Equivariant maps between representation spheres

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Abstract

Let G be a compact Lie group. We prove that if V and W are orthogonal G-representations such that $V^G = W^G = \{0\}$, then a G-equivariant map $S(V) \to S(W)$ exists provided that $\dim V^H \leq \dim W^H$ for any closed subgroup $H \subseteq G$. This result is complemented by a reinterpretation in terms of divisibility of certain Euler classes when G is a torus.

1 Introduction

A basic problem in the theory of transformation groups is to find necessary and sufficient conditions for the existence of a G-equivariant map between two G-spaces. Perhaps the most well-known result in the necessary direction is the celebrated Borsuk–Ulam theorem [1], which states that if V and W are two orthogonal fixed-point free $\mathbb{Z}_2$-representations, then the existence of a $\mathbb{Z}_2$-equivariant map $S(V) \to S(W)$ implies that $\dim V \leq \dim W$. This result has numerous and far reaching generalizations, see e.g. [9], [10] for an overview. One such generalization, particularly interesting from the point of view of this note, is:

Theorem 1.1 ([6]). *Let V and W be orthogonal representations of $G = (\mathbb{S}^1)^k$ or $G = (\mathbb{Z}_p)^\ell$, p a prime, such that $V^G = W^G = \{0\}$. If there exists a G-equivariant map $S(V) \to S(W)$, then*

$$\dim V^H \leq \dim W^H \text{ for any closed subgroup } H \subseteq G. \qquad (*)$$

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On the other hand, sufficient conditions for the existence of G-equivariant maps between representation spheres have not been investigated nearly extensively. This is our starting point: we prove in Corollary 3.3 that $(*)$ is sufficient for the existence of a G-equivariant map $S(V) \to S(W)$ for *any* compact Lie group G. It is not a new result in the sense that it can be extracted from the existing literature, see [3, Chapter II], although it is rather buried in the text. This, coupled with the fact that the second-named author has been inquired about converses to various versions of the Borsuk–Ulam theorem, makes us believe that it is worthwhile to carefully spell the details out.

A corollary to the discussion above is that if G is a torus or a p-torus, then $(*)$ is equivalent to the existence of a G-equivariant map $S(V) \to S(W)$. When G is a torus, we reinterpret this result in terms of divisibility of Euler classes of V^H and W^H in $H^*(BG;\mathbb{Z})$. This angle of research has been pursued previously in various guises, e.g. by Marzantowicz [8] (in the same setting, for G a compact Lie group) and Komiya [4], [5] (with K-theoretic Euler classes, for G an abelian compact Lie group). However, in each case only the necessary criteria were described.

2 Preliminaries

2.1 Notation

Let G be a compact Lie group. If $H \subseteq G$ is a closed subgroup, then NH denotes the normalizer of H in G and $WH = NH/H$ the Weil group of H. Given a G-space X, write $\mathcal{O}(X)$ for the set of isotropy groups of X. If $H \in \mathcal{O}(X)$, then (H) stands for its conjugacy class, referred to as an *orbit type*. There is a natural partial order on the set of orbit types of X, namely:

$$(H) \leq (K) \text{ if and only if } K \text{ is conjugate to a subgroup of } H.$$

Recall that a finite-dimensional *G-complex* is a G-space X that possesses a filtration

$$X_{(0)} \subseteq X_{(1)} \subseteq \cdots \subseteq X_{(n)} = X$$

by G-invariant subspaces, with $X_{(k+1)}$ obtained from $X_{(k)}$ by attaching equivariant cells $D^{k+1} \times G/H$ via G-equivariant maps $S^k \times G/H \to X_{(k)}$, $0 \leq k \leq n-1$. The space $X_{(k)}$ is called the *k-skeleton* of X and the integer n is the (*cellular*) *dimension* of X.

