Non-Weakly Supercyclic Classes of Weighted Composition Operators on Banach Spaces of Analytic Functions

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Abstract

We present a non-weak supercyclicity criterion for vectors in infinite dimensional Banach spaces. Also, we give sufficient conditions under which a class of weighted composition operators on a Banach space of analytic functions is not weakly supercyclic. In particular, we show that the semigroup of linear isometries on the spaces S^p ($p > 1$), is not weakly supercyclic. Moreover, we observe that every composition operator on some Banach space of analytic functions such as the disc algebra or the analytic Lipschitz space is not weakly supercyclic.

1 Introduction and preliminary results

Let X be a Banach space over the field of complex numbers and $\dim X > 1$. By $B(X)$ we mean the set of all bounded linear operators on X. A set $\Gamma \subseteq B(X)$ is hypercyclic (supercyclic) if there exists a vector $x \in X$ such that $O(x, \Gamma) = \{Tx : T \in \Gamma\}$ ($\mathbb{C}.O(x, \Gamma) = \{\lambda Tx : T \in \Gamma, \lambda \in \mathbb{C}\}$) is a dense subset of X. An operator $T \in B(X)$ is called hypercyclic (supercyclic) if the semigroup $\Gamma = \{T^n : n \in \mathbb{N}_0\}$ is hypercyclic (supercyclic). Here $\mathbb{N}_0 = \{0, 1, 2, 3, ...\}$. Similarly, by considering density in the weak topology instead of the norm topology,

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we can define weak hypercyclicity and weak supercyclicity. A set $\Gamma \subseteq B(X)$ is weakly hypercyclic (weakly supercyclic) if there exists a vector $x \in X$ such that $O(x,\Gamma) = \{Tx : T \in \Gamma\}$ ($\mathbb{C}.O(x,\Gamma) = \{\lambda Tx : T \in \Gamma, \lambda \in \mathbb{C}\}$) is a weakly dense subset of X. An operator $T \in B(X)$ is called weakly hypercyclic (weakly supercyclic) if the semigroup $\Gamma = \{T^n : n \in \mathbb{N}_0\}$ is weakly hypercyclic (weakly supercyclic).

Let $\mathbb{D}$ denote the open unit disc in the complex plane. By a Banach space of analytic functions we mean a Banach space consisting of analytic functions that contains the constant functions and the identity function such that the linear functional of point evaluation at λ defined by $e_\lambda(f) = f(\lambda)$ is bounded for every $\lambda \in \mathbb{D}$. By the Cauchy integral formula, it is not surprising that evaluation of the derivative at each point λ of the disc, which we denote by e'_λ, is a bounded linear functional. In the following, some examples of classical Banach spaces of analytic functions are presented. Note that the class of all analytic functions on a simply connected domain Ω of the complex plane $\mathbb{C}$ will be denoted by $H(\Omega)$, endowed with the compact open topology.

1. The space of all bounded analytic functions on $\mathbb{D}$, denoted by H^∞ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.

2. The Hardy space H^p, $1 \leq p < \infty$, consists of those functions f in $H(\mathbb{D})$ for which $\|f\|^p_{H^p} = \sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^p dm(z) < \infty$ where dm is the normalized arc length measure on $\mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

3. The space S^p, $1 \leq p < \infty$, consisting of all analytic functions f on $\mathbb{D}$ for which

$$\|f\| = \|f\|_\infty + \|f'\|_{H^p} < \infty.$$

4. The weighted Bergman space A^p_α ($1 \leq p < \infty$, $\alpha > -1$) is defined as the space of all f in $H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha (\alpha+1) dA(z)\right)^{\frac{1}{p}} < \infty,$$

where dA is the normalized Lebesgue area measure on $\mathbb{D}$. The space A^p_0 is called the Bergman space and is denoted by L^p_a.

5. The weighted Dirichlet-type space $\mathcal{D}^p_\alpha$ ($1 \leq p < \infty$, $-1 < \alpha < \infty$) is the space of $f \in H(\mathbb{D})$ such that $f' \in A^p_\alpha$ equipped with the norm

$$\|f\|_{\mathcal{D}^p_\alpha} = |f(0)| + \|f'\|_{A^p_\alpha} < \infty.$$

6. The space $H^\infty_{\nu_p}(\mathbb{D})$, $(p > 0)$ with the standard weights $\nu_p(z) = (1-|z|^2)^p$ consisting of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\nu_p} = \sup_{z\in\mathbb{D}} \nu_p(z)|f(z)| < \infty.$$

7. The analytic Besov space B^p, $1 < p < \infty$, is defined as the set of all analytic functions on the disc such that

$$\|f\|^p_{B^p} = |f(0)|^p + (p-1)\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{(p-2)}dA(z) < \infty.$$

8. The disc algebra $A(\mathbb{D})$, is the Banach space of functions that are continuous on the closed unit disc and analytic on the open unit disc, with the supremum norm.

