Numerical treatment of nonlocal boundary value problem with layer behaviour

Erkan Cimen   Musa Cakir

Abstract

This paper deals with the singularly perturbed nonlocal boundary value problem for a linear first order differential equation. For the numerical solution of this problem, we use a fitted difference scheme on a piecewise uniform Shishkin mesh. An error analysis shows that the method is almost first order convergent, in the discrete maximum norm, independently of the perturbation parameter. Numerical results are presented which illustrate the theoretical results.

1 Introduction

We consider nonlocal singularly perturbed boundary value problem

$$\varepsilon u'(x) + a(x)u(x) = f(x),\ x \in \Omega, \tag{1.1}$$

$$u(0) + \gamma u(l_1) = Au(l) + B,\ l_1 \in \Omega, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, γ, A, B are given constants, l_1 and l are given real numbers with $0 < l_1 < l$ and $\Omega = (0, l)$, $\bar{\Omega} = [0, l]$. $a(x)$ and $f(x)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\bar{\Omega}$ and moreover,

$$0 < \alpha \leqslant a(x) \leqslant a^* < \infty.$$

Received by the editors in June 2016 - In revised form in December 2016.
Communicated by K. in 't Hout.
2010 *Mathematics Subject Classification :* 34B10, 65L05, 65L11, 65L12, 65L20.
Key words and phrases : Singular perturbation, nonlocal condition, finite difference method, initial layer, uniform convergence.

A singularly perturbed equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and normally, boundary (or initial) layers occur in their solutions. Singularly perturbed differential equations arise very frequently in fluid mechanics and other branches of applied mathematics [1-6].

Nonlocal boundary value problems have also been studied extensively in the literature [7] such as the problem arising in hydrodynamics, transonic gas dynamics [8] and boundary control problem for vibrations of a string [9]. The existence and uniqueness of solutions of nonlocal problems, and also their numerical solution, have been considered in many papers [10-16] and the references cited therein. Some approaches to approximating this type of problem have also been considered in [13, 15]. However, the algorithms developed in the papers cited above are mainly concerned with regular cases (i.e. when boundary layers are absent). Amiraliyev and Cakir [17] gave ε-uniform numerical scheme on an arbitrary non-uniform mesh to solve reaction-diffusion singularly perturbed problem with nonlocal boundary condition. They proved that the scheme is first-order ε-uniform convergence. Cakir and Amiraliyev [18] proposed a uniform finite difference method on piecewise uniform Shishkin type mesh for solving singularly perturbed semilinear convection-diffusion three-point boundary value problem. Cakir [19] constructed a hybrid scheme on Shishkin type mesh for solving linear second order singularly perturbed boundary value problem with nonlocal boundary condition. The scheme is the combination of Samarskii's scheme and finite difference scheme. He proved that the hybrid difference scheme is second-order accurate. The solution $u(x)$ of the problems in [17-19] have generally boundary layers at $x = 0$ and $x = l$ points.

The use of classical numerical methods for solving such problems may give rise to difficulties when the perturbation parameter ε is small. Therefore, it is important to develop suitable numerical methods to these problems. The outcomes in this paper concern both the analytical results and numerical solutions study of first-order nonlocal singularly perturbed boundary value problem. We construct uniformly convergent difference scheme on a piecewise equidistant mesh for the problem (1.1)-(1.2). Then we present the error analysis for the approximate solution and prove the uniform convergence in the discrete maximum norm. Finally, the numerical results are presented to support the theory.

Henceforth, C denote the generic positive constants independent of ε and mesh parameter, but whose value is fixed.

2 Analytical Results

Here we show some properties of the solution of (1.1)-(1.2), which are needed in later sections for the analysis of appropriate numerical solution. We also adopt the convention that

$$\|g\|_\infty \equiv \|g\|_{\infty,\bar{\Omega}} = \max_{0\leqslant x\leqslant l} |g(x)| .$$

Lemma 2.1. *Let $a, f \in C^1(\overline{\Omega})$ and*

$$1 + \gamma e^{-\frac{a^* l_1}{\varepsilon}} - Ae^{-\frac{\alpha l}{\varepsilon}} \geqslant c_0 > 0. \tag{2.1}$$

Then for the solution $u(x)$ of the problem (1.1)-(1.2) the following estimates hold:

$$\|u\|_\infty \leqslant c_0^{-1}|B| + \alpha^{-1}\|f\|_\infty (1 + c_0^{-1}|\gamma| + c_0^{-1}|A|), \tag{2.2}$$

$$|u'(x)| \leqslant C(1 + \frac{1}{\varepsilon}e^{-\frac{\alpha x}{\varepsilon}}),\ x \in \overline{\Omega}. \tag{2.3}$$