Observe that if X is a G-complex and $H \subseteq G$ is a closed subgroup, then $X^H = \{x \in X \mid hx = x \text{ for any } h \in H\}$, the *H-fixed point set* of X, is a WH-complex, while $X^{(H)} = \bigcup_{K \in (H)} X^K$ is a G-subcomplex of X. The cellular dimension of $X^{(H)}$ as a G-complex is equal to the cellular dimension of X^H as a WH-complex. We denote this dimension by $d_H(X)$.

2.2 Euler classes calculus

Let $G \hookrightarrow EG \to BG$ be the universal principal G-bundle and V an orthogonal G-representation. The *Borel space* $EG \times_G V = (EG \times V)/G$, where the orbit space is taken with respect to the diagonal action, is a vector bundle with base space BG and fibre V. Provided that this bundle is R-orientable for some ring R, its Euler class, denoted $e(V)$, is called the *Euler class of* V (over R).

Let $G = \mathbb{T}^k = (\mathbb{S}^1)^k$. Recall that any non-trivial irreducible orthogonal representation of G is given by

$$V_{(\alpha_1,\ldots,\alpha_k)} = V_1^{\alpha_1} \otimes \cdots \otimes V_k^{\alpha_k},$$

where the tensor product is considered over the field of complex numbers, and:

- V_i stands for the irreducible complex G-representation corresponding to the projection $G \to \mathbb{S}^1$ onto the i-th coordinate, $1 \le i \le k$,
- V^j denotes the j-th tensor power of a representation V,
- $0 \le \alpha_i$ for any $1 \le i \le k$.

In particular, every non-trivial irreducible orthogonal G-representation is complex one-dimensional and admits complex structure. Consequently, the latter is also true for any orthogonal G-representation V without a trivial direct summand, and it follows that the corresponding vector bundle $EG \times_G V$ is integrally orientable.

Now recall that

$$H^*(BG;\mathbb{Z}) \cong \mathbb{Z}[t_1,\ldots,t_k],$$

where $t_i = e(V_i)$ for $1 \le i \le k$. Using the facts that $e(V \oplus W) = e(V)e(W)$ and, for one-dimensional representations, $e(V \otimes W) = e(V) + e(W)$, we see that the Euler class of $V = \bigoplus_\alpha r_\alpha V_\alpha$ is given by

$$e(V) = \prod_\alpha (\alpha_1 t_1 + \cdots + \alpha_k t_k)^{r_\alpha}.$$

In particular, $e(V) = 0$ if and only if V contains a trivial direct summand.

3 The existence of equivariant maps for compact Lie groups

Throughout this section G is a compact Lie group. We will be interested in the existence of G-equivariant maps between representation spheres. The main result of this section is Corollary 3.3, and the main ingredient in its proof is the following fact from equivariant obstruction theory.

Theorem 3.1 ([3, Chapter II, Proposition 3.15])**.** *Let $n \ge 1$ be an integer. Suppose that (X, A) is a relative G-complex with a free action on $X \setminus A$ and Y is an $(n-1)$-connected and n-simple G-space.*

(1) *Any G-equivariant map $A \to Y$ can be extended over the n-skeleton of X.*

(2) *Let $f_0, f_1 \colon A \to Y$ be G-equivariant maps and $\tilde{f}_0, \tilde{f}_1 \colon X_{(n)} \to Y$ their extensions. If f_0 and f_1 are G-homotopic, then there exists a G-homotopy between $\tilde{f}_0|_{X_{(n-1)}}$ and $\tilde{f}_1|_{X_{(n-1)}}$ extending the one between f_0 and f_1.*

As a matter of fact, Theorem 3.2 below is also formulated in [3, Chapter II], but its proof is spread throughout the text. We provide what we believe to be a more accessible treatment for the convenience of the reader.

Theorem 3.2. *Let X be a finite G-complex and Y a G-space such that $Y^{(H)}$ is non-empty for any minimal orbit type (H) of X.*

(1) *If Y^H is $(d_H(X)-1)$-connected and $d_H(X)$-simple for any $H \in \mathcal{O}(X)$, then there exists a G-equivariant map $X \to Y$.*

(2) *If Y^H is $d_H(X)$-connected and $(d_H(X)+1)$-simple for any $H \in \mathcal{O}(X)$, then any two G-equivariant maps $X \to Y$ are G-homotopic.*

Proof. (1) In order to construct a G-equivariant map $f \colon X \to Y$, we will proceed inductively with respect to partial order on the set of orbit types of X.