9. The analytic Lipschitz space $\mathrm{Lip}_\alpha(\mathbb{D})$, $(0 < \alpha \leq 1)$, is the set

$$\{f \text{ analytic in } \mathbb{D} \;:\; |f(z) - f(w)| = O(|z-w|^\alpha) \text{ for all } z, w \in \overline{\mathbb{D}}\},$$

with the norm

$$\|f\|_\alpha = |f(0)| + \sup\{\frac{|f(z)-f(w)|}{|z-w|^\alpha} \;:\; z \neq w \in \partial\mathbb{D}\}.$$

10. Let H be a Hilbert space whose vectors are functions analytic on $\mathbb{D}$ and the monomials $1, z, z^2, \ldots$ constitute a complete orthogonal set of non-zero vectors in H. Writing $\beta(j) = \|z^j\|$, $j \geq 1$ with the normalization $\beta(0) = 1$, the inner product on H given by

$$\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \beta(n)^2.$$

The space H is called the weighted Hardy space with weight $\beta = (\beta(n))_{n\geq 0}$ and will be denoted by $H^2(\beta, \mathbb{D})$. See Pages 14 and 16 of [8].

Let φ be an automorphism of the disc. Recall that φ is elliptic if it has one fixed point in the disc and the other in the complement of the closed disc. Moreover, any elliptic automorphism φ, with fixed point $a \in \mathbb{D}$ is conformally equivalent to λz where $\lambda = \varphi'(a)$ [8, Page 59]. For $c \in \mathbb{D}$, let $\varphi_c(z) = (c-z)/(1-\bar{c}z)$, $(z \in \mathbb{D})$. Throughout this paper, $\mathrm{Aut}(\mathbb{D})$ denotes the set of all automorphisms of $\mathbb{D}$. Recall that a multiplier of a Banach space of analytic functions X is an analytic function w on $\mathbb{D}$ such that $wX \subseteq X$. The set of all multipliers of X is denoted by $M(X)$. If w is a multiplier, then the multiplication operator M_w, defined by $M_w f = wf$, is bounded on X. In what follows, suppose that $w \in M(X)$ and φ is an analytic self map of $\mathbb{D}$ such that $(f \circ \varphi)(z) = f(\varphi(z))$ is in X for every $f \in X$. An application of the closed graph theorem shows that the weighted composition operator $C_{w,\varphi}$ defined by $C_{w,\varphi}(f)(z) = M_w C_\varphi(f)(z) = w(z)f(\varphi(z))$ is bounded. The map φ is called the composition map and w is called the weight. Throughout this paper, each self map of $\mathbb{D}$ is analytic.

In 1964, Forelli [11] showed that every isometry on H^p for $1 < p < \infty$ and $p \neq 2$ is a weighted composition operator. Recently, there has been a great interest in studying composition and weighted composition operators on the unit disc, polydisc, or the unit ball; see, for example, the monographs [8, 30], and the papers [4, 14].

Rolewicz [27] has shown that any scalar multiple λB of the unilateral backward shift B is hypercyclic on ℓ_p $(1 \leq p < \infty)$ whenever $|\lambda| > 1$. On the other hand, in Kitai's dissertation [20], the linear dynamics, as a branch of functional analysis, was born. Recently, hypercyclic and supercyclic operators have been the focus of much work in linear dynamics. The reader can see [2] to get more information about hypercyclic and supercyclic operators. Sufficient conditions under which an operator is not weakly supercyclic are given by Montes-Rodríguez and Shkarin in [23]. Furthermore, Shkarin has studied non-sequential weak supercyclicity and hypercyclicity of operators on a Banach space in [29]. Recently, Hedayatian and Faghih-Ahmadi have shown that every operator in the commutant of a cyclic convolution operator on the Hardy space H^p, $p \geq 1$ is not weakly supercyclic [16]. Moreover, several authors have studied the dynamics of a general weighted composition operator $C_{w,\varphi}$. Recently, the hypercyclicity and supercyclicity of composition operators on $H(\Omega)$ have been investigated with respect to the compact-open topology [5]. Also, the study of hypercyclicity and supercyclicity of composition operators on $H(\mathbb{D})$ with respect to the weak topology and to its corresponding compact-open topology, have been explored in [19, 31]. Rezaei [26] studies composition operators that are chaotic in the sense of Devaney.