Proof. From (1.1) we have

$$u(x) = u(0)\, e^{-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta} + \frac{1}{\varepsilon}\int_0^x f(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^x a(\eta)d\eta} d\xi, \tag{2.4}$$

from which, by setting the boundary condition $u(0) + \gamma u(l_1) = Au(l) + B$, we get

$$u(0) = \frac{B - \frac{\gamma}{\varepsilon}\int_0^{l_1} f(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^{l_1} a(\eta)d\eta} d\xi + \frac{A}{\varepsilon}\int_0^{l} f(\xi)e^{-\frac{1}{\varepsilon}\int_\xi^{l} a(\eta)d\eta} d\xi}{1 + \gamma e^{-\frac{1}{\varepsilon}\int_0^{l_1} a(\eta)d\eta} - Ae^{-\frac{1}{\varepsilon}\int_0^{l} a(\eta)d\eta}}. \tag{2.5}$$

For $\gamma > 0$, $A > 0$, we deduce that

$$1 + \gamma e^{-\frac{1}{\varepsilon}\int_0^{l_1} a(\eta)d\eta} - Ae^{-\frac{1}{\varepsilon}\int_0^{l} a(\eta)d\eta} \geqslant 1 + \gamma e^{-\frac{a^* l_1}{\varepsilon}} - Ae^{-\frac{\alpha l}{\varepsilon}} \geqslant c_0 > 0.$$

It then follows from (2.5) we get

$$|u(0)| \leqslant c_0^{-1}\{|B| + \frac{|\gamma|}{\varepsilon}\int_0^{l_1} |f(\xi)|\, e^{-\frac{1}{\varepsilon}\int_\xi^{l_1} a(\eta)d\eta} d\xi + \frac{|A|}{\varepsilon}\int_0^{l} |f(\xi)|\, e^{-\frac{1}{\varepsilon}\int_\xi^{l} a(\eta)d\eta} d\xi\}$$

$$\leqslant c_0^{-1}\{|B| + \frac{|\gamma|}{\varepsilon}\|f\|_\infty \int_0^{l_1} e^{-\frac{\alpha(l_1-\xi)}{\varepsilon}} d\xi + \frac{|A|}{\varepsilon}\|f\|_\infty \int_0^{l} e^{-\frac{\alpha(l-\xi)}{\varepsilon}} d\xi\}$$

$$\leqslant c_0^{-1}\{|B| + \alpha^{-1}|\gamma|\,\|f\|_\infty (1 - e^{-\frac{\alpha l_1}{\varepsilon}}) + \alpha^{-1}|A|\,\|f\|_\infty (1 - e^{-\frac{\alpha l}{\varepsilon}})\}. \tag{2.6}$$

Next from (2.4) we see that

$$|u(x)| \leqslant |u(0)|\, e^{-\frac{\alpha}{\varepsilon}x} + \alpha^{-1}\|f\|_\infty (1 - e^{-\frac{\alpha}{\varepsilon}x})$$

which complete the proof (2.2).

To prove (2.3), first we estimate $u'(0)$ such that from (1.1)

$$|u'(0)| \leqslant \frac{1}{\varepsilon}|f(0) - a(0)u(0)| \leqslant \frac{C}{\varepsilon}. \tag{2.7}$$

Now, differentiating (1.1) we have

$$\varepsilon u''(x) + a(x)u'(x) = F(x) \tag{2.8}$$

where

$$F(x) = f'(x) - a'(x)u(x).$$

Since $|F(x)| \leqslant C$, it now follows from (2.8), (2.7) that

$$\left|u'(x)\right| \leqslant \left|u'(0)\right| e^{-\frac{1}{\varepsilon}\int_0^x a(\eta)d\eta} + \frac{1}{\varepsilon}\int_0^x |F(\xi)|\, e^{-\frac{1}{\varepsilon}\int_\xi^x a(\eta)d\eta} d\xi$$

$$\leqslant C\{\varepsilon^{-1}e^{-\frac{\alpha x}{\varepsilon}} + \alpha^{-1}(1 - e^{-\frac{\alpha x}{\varepsilon}})\}$$

which proves (2.3). ■

3 Discrete Problem

In what follows, we denote by ω a nonuniform mesh on Ω :

$$\omega_N = \{0 < x_1 < ... < x_{N-1} < l, h_i = x_i - x_{i-1}\}$$

and $\bar{\omega}_N = \omega_N \cup \{x_0 = 0, x_N = l\}$. Before describing our numerical method, we introduce some notation for the mesh functions. For any mesh function $g(x)$, we use

$$g_i = g(x_i),\ g_{\bar{x},i} = (g_i - g_{i-1})/h_i,\ \|g\|_\infty \equiv \|g\|_{\infty,\bar{\omega}_N} = \max_{0\leqslant i\leqslant N} |g_i|\,.$$

