamics and we need to restrict the strategy change within the set $\mathcal{U}_i^d$. This simplifies the performance comparison. Now we proceed to consider the strategy change in the wider set $\mathcal{U}_i$. To get a tractable analysis, meanwhile, we need to introduce more structural information into the system model. In particular, we consider individual dynamics and costs involving both additive and multiplicative coupling. The corollary below may be proved by following the above argument in establishing Theorem 13 and we will not repeat the details here.

COROLLARY 14. *Assume the conditions in Theorem 13 hold, the individual dynamics may be decomposed into the form*

$$f_{a_i}(z_i, u_i, z_j) = f_{a_i}^0(z_i, u_i)g_{a_i}^0(z_j) + g_{a_i}^1(z_j),$$

and the cost is given in the form $J_i(u_i, u_{-i}) = E\int_0^T \big\{(1/n)\sum_{j=1}^n [L^0(z_i, u_i)h^0(z_j) + h^1(z_j)]\big\}dt$, $1 \le i \le n$. *Also, assume the individual controls* $(u_1, \cdots, u_n) = (\varphi_{a_1}, \cdots, \varphi_{a_n})$ *are specified by the NCE principle as in Theorem 13. Then* $(u_1, \cdots, u_n)$ *is an* ε*-Nash strategy with respect to* $\mathcal{U}^n$*, where* $\varepsilon \to 0$ *as* $n \to \infty$. □

We note that the result in Corollary 14 may be further extended to models with a cost integrand of the form $\Theta(z_i, u_i, (1/n)\sum_{j=1}^n h(z_j))$ for J_i. Hence, our NCE methodology can deal with many typical models with mean field coupling. It would be of interest to consider the asymptotic equilibrium analysis with respect to the strategy space $\mathcal{U}^n$ for more general nonlinear models than that appearing in Corollary 14. This will depend upon developing more sophisticated crossing perturbation analysis and will be investigated in future work.

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