Recall that isometries cannot be supercyclic on Banach spaces [1, 22]. On the other hand, they can be weakly supercyclic. For example, surjective linear isometries can be weakly supercyclic on $\ell^p(\mathbb{Z})$, $(p > 2)$ [2, Page 253], and a unitary Hilbert space operator also can be weakly supercyclic [3]. The set of all unitary operators on a Hilbert space is supercyclic [10, Page 183] and so is weakly supercyclic. Also, there are examples of supercyclic groups of isometries on Banach spaces. The group of all isometries on $L^p(\mu)$ $(1 \leq p < \infty)$ where μ is a homogeneous measure is supercyclic [13]. Moreover, for $1 \leq p < \infty$, the group of all isometries on the Banach space $L^p(X, \mu)$ is supercyclic, where X is the disjoint union of an uncountable family of copies of the interval $[0, 1]$ and μ is a certain measure on X [15]. It has been shown that the semigroup of linear isometries on the space S^p $(p > 1)$, the group of all surjective linear isometries on the Hardy space H^p and the Bergman space L^p_a $(1 < p < \infty, p \neq 2)$ are not supercyclic [24]. In Section 2, we give sufficient conditions for non-weak supercyclicity of vectors in an infinite dimensional Banach space. In particular, we observe that the semigroup of linear isometries on the space $S^p(p > 1)$ is not weakly supercyclic. In Section 3, we study the weak supercyclicity behavior of certain classes of weighted composition operators on some spaces of analytic functions on the unit disc $\mathbb{D}$. Also, we show that the class $\{C_\varphi : C_\varphi \text{ is an isometry}\}$ is not weakly supercyclic on the Hardy space H^p, the weighted Bergman space A^p_α $(1 \leq p < \infty)$, the analytic Besov space B^p $(2 < p < \infty)$, and the space $H^\infty_{v_p}(\mathbb{D})$ $(p > 0)$. Moreover, we observe that every composition operator on some Banach spaces of analytic functions such as the disc algebra or the analytic Lipschitz space is not weakly supercyclic. Furthermore, sufficient conditions are given for non-weak supercyclicity of a weighted composition operator.

2 Infinite dimensional reflexive Banach spaces

The angle-criterion for supercyclicity states that for a $T \in B(X)$ and $x \in X$, if there is a non-zero $x^* \in X^*$ such that

$$\limsup_{n\to\infty} \frac{|x^*(T^n x)|}{\|T^n x\| \|x^*\|} \neq 1$$

then x is not a supercyclic vector for T. Using this criterion it is shown that the classical Volterra operator and the composition operators associated with parabolic non-automorphisms of the unit disc $\mathbb{D}$ are not supercyclic, (see for example, Section 9.1 of [2]) . In this section, we give a non-weak supercyclicity criterion for vectors in an infinite dimensional reflexive Banach space which is, to some extent, similar to the angle-criterion but has a quite different method of proof. By applying this criterion we show that various operators are not weakly supercyclic.

Given a directed set I, a Banach space X, $x_i \in X$ $(i \in I)$ and $x \in X$, we use the expression $x_i \xrightarrow{w} x$ to denote that the net $(x_i)_{i\in I}$ converges to x with respect to the weak topology of X. Also, the weak operator topology (WOT) on $B(X)$ is the one in which a net (T_α) converges to T if and only if $T_\alpha(x) \longrightarrow T(x)$ weakly for all $x \in X$.

Theorem 1. *Let X be an infinite dimensional reflexive Banach space, Γ be a subset of $B(X)$ and x be a non-zero vector in X. If there exists a linear functional $x^* \in X^*$ such that*

$$c = \inf\left\{\frac{|x^*(Tx)|}{\|T\|} : \; T \in \Gamma\right\} > 0,$$

then x is not a weakly supercyclic vector for Γ. In addition, the set of all weakly supercyclic vectors for Γ is not norm dense in X.

Proof. Since the weak supercyclicity of $\{\frac{T}{\|T\|} : T \in \Gamma\}$ is equivalent to the weak supercyclicity of Γ, one can assume that $\|T\| = 1$ for all $T \in \Gamma$. Suppose, on the contrary, that x is a weakly supercyclic vector for Γ. Therefore, for any $y \in X$ there are two nets $(\alpha_i)_i$ in $\mathbb{C}$ and $(T_i)_i$ in Γ such that $\alpha_i T_i(x) \xrightarrow{w} y$. Thus, there exists j so that

$$|x^*(\alpha_i T_i(x)) - x^*(y)| < 1 \;\text{ for all } i > j,$$

which in turn implies that

$$|\alpha_i| \leq c^{-1}(|x^*(y)| + 1) \text{ for all } i > j.$$

By passing to a subnet if necessary, we assume that $\lim_i \alpha_i = \alpha$ exists. Since the unit ball of $B(X)$ is WOT-compact, $(T_i)_i$ has a WOT-convergent subnet $(T_{i_k})_k$ with limit $T \in \overline{\Gamma}^{WOT}$. So we observe that $\alpha_{i_k} T_{i_k}(x) \xrightarrow{w} \alpha T x$ and hence $\alpha T x = y$, which implies that

$$X = \left\{ \alpha T x : \alpha \in \mathbb{C},\ T \in \overline{\Gamma}^{WOT} \right\}.$$

Since the weak closure of the unit sphere of X is ball(X), there exists a net $(x_\beta)_\beta$ of unit vectors weakly converging to zero. On the other hand, for each β there is an operator $S_\beta \in \overline{\Gamma}^{WOT}$ and $\lambda_\beta \in \mathbb{C}$ such that $\lambda_\beta S_\beta(x) = x_\beta$. Thus

$$c|\lambda_\beta| \leq |\lambda_\beta||x^*(S_\beta(x))| = |x^*(x_\beta)|$$

and

$$1 = \| \lambda_\beta S_\beta(x) \| \leq |\lambda_\beta| \, \| x \|$$

show that

$$\frac{1}{\|x\|} \leq |\lambda_\beta| \leq c^{-1}\|x^*\|.$$

It follows that there exist subnets $(\lambda_{\beta_i})_i$ and $(S_{\beta_i})_i$ such that $\lambda_{\beta_i} \longrightarrow \lambda$, for some non-zero constant λ, and $S_{\beta_i} \xrightarrow{WOT} S \in \overline{\Gamma}^{WOT}$. Hence

$$0 < |\lambda||c| \leq |\lambda||x^*(Sf)| = \lim_i |\lambda_{\beta_i} x^*(S_{\beta_i} x)| = \lim_i |x^*(x_{\beta_i})| = 0$$

which is absurd.