To obtain difference approximation for (1.1), we integrate (1.1) over (x_{i-1}, x_i):

$$\varepsilon u_{\bar{x},i} + h_i^{-1}\int_{x_{i-1}}^{x_i} a(x)u(x)dx = h_i^{-1}\int_{x_{i-1}}^{x_i} f(x)dx,$$

which yields relation

$$\varepsilon u_{\bar{x},i} + a_i u_i + R_i = f_i,\ 1 \leqslant i \leqslant N, \tag{3.1}$$

with the local truncation error

$$R_i = h_i^{-1}\int_{x_{i-1}}^{x_i} (x - x_{i-1})\frac{d}{dx}(f(x) - a(x)u(x))dx. \tag{3.2}$$

It is now necessary to define an approximation for the boundary condition of (1.2) . Let x_{N_0} be the mesh point nearest to l_1. Then, using Taylor formula with respect to x_{N_0}, we can write

$$u(x) = u(x_{N_0}) + (x - x_{N_0})u'(\xi),\ \xi \in (x_{N_0}, l_1). \tag{3.3}$$

Substituting $x = l_1$ into (3.3), for the boundary condition of (1.2), we obtain

$$u_0 + \gamma u_{N_0} + r(l_1) = Au_N + B \tag{3.4}$$

where

$$r(l_1) = (l_1 - x_{N_0})u'(\xi),\ \xi \in (x_{N_0}, l_1) \tag{3.5}$$

As a consequence of (3.1) and (3.4), we propose the following difference scheme for approximating (1.1)-(1.2):

$$\varepsilon y_{\bar{x},i} + a_i y_i = f_i,\ 1 \leqslant i \leqslant N, \tag{3.6}$$

$$y_0 + \gamma y_{N_0} = A y_N + B. \tag{3.7}$$

The difference scheme (3.6)-(3.7), in order to be ε-uniform convergent, we will use the Shishkin mesh. For an even number N, the piecewise uniform mesh takes $N/2$ points in the interval $[0,\sigma]$ and also $N/2$ points in the interval $[\sigma, l]$, where the transition point σ, which separates the fine and coarse portions of the mesh, is obtained by taking

$$\sigma = \min\left\{ l/2, \alpha^{-1}\varepsilon \ln N \right\}.$$

In practice, one usually has $\sigma \ll l$, so the mesh is fine on $[0,\sigma]$ and coarse on $[\sigma, l]$. Hence, if we denote by $h^{(1)}$ and $h^{(2)}$ the stepsizes in $[0,\sigma]$ and $[\sigma, l]$, respectively, we have

$$h^{(1)} = 2\sigma N^{-1},\ h^{(2)} = 2(l-\sigma)N^{-1},$$

$$h^{(1)} \leqslant lN^{-1},\ lN^{-1} \leqslant h^{(2)} \leqslant 2lN^{-1},\ h^{(1)} + h^{(2)} = 2lN^{-1},$$

so

$$\overline{\omega}_N = \begin{cases} x_i = ih^{(1)},\ i = 0,1,...,N/2, \\ x_i = \sigma + (i - N/2)h^{(2)},\ i = N/2+1,...,N. \end{cases} \tag{3.8}$$

In the rest of the paper we only consider this mesh.

4 Error analysis

Let $z_i = y_i - u_i$. Then the error in the numerical solution satisfies

$$\varepsilon z_{\bar{x},i} + a_i z_i = R_i,\ 1 \leqslant i \leqslant N, \tag{4.1}$$

$$z_0 + \gamma z_{N_0} = A z_N + r. \tag{4.2}$$

where the truncation error R_i and r are given by (3.2) and (3.5) respectively.