If H is a representative of a minimal orbit type of X, then X^H is a free WH-complex. Define a WH-equivariant map $(X^H)_{(0)} \to Y^H$ by sending the 0-cells of X^H to an arbitrary orbit of Y^H and extend it to a map $f^H \colon X^H \to Y^H$ by means of Theorem 3.1. Since $X^{(H)}$ has a single orbit type, it is G-homeomorphic to $(G/H) \times_{WH} X^H$ by [2, Chapter II, Corollary 5.11]. We can therefore saturate f^H to obtain a G-equivariant map $X^{(H)} \to Y^{(H)}$ via the composition

$$X^{(H)} \approx (G/H) \times_{WH} X^H \to (G/H) \times_{WH} Y^H \to Y^{(H)},$$

where the last map is given by $[gH, y] \mapsto gy$ (see [2, Chapter II, Corollary 5.12]).

It is straightforward to see that any two distinct minimal orbit types (H_i), (H_j) have $X^{(H_i)} \cap X^{(H_j)} = \varnothing$, thus the above procedure yields a G-equivariant map

$$\bigcup_{(H)} X^{(H)} \to \bigcup_{(H)} Y^{(H)},$$

where (H) runs over all minimal orbit types of X.

Now choose $K \in \mathcal{O}(X)$ and assume inductively that f is defined on a subcomplex

$$X^{<(K)} = \bigcup_{(H)<(K)} X^{(H)}.$$

By construction, f takes values in $Y^{<(K)}$. In view of [3, Chapter I, Proposition 7.4], G-extensions of $X^{<(K)} \to Y^{<(K)}$ to $X^{(K)} \to Y^{(K)}$ are in one-to-one correspondence with WK-extensions of $X^{<K} \to Y^{<K}$ to $X^K \to Y^K$. However, the WK-action on $X^K \setminus X^{<K}$ is free, hence Theorem 3.1 applied to the relative complex $\left(X^K, X^{<K}\right)$ results in a WK-equivariant map $X^K \to Y^K$. There are only finitely many orbit types, hence this process stops after a finite number of steps, producing a G-equivariant map $X \to Y$.

(2) Let $H \in \mathcal{O}(X)$ be a representative of a minimal orbit type. Since Y^H is path-connected, any two NH-equivariant maps $WH \to Y^H$ are G-homotopic. Therefore any two G-equivariant maps $(X^H)_{(0)} \to Y^H$ are also G-homotopic. It now suffices to successively apply the second part of Theorem 3.1 just as above. ∎

As a consequence, we obtain the following corollary.

Corollary 3.3. *Let V, W be two orthogonal G-representations with $V^G = W^G = \{0\}$.*

(1) If $\dim(V^H) \leq \dim(W^H)$ for any $H \in \mathcal{O}\big(S(V)\big)$, then there exists a G-equivariant map $S(V) \to S(W)$.

(2) If, additionally, G is connected and for any $H \in \mathcal{O}\big(S(V)\big)$ we have $\dim WH > 0$, then any two G-equivariant maps $S(V) \to S(W)$ are G-homotopic.

Proof. Let $H \in \mathcal{O}\big(S(V)\big)$ and note that the cellular dimension of the G-complex $S(V)^{(H)}$ is at most $\dim V^H - 1$, since this dimension is equal to the dimension of the orbit space $S(V)^{(H)}/G = S(V)^H /\, WH$. On the other hand, the space $S(W)^H$ is non-empty, simple and $(\dim W^H - 2)$-connected. Since $\dim V^H \leq \dim W^H$, applying Theorem 3.2 concludes the proof. ∎

4 Torus equivariant maps between representation spheres

4.1

Let $G = \mathbb{T}^k$. Unless otherwise stated, V and W are assumed to be orthogonal G-representations such that $V^G = W^G = \{0\}$. Given a decomposition of V into irreducible components, say $V = \bigoplus_{\alpha \in \mathcal{A}} r_\alpha V_\alpha$, we introduce the following notation. For any $\alpha \in \mathcal{A}$,

- K_α denotes the kernel of V_α, and
- $\mathbb{T}_\alpha$ the connected component of identity of K_α.