For the next part put $V = \{y \in X : \|y - x\| < c/2\|x^*\|\}$. Therefore, for $y \in V$ and $T \in \Gamma$

$$\begin{aligned} |x^*(Ty)| &\geq |x^*(Tx)| - |x^*(Ty - Tx)| \\ &\geq c\|T\|/2. \end{aligned}$$

Thus, y is not a weakly supercyclic vector for Γ. ∎

Corollary 1. *Let X be an infinite dimensional reflexive Banach space, Γ be a bounded subset of $B(X)$ and x be a non-zero vector in X. If there exists a linear functional $x^* \in X^*$ such that*

$$c = \inf\{|x^*(Tx)| : \ T \in \Gamma\} > 0,$$

then x is not a weakly supercyclic vector for Γ. Moreover, the set of all weakly supercyclic vectors for Γ is not norm dense in X.

In the preceding theorem let x be the constant function 1 and x^* be the evaluation functional at a. Then we have the following corollaries.

Corollary 2. *Let X be an infinite dimensional reflexive Banach space of analytic functions on the open unit disc $\mathbb{D}$ and Γ be a class of weighted composition operators on X which is bounded. Suppose that there is $a \in \mathbb{D}$ and $\varepsilon > 0$ such that $|w(a)| > \varepsilon$ for all weighted maps w such that $C_{w,\varphi} \in \Gamma$, for some composition map φ. Then the set of weakly supercyclic vectors for Γ is not norm dense in X.*

Corollary 3. *The set of weakly supercyclic vectors for any bounded set of composition operators on an infinite dimensional reflexive Banach space of analytic functions on $\mathbb{D}$ is not norm dense in X.*

Let φ be a self map on $\mathbb{D}$. For a positive integer n, the nth iterate of φ is denoted by φ_n and φ_0 is the identity function. Also, for a weighted composition operator $C_{w,\varphi}$ on a Banach space of analytic function X, we have

$$C^n_{w,\varphi}(f) = \Pi_{j=0}^{n-1} w \circ \varphi_j \,.\, (f \circ \varphi_n),$$

for all $f \in X$ and $n \geq 1$. Also, recall that an operator T is a power bounded operator, if there exists a positive number M such that $\|T^n\| \leq M$ for all $n \geq 0$. Taking $\Gamma = \{C^n_{w,\varphi} : n \geq 0\}$ in Corollary 1, with x^* the evaluation functional at a we obtain the following result.

Corollary 4. *Let $C_{w,\varphi}$ be a weighted composition operator on an infinite dimensional reflexive Banach space of analytic functions. Suppose that φ has a fixed point $a \in \mathbb{D}$ such that $|w(a)| \geq 1$.*

1. *If $C_{w,\varphi}$ is power bounded then it is not weakly supercyclic.*

2. *If $C_{w,\varphi}$ is weakly supercyclic then $\|C_{w,\varphi}\| > 1$.*

Proposition 1. *Let X be an infinite dimensional reflexive Banach space and T be a power bounded operator on X which has a non-zero fixed point. Then T is not weakly supercyclic.*

Proof. Assume on the contrary that T is weakly supercyclic. By [28, Proposition 2.1] the set of weakly supercyclic vectors for T is norm dense in X. Let x be a non-zero fixed point of T. By the Hahn-Banach Theorem there is $x^* \in X^*$ such that $x^*(x) \neq 0$. Hence $|x^*(T^n x)| = |x^*(x)| > 0$ for all $n \geq 0$ and by Theorem 1 we have a contradiction. ∎

Since $C_\varphi 1 = 1$ we have the following result.