Lemma 4.1. *If $a(x), f(x) \in C^1(\bar{\Omega})$, then for the truncation error R_i we have*

$$\|R\|_{\infty,\omega_N} \leqslant CN^{-1}\ln N, \tag{4.3}$$

$$|r| \leqslant CN^{-1}\ln N. \tag{4.4}$$

Proof. From explicit expression (3.2) for R_i, on an arbitrary mesh we have

$$|R_i| \leqslant h_i^{-1} \int\limits_{x_{i-1}}^{x_i} (x - x_{i-1}) \left| \frac{d}{dx}(f(x) - a(x)u(x)) \right| dx$$

$$\leqslant Ch_i^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_{i-1})(1+|u'(x)|)dx,\ 1\leqslant i\leqslant N.$$

This inequality together with (2.3) enables us to write

$$|R_i|\leqslant C\{h_i+h_i^{-1}\varepsilon^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_{i-1})e^{-\alpha x/\varepsilon}dx\},\ 1\leqslant i\leqslant N, \tag{4.5}$$

in which

$$h_i=\begin{cases} h^{(1)}, & 1\leqslant i\leqslant N/2,\\ h^{(2)}, & N/2+1\leqslant i\leqslant N.\end{cases}$$

We consider first the case $\sigma=l/2$, and so $l/2<\alpha^{-1}\varepsilon\ln N$ and $h^{(1)}=h^{(2)}=h=lN^{-1}$. Hereby, since

$$h_i^{-1}\varepsilon^{-1}\int\limits_{x_{i-1}}^{x_i}(x-x_{i-1})e^{-\alpha x/\varepsilon}dx\leqslant\varepsilon^{-1}h\leqslant\frac{2\ln N}{\alpha l}\frac{l}{N}=2\alpha^{-1}N^{-1}\ln N,$$

it follows from (4.5) that

$$|R_i|\leqslant CN^{-1}\ln N,\ 1\leqslant i\leqslant N.$$

We now consider the case $\sigma=\alpha^{-1}\varepsilon\ln N$ and estimate R_i on $[0,\sigma]$ and $[\sigma,l]$ separately. In the layer region $[0,\sigma]$, inequality (4.5) reduces to

$$|R_i|\leqslant C(1+\varepsilon^{-1})h^{(1)}=C(1+\varepsilon^{-1})\frac{\alpha^{-1}\varepsilon\ln N}{N/2},\ 1\leqslant i\leqslant N/2.$$

Hence,

$$|R_i|\leqslant CN^{-1}\ln N,\ 1\leqslant i\leqslant N/2.$$

It remains to estimate R_i for $N/2+1\leqslant i\leqslant N$. In this case we are able to write (4.5) as

$$|R_i|\leqslant C\left\{h^{(2)}+\alpha^{-1}(e^{-\frac{\alpha x_{i-1}}{\varepsilon}}-e^{-\frac{\alpha x_i}{\varepsilon}})\right\},\ N/2+1\leqslant i\leqslant N. \tag{4.6}$$

Since $x_i=\alpha^{-1}\varepsilon\ln N+(i-N/2)h^{(2)}$ it follows that:

$$e^{-\frac{\alpha x_{i-1}}{\varepsilon}}-e^{-\frac{\alpha x_i}{\varepsilon}}=\frac{1}{N}e^{-\frac{\alpha(i-1-N/2)h^{(2)}}{\varepsilon}}(1-e^{-\frac{\alpha h^{(2)}}{\varepsilon}})<N^{-1}$$

and this together with (4.6) to give the bound

$$|R_i|\leqslant CN^{-1}.$$

Next, we estimate the remainder term r. Suppose that $l_1\in[0,\sigma]$, and from Lemma 2.1., the first derivative of u on this interval is bounded. From (3.5), we obtain

$$|r|=|l_1-x_{N_0}|\,|u'(\xi)|\leqslant Ch^{(1)}\leqslant CN^{-1}\ln N.$$

Similarly, for $l_1\in[\sigma,l]$, from (3.5), we get

$$|r|=|l_1-x_{N_0}|\,|u'(\xi)|\leqslant Ch^{(2)}\leqslant CN^{-1}\ln N.$$

Thus, the proof is completed. ■

Lemma 4.2. *Let z_i be the solution (4.1)-(4.2) and the problem data of (1.1)-(1.2) are such that*

$$1+\gamma\mu^*(1+a^*\rho_1)^{-N/2}-A[(1+\alpha\rho_1)(1+\alpha\rho_2)]^{-N/2}\geqslant c_*>0,$$

where

$$\mu^*=\begin{cases}(1+a^*\rho_2)^{N/2-N_0}, & N/2\leqslant N_0,\\ (1+a^*\rho_2)^{-N/2}, & N/2>N_0,\end{cases},\rho_k=h^{(k)}/\varepsilon,\ k=1,2.$$

Then the estimate

$$\|z\|_{\infty,\omega_N}\leqslant C(\|R\|_{\infty,\omega_N}+|r|). \tag{4.7}$$

holds.