Then K_α is a $(k-1)$-dimensional subgroup of G and $\mathbb{T}_\alpha$ is a $(k-1)$-dimensional torus. Furthermore, let m_α be the index of $\mathbb{T}_\alpha$ in K_α. The number m_α is in fact the greatest common divisor of the k-tuple $\alpha = (\alpha_1, \ldots, \alpha_k)$. In particular, it indicates whether V_α is a tensor power of another irreducible G-representation $V_{\tilde{\alpha}}$, where $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ and $\alpha = m_\alpha \tilde{\alpha}$. Let $\tilde{\mathcal{A}} = \{\tilde{\alpha} \mid \alpha \in \mathcal{A}\}$ and, for $\tilde{\alpha} \in \tilde{\mathcal{A}}$, define $\mathcal{H}^V_{\tilde{\alpha}} = \{\alpha \in \mathcal{A} \mid m_\alpha \tilde{\alpha} = \alpha\}$. Geometrically, $\mathcal{H}^V_{\tilde{\alpha}}$ corresponds to the set of $\alpha \in \mathcal{A}$ such that $\mathbb{T}_\alpha = \mathbb{T}_{\tilde{\alpha}}$.

Proposition 4.1. *Let $V = \bigoplus_{\alpha \in \mathcal{A}} r_\alpha V_\alpha$ and $W = \bigoplus_{\beta \in \mathcal{B}} q_\beta V_\beta$ be orthogonal G-representations such that $\dim V < \dim W$. Any G-equivariant map $S(V) \to S(W)$ can be extended to a G-equivariant map $S(V') \to S(W)$, where V' is an orthogonal G-representation such that $V \subseteq V'$ and $\dim V' = \dim W$.*

Proof. Let $S(V) \to S(W)$ be a G-equivariant map. Note that since $V^G = W^G = \{0\}$, we have $V = \bigoplus_{\tilde{\alpha} \in \tilde{\mathcal{A}}} V^{\mathbb{T}_{\tilde{\alpha}}}$ and $W = \bigoplus_{\tilde{\beta} \in \tilde{\mathcal{B}}} W^{\mathbb{T}_{\tilde{\beta}}}$. In view of Theorem 1.1, $\dim V^{\mathbb{T}_{\tilde{\alpha}}} \leq \dim W^{\mathbb{T}_{\tilde{\alpha}}}$ for any $\tilde{\alpha} \in \tilde{\mathcal{A}}$, which shows that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$. Furthermore,

$$\sum_{\tilde{\alpha} \in \tilde{\mathcal{A}}} \dim V^{\mathbb{T}_{\tilde{\alpha}}} = \dim V < \dim W = \sum_{\tilde{\beta} \in \tilde{\mathcal{B}}} \dim W^{\mathbb{T}_{\tilde{\beta}}}.$$

Consequently, $\tilde{\mathcal{A}} \subsetneq \tilde{\mathcal{B}}$ or $\dim W^{\mathbb{T}_{\tilde{\alpha}}} - \dim V^{\mathbb{T}_{\tilde{\alpha}}} = d_{\tilde{\alpha}} > 0$ for some $\tilde{\alpha} \in \tilde{\mathcal{A}}_1 \subseteq \tilde{\mathcal{A}}$. Set

$$V' = V \oplus \Big(\bigoplus_{\tilde{\alpha} \in \tilde{\mathcal{A}}_1} d_{\tilde{\alpha}} V_{\tilde{\alpha}} \Big) \oplus \Big(\bigoplus_{\tilde{\beta} \in \tilde{\mathcal{B}} \setminus \tilde{\mathcal{A}}} (\dim W^{\mathbb{T}_{\tilde{\beta}}}) V_{\tilde{\beta}} \Big).$$