Corollary 5. *Suppose that X is an infinite dimensional reflexive Banach space of analytic functions. If C_φ is weakly supercyclic on X then $\|C_\varphi\| > 1$.*

Corollary 6. *Let X be the Hardy space H^p or the Bergman space L^p_a, $(1 < p < \infty)$. If φ, not the identity and not an elliptic automorphism, has a fixed point $a \in \mathbb{D}$, then C_φ is not weakly supercyclic.*

Proof. It is known that $\lim_{n\to\infty} \varphi_n(0) = a$ (see page 59 of [8]). Moreover, by Corollary 3.7 of [8] and Theorem 10.3.2 of [32],

$$\|C^n_\varphi\| \leq \left(\frac{1 + |\varphi_n(0)|}{1 - |\varphi_n(0)|}\right)^\beta,$$

hence

$$\lim_{n\longrightarrow\infty} \|C^n_\varphi\| \leq \left(\frac{1 + |a|}{1 - |a|}\right)^\beta,$$

where $\beta = \frac{1}{p}$ for the space H^p and $\beta = \frac{2}{p}$ for the space L^p_a. Thus, the result follows from Proposition 1 or Corollary 4. ∎

Definition 1. *Let X be a Banach space of analytic functions on $\mathbb{D}$. A net $(T_i)_i$ in $B(X)$, is said to converge pointwise evaluation to $T \in B(X)$ if $\lim_{i\to\infty}(T_i f)(z) = (Tf)(z)$ for all $f \in X$ and all $z \in \mathbb{D}$; this property is denoted by $T_i \xrightarrow{p.w.e.} T$.*

Theorem 2. *Let X be an infinite dimensional Banach space of analytic functions on $\mathbb{D}$ and $f \in X$. Suppose that Γ is a subset of $B(X)$ such that each net $(T_i)_i$ in Γ has a subnet which converges pointwise evaluation to some $T \in \Gamma$ and moreover, Γ is bounded away from 0 and ∞, i.e., there exist two positive constants c_1 and c_2 such that $c_1 \le ||T|| \le c_2$ for all $T \in \Gamma$. If there exists $\lambda \in \mathbb{D}$ such that*

$$c = \inf\left\{\frac{|(Tf)(\lambda)|}{\|T\|} : T \in \Gamma\right\} > 0,$$

then f is not a weakly supercyclic vector for Γ. In addition, the set of weakly supercyclic vectors for Γ is not norm dense in X.

Proof. Suppose that on the contrary, f is a weakly supercyclic vector for Γ. Therefore, for any $g \in X$ there are two nets $(\alpha_i)_i$ in $\mathbb{C}$ and $(T_i)_i$ in Γ such that $\alpha_i T_i(f) \xrightarrow{w} g$. Thus, there exists j so that

$$|(\alpha_i T_i(f))(\lambda) - g(\lambda)| < 1 \quad \text{for all} \quad i > j$$

which in turn implies that

$$|\alpha_i| \le (c_1 c)^{-1}\left(|g(\lambda)| + 1\right) \quad \text{for all} \quad i > j.$$

By passing to a subnet if necessary, we assume that $\lim_i \alpha_i = \alpha$ exists. By hypothesis there is a subnet $(T_{i_k})_k$ so that $T_{i_k} \xrightarrow{p.w.e.} T$ for some $T \in \Gamma$. So we observe that $(\alpha_{i_k} T_{i_k}(f))(z) \to (\alpha Tf)(z)$ for all $z \in \mathbb{D}$ and hence $\alpha Tf = g$, which implies that

$$X = \{\alpha Tf : \alpha \in \mathbb{C}, T \in \Gamma\}.$$

Since the weak closure of the unit sphere of X is ball(X), there exists a net $(f_\beta)_\beta$ of unit vectors weakly converging to zero. On the other hand, for each β there is an operator $S_\beta \in \Gamma$ and $\lambda_\beta \in \mathbb{C}$ such that $\lambda_\beta S_\beta(f) = f_\beta$. Thus

$$cc_1|\lambda_\beta| \le c|\lambda_\beta|||S_\beta|| \le |\lambda_\beta||S_\beta(f)(\lambda)| = |f_\beta(\lambda)| = |e_\lambda(f_\beta)|$$

which implies that

$$\frac{1}{c_2\|f\|} \le |\lambda_\beta| \le \frac{\|e_\lambda\|}{c_1 c}.$$

It follows that there exist subnets (λ_{β_i}) and $(S_{\beta_i})_i$ such that $\lambda_{\beta_i} \to \lambda'$, for some non-zero constant λ', and $S_{\beta_i} \xrightarrow{p.w.e.} S \in \Gamma$. Hence

$$0 < c_1|\lambda'||c| \le |\lambda'||(Sf)(\lambda)| = \lim_i |\lambda_{\beta_i}(S_{\beta_i}f)(\lambda)| = \lim_i |f_{\beta_i}(\lambda)| = \lim_i |e_\lambda(f_{\beta_i})| = 0$$

which is absurd. For the next part put $V = \{g \in X : \|g - f\| < c/2\|e_\lambda\|\}$. Therefore, for $g \in V$ and $T \in \Gamma$

$$\begin{aligned} |Tg(\lambda)| &\geq |Tf(\lambda)| - |Tg(\lambda) - Tf(\lambda)| \\ &\geq c\|T\|/2. \end{aligned}$$