Proof. From (4.1) we have

$$z_i=\frac{\varepsilon}{\varepsilon+a_ih_i}z_{i-1}+\frac{h_iR_i}{\varepsilon+a_ih_i}.$$

Solving the first-order difference equation with respect to z_i, we get

$$z_i=z_0Q_i+\sum_{k=1}^{i}\varphi_kQ_{i-k}, \tag{4.8}$$

where

$$Q_{i-k}=\begin{cases}1, & k=i,\\ \prod_{j=k+1}^{i}\frac{\varepsilon}{\varepsilon+a_jh_j}, & 0\leqslant k\leqslant i-1,\end{cases}\quad \varphi_i=\frac{h_iR_i}{\varepsilon+a_ih_i}.$$

From (4.2) and (4.8) we then obtain

$$\begin{aligned} z_0 &= \frac{A\sum_{k=1}^{N}\varphi_kQ_{N-k}-\gamma\sum_{k=1}^{N_0}\varphi_kQ_{N_0-k}-r}{1+\gamma Q_{N_0}-AQ_N}\\ &= \frac{A\sum_{k=1}^{N}\frac{h_kR_k}{\varepsilon+a_kh_k}Q_{N-k}-\gamma\sum_{k=1}^{N_0}\frac{h_kR_k}{\varepsilon+a_kh_k}Q_{N_0-k}-r}{1+\gamma Q_{N_0}-AQ_N}. \end{aligned} \tag{4.9}$$

It is not difficult to see that the absolute value of the numerator (4.9) is bounded by $C(\|R\|_{\infty,\omega_N}+|r|)$ and for $\gamma>0$, $A>0$ the denominator is bounded by c_* thereby will be

$$|z_0|\leqslant C(\|R\|_{\infty,\omega_N}+|r|). \tag{4.10}$$

Now, applying the discrete maximum principle to (4.1) and (4.9), we have

$$\|z\|_{\infty,\omega_N}\leqslant|z_0|+\alpha^{-1}\|R\|_{\infty,\omega_N}$$

which along with (4.10) leads to (4.7). ∎

Combining the previous lemmas yield the main result of the paper.

Theorem 4.3. *Let u be the solution of (1.1)-(1.2) and y the solution of (3.6)-(3.7).Then under hypotheses Lemmas 4.1 and 4.2*

$$\|y-u\|_{\infty,\bar{\omega}_N}\leqslant CN^{-1}\ln N.$$

5 Numerical results

We consider the test problems:

Example 5.1.

$$\varepsilon u'(x) + 4u(x) = 4,\ 0 < x < 1,$$

$$u(0) + \frac{1}{2}u(\frac{1}{2}) = 2u(1) - 1.$$

The exact solution is given by

$$u(x) = 1 - \frac{e^{-4x/\varepsilon}}{2 + e^{-2/\varepsilon} - 4e^{-4/\varepsilon}}.$$

Example 5.2.

$$\varepsilon u'(x) + (2x+6)u(x) = 2x + 6,\ 0 < x < 1,$$

$$u(0) + \frac{1}{3}u(\frac{1}{4}) = u(1).$$

The exact solution is given by

$$u(x) = 1 - \frac{e^{-(x+3)^2/\varepsilon}}{3e^{-9/\varepsilon} + e^{-169/(16\varepsilon)} - 3e^{-16/\varepsilon}}.$$

Example 5.3.

$$\varepsilon u'(x) + 2(x+4)u(x) = (2x+8)\cosh(\frac{x}{2}) + \frac{\varepsilon}{2}\sinh(\frac{x}{2}),\ 0 < x < 1,$$

$$u(0) + \frac{1}{6}u(\frac{1}{4}) = \frac{3}{2}u(1) - \frac{5}{4}.$$

The exact solution is given by

$$u(x) = \cosh(\frac{x}{2}) - \frac{(13.5 + \cosh(\frac{1}{8}) - 9\cosh(\frac{1}{2}))e^{-(x+4)^2/\varepsilon}}{6e^{-16/\varepsilon} + e^{-289/(16\varepsilon)} - 9e^{-25/\varepsilon}}.$$

We define the exact error e_ε^N and the computed parameter-uniform maximum pointwise error e^N as follows:

$$e_\varepsilon^N = \|y - u\|_{\infty,\bar{\omega}},\ e^N = \max_\varepsilon e_\varepsilon^N,$$

where y is the numerical approximation to u for various values of N and ε. We also define the computed parameter-uniform rate of convergence to be

$$p^N = \log_2\left(e^N/e^{2N}\right).$$

Particular values of ε for which we solve the test problems. The resulting errors e^N and the corresponding numbers p^N are listed in Tables 1-3.