Then $\dim V' = \dim W$ and $\dim (V')^H \leq \dim W^H$ for any $H \in \mathcal{O}\big(S(V')\big)$. Indeed, if H properly contains a $(k-1)$-dimensional torus, then $\dim (V')^H = \dim V^H$. Otherwise, since $\tilde{\mathcal{A}}' = \tilde{\mathcal{B}}$,

$$(V')^H = \Big(\bigoplus_{\tilde{\beta} \in \tilde{\mathcal{B}}} (V')^{\mathbb{T}_{\tilde{\beta}}} \Big)^H = \bigoplus_{\tilde{\beta} \in \tilde{\mathcal{B}}} \big((V')^{\mathbb{T}_{\tilde{\beta}}} \big)^H = \bigoplus_{\substack{\tilde{\beta} \,:\, \tilde{\beta} \in \tilde{\mathcal{B}} \\ H \subseteq \mathbb{T}_{\tilde{\beta}}}} (V')^{\mathbb{T}_{\tilde{\beta}}}.$$

But $\dim (V')^{\mathbb{T}_{\tilde{\beta}}} = \dim W^{\mathbb{T}_{\tilde{\beta}}}$ for any $\tilde{\beta} \in \tilde{\mathcal{B}}$ by construction, hence $\dim (V')^H = \dim W^H$. The existence of a G-equivariant map $S(V') \to S(W)$ now follows from Corollary 3.3. ■

Lemma 4.2. *If there exists a G-equivariant map $S(V) \to S(W)$, then $e(V)$ divides $e(W)$ in $H^*(BG; \mathbb{Z})$.*

Proof. In view of Theorem 1.1, $\dim V \leq \dim W$. If $\dim V < \dim W$, use Proposition 4.1 to obtain a G-equivariant map $S(V') \to S(W)$, where V' is an orthogonal G-representation such that $V \subseteq V'$ and $\dim V' = \dim W$. In view of [8, Proposition 1.8], $e(V')$ divides $e(W)$. Since $e(V') = e(V)e(V^{\perp})$, where $V^{\perp}$ is the orthogonal complement of V in V', we see that $e(V)$ also divides $e(W)$. ■

Theorem 4.3. *Let $V = \bigoplus_{\alpha \in \mathcal{A}} r_\alpha V_\alpha$ and $W = \bigoplus_{\beta \in \mathcal{B}} q_\beta V_\beta$ be orthogonal G-representations. The following conditions are equivalent.*

1. *There exists a G-equivariant map $S(V) \to S(W)$.*
2. *For any $H \in \mathcal{O}\big(S(V)\big)$, the Euler class of V^H divides the Euler class of W^H in $H^*(BG; \mathbb{Z})$.*
3. *For any $H \in \mathcal{O}\big(S(V)\big)$, $\dim V^H \leq \dim W^H$.*
4. *For any $(k-1)$-dimensional isotropy subgroup $H \in \mathcal{O}\big(S(V)\big)$, $\dim V^H \leq \dim W^H$.*
5. *For any $\tilde{\alpha} \in \tilde{\mathcal{A}}$ and any $m \in \mathbb{N}$,*

$$\sum_{\substack{\alpha \,:\, \alpha \in \mathcal{H}^V_{\tilde{\alpha}} \\ m \mid m_\alpha}} r_\alpha \leq \sum_{\substack{\beta \,:\, \beta \in \mathcal{H}^W_{\tilde{\alpha}} \\ m \mid m_\beta}} q_\beta.$$

Proof. "(1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1)". Let $S(V) \to S(W)$ be a G-equivariant map. Choose a subgroup $H \in \mathcal{O}\big(S(V)\big)$ and restrict to a G-equivariant map $S(V^H) \to S(W^H)$. It follows from Lemma 4.2 that $e(V^H)$ divides $e(W^H)$ in $H^*(BG;\mathbb{Z})$. If this happens, then $\deg e(V^H) \leq \deg e(W^H)$, which directly translates into $\dim V^H \leq \dim W^H$. This last condition for any $H \in \mathcal{O}\big(S(V)\big)$ implies the existence of a G-equivariant map $S(V) \to S(W)$ via Corollary 3.3.