Thus, g is not a weakly supercyclic vector for Γ. ∎

There exists a norm equivalent to the original norm of a Banach space X such that the group of all surjective linear isometries on X with the new norm consists only of unimodular scalars of the identity [18]. Thus, one can find a Banach space on which the group of surjective linear isometries is not supercyclic; even stronger is not weakly supercyclic. Recall that the big Bloch space $\mathcal{B}$ is the set of all analytic functions f on $\mathbb{D}$ such that $f(0) = 0$ and

$$\|f\| = \sup\{|f'(z)|(1 - |z^2|) : z \in \mathbb{D}\} < \infty.$$

Also, the little Bloch space $\mathcal{B}_0$, is the subspace of $\mathcal{B}$ spanned by the polynomials. The set of all linear isometries on the little Bloch space $\mathcal{B}_0$ and the set of all surjective linear isometries on the big Bloch space $\mathcal{B}$ has the form $\Gamma = \{\lambda(C_\varphi - Z_\varphi) :$ φ is a rotation, $|\lambda| = 1\}$ where $Z_\varphi(f) = f(\varphi(0))$ [9]. Since e_0 is continuous, there is no weakly supercyclic vector for this spaces. Therefore, the semigroup of linear isometries on the little Bloch space $\mathcal{B}_0$ and the group of surjective linear isometries on the big Bloch space $\mathcal{B}$ are not weakly supercyclic. On the other hand, the group of all unitary operators on a Hilbert space is supercyclic. Thus, it is natural to investigate whether the semigroup of linear isometries or the group of surjective linear isometries on a Banach space is weakly supercyclic or not.

Proposition 2. *The semigroup of linear isometries on the space S^p, $p > 1$, is not weakly supercyclic.*

Proof. The set of all isometries on S^p, $p > 1$ is of the form

$$\Gamma = \{\beta C_\varphi : \varphi \text{ is a rotation and } |\beta| = 1\}$$

[25]. Suppose on the contrary that f is a weakly supercyclic vector of Γ. Since point evaluations are continuous, $\{\alpha f(0) : \alpha \in \mathbb{C}\}$ is dense in $\mathbb{C}$. Thus $f(0) \neq 0$. Let $T_i = (\beta_i C_{\varphi_i})_i$ be an arbitrary net in Γ, where $|\beta_i| = 1$ and $\varphi_i(z) = \lambda_i z$ for some λ_i with $|\lambda_i| = 1$ and each $z \in \mathbb{D}$. There exist two subnets $(\beta_{i_j})_j$ and $(\lambda_{i_j})_j$ such that $\beta_{i_j} \longrightarrow \beta$ and $\lambda_{i_j} \longrightarrow \lambda$ for some numbers $|\beta| = 1$ and $|\lambda| = 1$. Hence $T_{i_j} \overset{p.w.e.}{\longrightarrow} T$ where $T = \beta C_\varphi$ and $\varphi(z) = \lambda z$ for each $z \in \mathbb{D}$. Put $\lambda = 0$ in Theorem 2, thus we get a contradiction. ∎

3 Classes of weighted composition operators

In this section, we give necessary conditions for weak supercyclicity of certain classes of weighted composition operators on a Banach space of analytic functions.

Theorem 3. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $\Gamma \subseteq B(X)$ be a class of weighted composition operators on X such that for two points; $a, b \in \mathbb{D}$, $\Gamma \subseteq \{C_{w,\varphi} : \ \varphi(a) = b, \ w(a) \neq 0\}$. If Γ is weakly supercyclic, then the sets*

$$A = \{\frac{w(c)}{w(a)} : \exists \ \varphi \text{ such that } C_{w,\varphi} \in \Gamma\},$$

and

$$B = \{\frac{w'(a)}{w(a)} : \exists \ \varphi \text{ such that } C_{w,\varphi} \in \Gamma\},$$

are unbounded for every $c \in \mathbb{D}, \ c \neq a$.

Proof. Suppose that f is a weakly supercyclic vector of Γ. Since point evaluations are continuous, $\{\alpha w(a) f(b) : \alpha \in \mathbb{C}, \ C_{w,\varphi} \in \Gamma\}$ is dense in $\mathbb{C}$. Thus $f(b) \neq 0$. Let $g(z) = z - a$, $\varepsilon > 0$ and $c \in \mathbb{D}, \ c \neq a$. Put

$$U_g = \{h \in X : |e_a(h - g)| < \varepsilon, |e_c(h - g)| < \varepsilon\}$$

a weak neighborhood of g. There exist $\alpha_0 \in \mathbb{C}$ and $w_0 C_{\varphi_0} \in \Gamma$ such that

$$|\alpha_0 (w_0 C_{\varphi_0} f)(a)| < \varepsilon,$$

and

$$|\alpha_0 (w_0 C_{\varphi_0} f)(c) - (c - a)| < \varepsilon.$$

Since, for any self map φ of $\mathbb{D}$, $\varphi_b \varphi \varphi_a(0) = 0$ the Schwarz lemma implies that

$$|\varphi_b \varphi \varphi_a(z)| \leq |z| \quad (z \in \mathbb{D});$$

thus,

$$|\varphi_b \varphi(z)| \leq |\varphi_a(z)| < 1 \quad (z \in \mathbb{D}).$$

Therefore, there is a constant M such that

$$\sup_{\varphi} \{|f(\varphi(c))| : \ \varphi \text{ is a self map of } \mathbb{D} \text{ with } \varphi(a) = b\} \leq M,$$

since $f \circ \varphi = (f \circ \varphi_b) \circ (\varphi_b \circ \varphi)$. Hence

$$\begin{aligned} |c - a| &\leq |\alpha_0 (w_0 C_{\varphi_0} f)(c) - (c - a)| + |\alpha_0 (w_0(c) f(\varphi_0(c))| \\ &< \varepsilon + \frac{\varepsilon |w_0(c)| M}{|w_0(a) f(b)|}. \end{aligned}$$