Table 1

Exact errors e_ε^N and convergence rates p^N on ω_N of the Example 5.1

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-2}	0.0134246	0.0077302	0.0043916	0.0024792	0.0014118
	0.80	0.82	0.82	0.81	
2^{-4}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
2^{-6}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
2^{-8}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
2^{-10}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
2^{-12}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
2^{-14}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
2^{-16}	0.0133740	0.0076837	0.0043427	0.0024219	0.0013361
	0.80	0.82	0.84	0.86	
e^N	0.0134246	0.0077302	0.0043916	0.0024792	0.0014118
p^N	0.80	0.82	0.82	0.81	

Table 2

Exact errors e_ε^N and convergence rates p^N on ω_N of the Example 5.2

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-2}	0.0092455	0.0052731	0.0029896	0.0016945	0.0009702
	0.81	0.82	0.82	0.80	
2^{-4}	0.0093547	0.0053032	0.0029713	0.0016473	0.0009048
	0.82	0.84	0.85	0.86	
2^{-6}	0.0094089	0.0053359	0.0029907	0.0016585	0.0009114
	0.82	0.84	0.85	0.86	
2^{-8}	0.0094226	0.0053442	0.0029956	0.0016614	0.0009131
	0.82	0.84	0.85	0.86	
2^{-10}	0.0094260	0.0053463	0.0029969	0.0016622	0.0009135
	0.82	0.84	0.85	0.86	
2^{-12}	0.0094268	0.0053468	0.0029972	0.0016623	0.0009136
	0.82	0.84	0.85	0.86	
2^{-14}	0.0094271	0.0053470	0.0029973	0.0016624	0.0009137
	0.82	0.84	0.85	0.86	
2^{-16}	0.0094272	0.0053470	0.0029973	0.0016624	0.0009137
	0.82	0.84	0.85	0.86	
e^N	0.0094272	0.0053470	0.0029973	0.0016945	0.0009702
p^N	0.81	0.82	0.82	0.80	

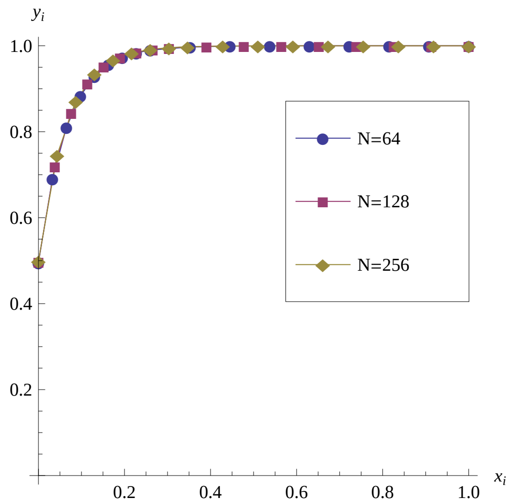

Figure 1: Numerical solution of Example 5.1 for $\varepsilon = 2^{-2}$

Table 3
Exact errors e_ε^N and convergence rates p^N on ω_N of the Example 5.3

ε	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
2^{-2}	0.0162093	0.0094311	0.0052590	0.0027941	0.0013880
	0.78	0.84	0.91	1.01	
2^{-4}	0.0163087	0.0094805	0.0052808	0.0028008	0.0013872
	0.78	0.84	0.91	1.01	
2^{-6}	0.0163363	0.0094956	0.0052889	0.0028050	0.0013891
	0.78	0.84	0.91	1.01	
2^{-8}	0.0163432	0.0094993	0.0052909	0.0028060	0.0013896
	0.78	0.84	0.91	1.01	
2^{-10}	0.0163449	0.0095003	0.0052914	0.0028063	0.0013898
	0.78	0.84	0.91	1.01	
2^{-12}	0.0163453	0.0095005	0.0052915	0.0028063	0.0013898
	0.78	0.84	0.91	1.01	
2^{-14}	0.0163454	0.0095005	0.0052915	0.0028063	0.0013898
	0.78	0.84	0.91	1.01	
2^{-16}	0.0163455	0.0095006	0.0052915	0.0028063	0.0013898
	0.78	0.84	0.91	1.01	
e^N	0.0163455	0.0095006	0.0052915	0.0028063	0.0013898
p^N	0.78	0.84	0.91	1.01	