"(4) $\Rightarrow$ (3)". Assume without loss of generality that $k \geq 2$. As exhibited in the proof of Proposition 4.1, for $H \in \mathcal{O}\big(S(V)\big)$ at most $(k-2)$-dimensional,

$$V^H = \bigoplus_{\substack{\tilde\alpha\,:\,\tilde\alpha\in\tilde{\mathcal{A}}\\ H\subseteq \mathbb{T}_{\tilde\alpha}}} V^{\mathbb{T}_{\tilde\alpha}} \text{ and } W^H = \bigoplus_{\substack{\tilde\beta\,:\,\tilde\beta\in\tilde{\mathcal{B}}\\ H\subseteq \mathbb{T}_{\tilde\beta}}} W^{\mathbb{T}_{\tilde\beta}}.$$

Since $\dim V^{K_\alpha} \leq \dim W^{K_\alpha}$ for any $\alpha \in \mathcal{A}$, we infer that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$. Thus in order to wrap this part of the proof up, it suffices to observe that $\dim V^{\mathbb{T}_{\tilde\alpha}} \leq \dim W^{\mathbb{T}_{\tilde\alpha}}$ for any $\tilde\alpha \in \tilde{\mathcal{A}}$. Indeed, $H = \bigcap_{\alpha\in\mathcal{H}^V_{\tilde\alpha}} K_\alpha$ is a $(k-1)$-dimensional isotropy of $S(V)$ such that $V^{\mathbb{T}_{\tilde\alpha}} = V^H$, hence

$$\dim V^{\mathbb{T}_{\tilde\alpha}} = \dim V^H \leq \dim W^H \leq \dim W^{\mathbb{T}_{\tilde\alpha}}.$$

"(4) $\Leftrightarrow$ (5)". Note that if we view $V^{\mathbb{T}_{\tilde\alpha}}$ and $W^{\mathbb{T}_{\tilde\alpha}}$ as representations of $\mathbb{S}^1 = G/\mathbb{T}_{\tilde\alpha}$, then (4) can be rephrased as $\dim (V^{\mathbb{T}_{\tilde\alpha}})^{\mathbb{Z}_m} \leq \dim (W^{\mathbb{T}_{\tilde\alpha}})^{\mathbb{Z}_m}$ for any $\tilde\alpha \in \mathcal{A}$ and $m \in \mathbb{N}$. But

$$(V^{\mathbb{T}_{\tilde\alpha}})^{\mathbb{Z}_m} = \bigoplus_{\substack{\alpha\,:\,\alpha\in\mathcal{H}^V_{\tilde\alpha}\\ m\,|\,m_\alpha}} r_\alpha V_\alpha,$$

which shows that

$$\dim (V^{\mathbb{T}_{\tilde\alpha}})^{\mathbb{Z}_m} = \sum_{\substack{\alpha\,:\,\alpha\in\mathcal{H}^V_{\tilde\alpha}\\ m\,|\,m_\alpha}} r_\alpha.$$