Now if the set $\{\dfrac{w(c)}{w(a)} : \exists\ \varphi$ such that $C_{w,\varphi} \in \Gamma\}$ is bounded then $c = a$ which is a contradiction. On the other hand, for $g(z) = z - a$ one can find a net $(\alpha_i C_{w_i,\varphi_i}(f))_{i\in I}$ in $\mathbb{C}.\Gamma(f)$ such that

$$\alpha_i C_{w_i,\varphi_i}(f) \xrightarrow{w} g.$$

Therefore,

$$\alpha_i w_i(a) f(b) \longrightarrow 0.$$

Moreover, since e'_a is continuous, we have

$$\alpha_i (w'_i(a) f(b) + w_i(a)\varphi'_i(a) f'(b)) \longrightarrow 1.$$

But an application of the Cauchy integral formula shows that there is a constant M so that $|\varphi'_i(a)| \leq M$, for all i; hence

$$\alpha_i w'_i(a) f(b) \longrightarrow 1.$$

Thus, $w'_i(a)/w_i(a) \longrightarrow \infty$, which implies that the set $\{\dfrac{w'(a)}{w(a)} : \exists\ \varphi$ such that $C_{w,\varphi} \in \Gamma\}$ is unbounded. ■

As a consequence of the preceding theorem, suppose that $C_{w,\varphi}$ is weakly supercyclic on a Banach space of analytic functions X. If $a \in \mathbb{D}$ is a fixed point of φ and $w(a) \neq 0$ then the sequence $\left(\frac{\Pi_{j=0}^{n-1} w\circ\varphi_j(c)}{w^n(a)}\right)_n$ is unbounded for every $c \in \mathbb{D} - \{a\}$. This improves Theorem 2.2 of [19].

Also, Corollary 6 can be extended as follows.

Corollary 7. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $a, b \in \mathbb{D}$. Then the semigroup $\Gamma = \{C_\varphi : \ \varphi(a) = b\}$ is not weakly supercyclic. In particular, if φ has a fixed point then C_φ is not weakly supercyclic.*

In addition, the semigroup $\Gamma = \{C_\varphi : \ C_\varphi \text{ is an isometry}\}$ is not weakly supercyclic on the Hardy space H^p, the weighted Bergman space A^p_α $(1 \leq p < \infty)$, the analytic Besov space B^p $(2 < p < \infty)$ and the space $H^\infty_{\nu_p}(\mathbb{D})$ $(p > 0)$. Moreover, the semigroup $\Gamma = \{C_\varphi : \varphi \in \mathrm{Aut}(\mathbb{D}) \text{ and } C_\varphi \text{ is an isometry}\}$ is not weakly supercyclic on the weighted Dirichlet-type space $\mathcal{D}^p_\alpha$ $(1 \leq p < \infty,\ -1 < \alpha < \infty)$. For all of these spaces $\Gamma = \{C_\varphi : \varphi \text{ is a rotation}\}$. The relevant references are, respectively, [21, Theorems 1.3 and 1.4], [6, Corollary 12] and [12, Corollary 2.3]. Thus, the result follows from the previous corollary.

Proposition 3. *Let X be a Banach space of analytic functions on $\mathbb{D}$ and $T = C_{w,\varphi}$ for which φ is not the identity and not an elliptic automorphism, has a fixed point $a \in \mathbb{D}$, such that $w(a) \neq 0$. Then T is not weakly supercyclic.*

Proof. Since

$$C^n_{w,\varphi}(f) = \Pi_{j=0}^{n-1} w \circ \varphi_j \,.\, (f \circ \varphi_n),$$

for all $f \in X$ and $n \geq 1$, $W_n = \Pi_{j=0}^{n-1} w \circ \varphi_j$ is the weight of T^n for $n \geq 1$. Moreover,

$$W_n'(a) = w'(a)w^{n-1}(a)(1 + \sum_{j=1}^{n-1} \varphi'^j(a)),$$

and

$$W_n(a) = w^n(a);$$

thus,

$$\frac{W_n'(a)}{W_n(a)} = \frac{w'(a)}{w(a)} \left(1 + \sum_{j=1}^{n-1} \varphi'^j(a)\right).$$

On the other hand, $|\varphi'(a)| < 1$ [8, Page 59]. Hence $\{\frac{W_n'(a)}{W_n(a)} : n \in \mathbb{N}\}$ is a bounded set and the result follows from Theorem 3. ∎

Recall that the group of surjective linear isometries on the Hardy space H^p or the Bergman space L_a^p $(1 < p < \infty, p \neq 2)$ have the form of weighted composition operators; i.e., $\{\mu(\varphi')^\beta C_\varphi : |\mu| = 1, \varphi \in \mathrm{Aut}(\mathbb{D})\}$ where $\beta = \frac{1}{p}$ or $\beta = \frac{2}{p}$, respectively ([9],[17]).