6 Conclusion

The singularly perturbed boundary-value problem for a linear first order differential equation with nonlocal condition is considered. To solve this problem, a uniform finite difference method on a special piecewise uniform mesh (also known as Shishkin mesh) is presented. First order convergence except for a loga-

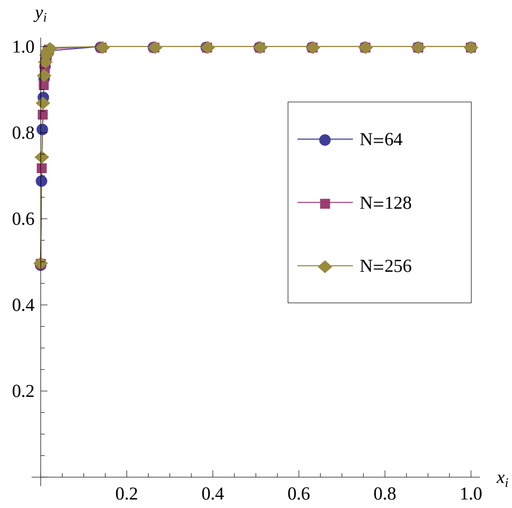

Figure 2: Numerical solution of Example 5.1 for $\varepsilon = 2^{-6}$

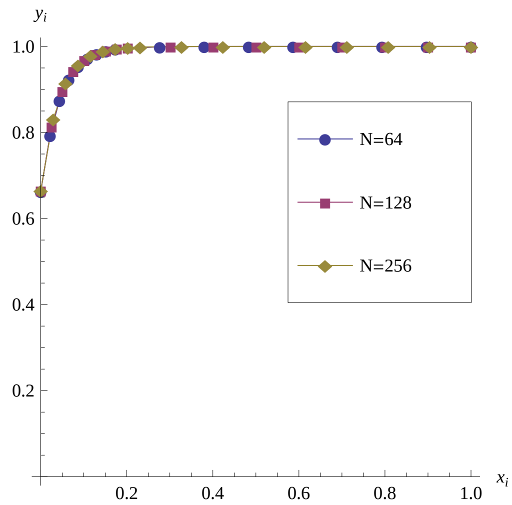

Figure 3: Numerical solution of Example 5.2 for $\varepsilon = 2^{-2}$

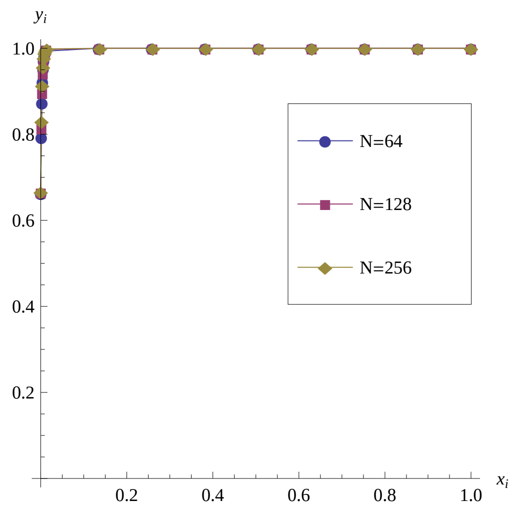

Figure 4: Numerical solution of Example 5.2 for $\varepsilon = 2^{-6}$

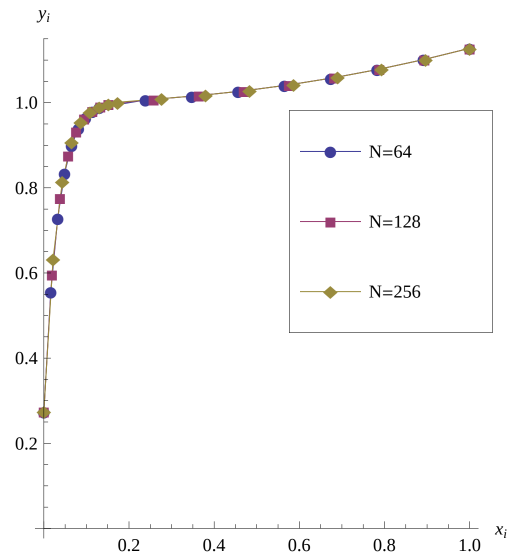

Figure 5: Numerical solution of Example 5.3 for $\varepsilon = 2^{-2}$

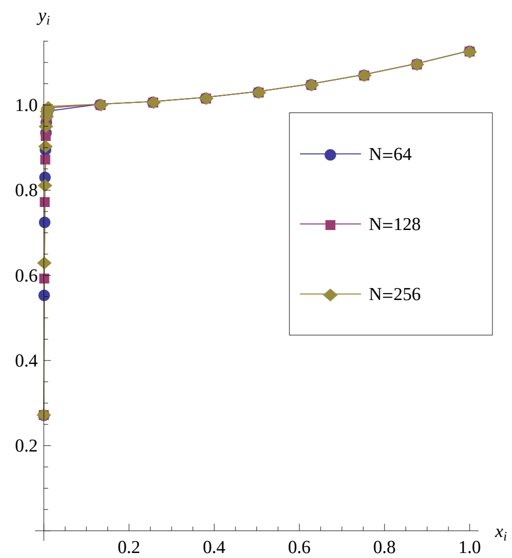

Figure 6: Numerical solution of Example 5.3 for $\varepsilon = 2^{-6}$

rithmic factor, in the discrete maximum norm, independently of the perturbation parameter is obtained. The exact errors and rates of convergence are tabulated in Tables for the considered test problems examples in support of the theoretical results. The graphs of the numerical solution of the examples for different values of perturbation parameter are plotted in Figs. 1-6. It is easy to see that the test problems have a boundary layer only at $x = 0$.

Acknowledgments

The authors are grateful to the referee for his/her valuable comments and suggestions which helped improve the quality of manuscript.

References

[1] A. H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1993.

[2] R. E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations*, Springer Verlag, New York, 1991.

[3] E. R. Doolan, J. J. H. Miller, W. H. A. Schilders, *Uniform Numerical Methods for Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.

[4] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*, World Scientific, Singapore, 1996.

[5] H. G. Roos, M. Stynes, L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations*, Springer-Verlag, Berlin, 2008.

[6] P. A. Farrel, A. F. Hegarty, J. J. H. Miller, E. O'Riordan, G. I. Shishkin, *Robust Computational Techniques for Boundary Layers*, Chapman Hall/CRC, New York, 2000.

[7] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal cauchy problem*, J. Math. Anal. Appl., **162** (1991), pp.494–501.