An analogous thing happens for W, which concludes the proof. ∎

Remark 4.4. The implication "(4) $\Rightarrow$ (3)" can be seen in a more geometrical manner. As observed above, (4) amounts precisely to the condition $(*)$ for $V^{\mathbb{T}_{\tilde\alpha}}$ and $W^{\mathbb{T}_{\tilde\alpha}}$ viewed as representations of $\mathbb{S}^1 = G/\mathbb{T}_{\tilde\alpha}$, for any $\tilde\alpha \in \tilde{\mathcal{A}}$. Therefore Corollary 3.3 implies the existence of an $\mathbb{S}^1$-equivariant map $f^{\mathbb{T}_{\tilde\alpha}}\colon S(V^{\mathbb{T}_{\tilde\alpha}}) \to S(W^{\mathbb{T}_{\tilde\alpha}})$, which can be considered as a G-equivariant map. Consequently, the join construction

$$S(V) = S\Big(\bigoplus_{\tilde\alpha\in\tilde{\mathcal{A}}} V^{\mathbb{T}_{\tilde\alpha}}\Big) = \mathop{\ast}_{\tilde\alpha\in\tilde{\mathcal{A}}} S(V^{\mathbb{T}_{\tilde\alpha}}) \longrightarrow \mathop{\ast}_{\tilde\alpha\in\tilde{\mathcal{A}}} S(W^{\mathbb{T}_{\tilde\alpha}}) = S\Big(\bigoplus_{\tilde\alpha\in\tilde{\mathcal{A}}} W^{\mathbb{T}_{\tilde\alpha}}\Big) \subseteq S(W)$$

yields the desired G-equivariant map.

On a related note, the implication "(5) $\Rightarrow$ (2)" is a purely algebraic fact and can be derived directly, without any geometrical interpretation. We would like to thank A. Schinzel for suggesting the following argument to us.

Suppose that (5) is satisfied and fix $\tilde\alpha \in \tilde{\mathcal{A}}$. We will show that

$$e(V^{\mathbb{T}_{\tilde\alpha}}) = \prod_{\alpha\in\mathcal{H}^V_{\tilde\alpha}} (\alpha_1 t_1 + \cdots + \alpha_k t_k)^{r_\alpha} = \Big(\prod_{\alpha\in\mathcal{H}^V_{\tilde\alpha}} m_\alpha^{r_\alpha}\Big)(\tilde\alpha_1 t_1 + \cdots + \tilde\alpha_k t_k)^{\sum_{\alpha\in\mathcal{H}^V_{\tilde\alpha}} r_\alpha}$$

divides

$$e(W^{\mathbb{T}_{\tilde{\alpha}}}) = \Big(\prod_{\beta\in\mathcal{H}^W_{\tilde{\alpha}}} m_\beta^{q_\beta}\Big)(\tilde{\alpha}_1 t_1 + \cdots + \tilde{\alpha}_k t_k)^{\sum_{\beta\in\mathcal{H}^W_{\tilde{\alpha}}} q_\beta}.$$

If $m = 1$, then (5) yields $\sum_{\alpha\in\mathcal{H}^V_{\tilde{\alpha}}} r_\alpha \leq \sum_{\beta\in\mathcal{H}^W_{\tilde{\alpha}}} q_\beta$, thus it suffices to prove that $\prod_{\alpha\in\mathcal{H}^V_{\tilde{\alpha}}} m_\alpha^{r_\alpha}$ divides $\prod_{\beta\in\mathcal{H}^W_{\tilde{\alpha}}} m_\beta^{q_\beta}$.

Let n be the highest power of a prime p appearing in any of m_α's. Observe that p appears in $\prod_{\alpha\in\mathcal{H}^V_{\tilde{\alpha}}} m_\alpha^{r_\alpha}$ with the power

$$\begin{aligned} M &= \sum_{\substack{p \mid m_\alpha \\ p^2 \nmid m_\alpha}} r_\alpha + 2 \sum_{\substack{p^2 \mid m_\alpha \\ p^3 \nmid m_\alpha}} r_\alpha + \cdots + n \sum_{p^n \mid m_\alpha} r_\alpha \\ &= \sum_{p \mid m_\alpha} r_\alpha + \sum_{\substack{p^2 \mid m_\alpha \\ p^3 \nmid m_\alpha}} r_\alpha + 2 \sum_{\substack{p^3 \mid m_\alpha \\ p^4 \nmid m_\alpha}} r_\alpha + \cdots + (n-1) \sum_{p^n \mid m_\alpha} r_\alpha = \cdots \\ &= \sum_{p \mid m_\alpha} r_\alpha + \sum_{p^2 \mid m_\alpha} r_\alpha + \cdots + \sum_{p^n \mid m_\alpha} r_\alpha, \end{aligned}$$

where α varies over $\mathcal{H}^V_{\tilde{\alpha}}$. Likewise, if m is the highest power of p appearing in any of m_β's, then p appears in $\prod_{\beta\in\mathcal{H}^W_{\tilde{\alpha}}} m_\beta^{q_\beta}$ with the power

$$N = \sum_{p \mid m_\beta} q_\beta + \sum_{p^2 \mid m_\beta} q_\beta + \cdots + \sum_{p^m \mid m_\beta} q_\beta,$$

where β varies over $\mathcal{H}^W_{\tilde{\alpha}}$. By assumption, for any $i \geq 0$,

$$\sum_{\substack{\alpha\,:\,\alpha\in\mathcal{H}^V_{\tilde{\alpha}} \\ p^i \mid m_\alpha}} r_\alpha \leq \sum_{\substack{\beta\,:\,\beta\in\mathcal{H}^W_{\tilde{\alpha}} \\ p^i \mid m_\beta}} q_\beta,$$

hence $M \leq N$. This shows that, for any prime p, the power of p which appears in the decomposition of $e(V^{\mathbb{T}_{\tilde{\alpha}}})$ does not exceed the one which appears in the decomposition of $e(W^{\mathbb{T}_{\tilde{\alpha}}})$. Therefore $e(V^{\mathbb{T}_{\tilde{\alpha}}})$ divides $e(W^{\mathbb{T}_{\tilde{\alpha}}})$. Consequently, using the fact that $\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{B}}$, $e(V) = \prod_{\alpha\in\tilde{\mathcal{A}}} e(V^{\mathbb{T}_{\tilde{\alpha}}})$ divides $e(W) = \prod_{\beta\in\tilde{\mathcal{B}}} e(W^{\mathbb{T}_{\tilde{\beta}}})$. A similar argument shows that the same thing happens for $e(V^H)$ and $e(W^H)$.

The second-named author asked the following question in [8, Problem 2.6]. Given orthogonal $\mathbb{S}^1$-representations V and W with $V^{\mathbb{S}^1} = W^{\mathbb{S}^1} = \{0\}$, is divisibility of $e(W)$ by $e(V)$ sufficient for the existence of an $\mathbb{S}^1$-equivariant map $S(V) \to S(W)$? The following example shows that the answer is negative in general.

Example 4.5. Let V_1 be the one-dimensional fixed-point free $\mathbb{S}^1$-representation. Define $V = 2V_1^3 \oplus V_1^5$ and $W = V_1^{18} \oplus 2V_1^5$. Then $e(V) = 45t^3$ divides $e(W) = 450t^3$, but the existence of an $\mathbb{S}^1$-equivariant map $S(V) \to S(W)$ would violate Theorem 4.3, as $\dim V^{\mathbb{Z}_3} = 2 > 1 = \dim W^{\mathbb{Z}_3}$.

4.2

It is known that if a group G is not an extension of a finite p-group of exponent p by a torus, then G does not have the strong Borsuk–Ulam property, see [7]. It is an open problem whether every such extension enjoys the strong Borsuk–Ulam property; this is not even clear in the case $G = \mathbb{T}^k \times (\mathbb{Z}_p)^\ell$. (We note that the proof of [7, Lemma 1.2] is incomplete and thus does not settle this last problem.)

Conjecture. *A compact Lie group G has the strong Borsuk–Ulam property if and only if $G = \mathbb{T}^k \times (\mathbb{Z}_p)^\ell$.*

It remains to be verified that the following classes of groups do not have this property:

- non-abelian finite groups with exponent p, and
- non-trivial extensions $0 \to \mathbb{T}^k \to G \to (\mathbb{Z}_p)^\ell \to 0$.

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