[8] S.N. Glazatov, *Nonlocal boundary value problems for linear and nonlinear equations of variable type*, Sobolev Institue of Mathematics SB RAS, **46** (1998), pp.1–26.

[9] V.A. Il'in, *Optimization of the boundary control by a displacement or by an elastic force on one end of a string under a model nonlocal boundary condition*, Proc. Steklov Inst. Math., **268** (2010), pp.117–129.

[10] T. Jankowski, *Multipoint boundary value problems for ODEs. Part I*, Appl. Anal., **80** (2001), pp.395–407.

[11] T. Jankowski, *Multipoint boundary value problems for ODEs. Part II*, Czec. Math. J., **54**, no.129 (2004), pp.843–854.

[12] T. Jankowski, *Existence of solutions of differential equations with nonlinear multipoint boundary conditions*, Comput. Math. Appl., **47** (2004), pp.1095–1103.

[13] S.K. Ntouyas, *Nonlocal initial and boundary value problems: A survey*, In: Cañada A, Drábek P, Fonda A, editors. *Handbook of Differential Equations: Ordinary Differential Equations Vol. 2.*, Elsevier, Amsterdam, 2005, pp. 461–558.

[14] M. Benchohra, S.K. Ntouyas, *Existence of solutions of nonlinear differential equations with nonlocal conditions*, J. Math. Anal. Appl., **252** (2000), pp.477–483.

[15] J.J. Nieto, R.R. Lopez, *Green's function for first-order multipoint boundary value problems and applications to the existence of solutions with constant sign*, J. Math. Anal. Appl., **388** (2012), pp.952–963.

[16] A. Stikonas, *A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions*, Nonlinear Analysis: Model Control, **19** 2014, pp.301–334.

[17] M. Cakir, G.M. Amiraliyev, *Numerical solution of the singularly perturbed problem with nonlocal boundary condition*, Appl. Math. Mech. Eng. Edt., **23** (2002), pp.755–764.

[18] M. Cakir, G.M. Amiraliyev, *Numerical solution of a singularly perturbed three-point boundary value problem*, Int. J. Comput. Math., **84** (2007), pp.1465–1481.

[19] M. Cakir, *Uniform second-order difference method for a singularly perturbed three-point boundary value problem*, Adv. Diff. Eqs., **2010** (2010), Article ID 102484, 13 pages.

Department of Mathematics, Faculty of Education,
Yuzuncu Yil University,
65080, Van, Turkey
email : cimenerkan@hotmail.com

Department of Mathematics, Faculty of Science,
Yuzuncu Yil University,
65080, Van, Turkey
email : cakirmusa@hotmail.com