Corollary 8. *Let X be a Banach space of analytic functions on $\mathbb{D}$, $a \in \mathbb{D}$ and $\beta > 0$. If $\Gamma \subseteq \{\mu(\varphi')^\beta C_\varphi : |\mu| = 1, \varphi \in Aut(\mathbb{D}), \varphi(a) = a\}$ then Γ is not weakly supercyclic.*

Proof. Suppose that $\mu(\varphi')^\beta C_\varphi \in \Gamma$ and put $w = \mu(\varphi')^\beta$. Observe that there are $c \in \mathbb{D}$ and λ with $|\lambda| = 1$ such that $\varphi = \lambda\varphi_c$. Thus

$$|w(a)| = |(|c|^2 - 1)^\beta (1 - \bar{c}a)^{-2\beta}|$$

and

$$|w'(a)| = |2\beta\bar{c}(|c|^2 - 1)^\beta (1 - \bar{c}a)^{-2\beta-1}|.$$

Therefore,

$$\sup\{\frac{|w'(a)|}{|w(a)|} : \exists\ \varphi \text{ such that } C_{w,\varphi} \in \Gamma\} < \infty$$

and the result follows from Theorem 3. ∎

Corollary 9. *Let X be the Hardy space H^p or the Bergman space L_a^p $(1 < p < \infty, p \neq 2)$ and $T = \mu(\varphi')^\beta C_\varphi$ be a surjective linear isometry on X for some $\varphi \in Aut(\mathbb{D})$ and $|\mu| = 1$ where $\beta = \frac{1}{p}$ for the space H^p and $\beta = \frac{2}{p}$ for the space L_a^p. If T is weakly supercyclic then φ is not an elliptic automorphism.*

Note that the weighted composition operator $C_{w,\varphi}$ is unitary on $H^2(\mathbb{D})$ if and only if φ is an automorphism of $\mathbb{D}$ and $w = \mu(\varphi')^{\frac{1}{2}}$ for some $|\mu| = 1$ [7, Theorem 6]. Therefore, if $C_{w,\varphi}$ is weakly supercyclic then φ is not an elliptic automorphism.

Let Y be a Banach space of analytic functions on $\mathbb{D}$ such that every element in Y has a continuous extension on $\overline{\mathbb{D}}$ and for every $\lambda \in \partial\mathbb{D}$ the linear functional of point evaluation at λ, e_λ is bounded. The disc algebra $A(\mathbb{D})$, the analytic Lipschitz $\mathrm{Lip}_\alpha(\mathbb{D})$, $(0 < \alpha \leq 1)$, S^p, $(p \geq 1)$ and $H^2(\beta, \mathbb{D})$ when $\sum \beta(n)^{-2} < \infty$, are some examples of such spaces [8, Pages 28 and 177].

The proof of the following proposition is the same as the proof of Theorem 3. Also, we assume that the function φ is continuous on $\overline{\mathbb{D}}$ and note that $w = C_{w,\varphi}1 \in Y$; therefore, w is continuous on $\overline{\mathbb{D}}$.

Proposition 4. *Let $\Gamma \subseteq B(Y)$ be a class of weighted composition operators on Y such that for two points $a, b \in \overline{\mathbb{D}}$, $\Gamma \subseteq \{C_{w,\varphi} : \varphi(a) = b,\ w(a) \neq 0\}$. If Γ is weakly supercyclic, then the set $\{\frac{w(c)}{w(a)} : \exists\ \varphi$ such that $C_{w,\varphi} \in \Gamma\}$ is unbounded for every $c \in \overline{\mathbb{D}}$, $c \neq a$.*

Corollary 10. *Let $a, b \in \overline{\mathbb{D}}$ and $\Gamma = \{C_\varphi : \varphi(a) = b\}$. Then Γ is not weakly supercyclic. In particular, by the Brouwer's fixed-point theorem every composition operator on Y is not weakly supercyclic.*

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References

[1] S. I. Ansari and P. S. Bourdon. Some properties of cyclic operators, Acta Sci. Math. (Szeged) **63** (1997) 195-207.

[2] F. Bayart and E. Matheron, Dynamics of linear operators, Cambridge University Press 2009.

[3] F. Bayart and E. Matheron. Hyponormal operators, weighted shifts, and weak forms of supercyclicity, Proc. Edinb. Math. Soc. **49** (2006) 1-15.

[4] L. Bernal-González, A. Bonilla, and M. C. Calderón-Moreno, Compositional hypercyclicity equals supercyclicity, Houston J. Math. **33** (2007) 581-591.

[5] J. Bès, Dynamics of weighted composition operators, Complex Anal. Oper. Theory **8** (2014) 159-176.

[6] J. Bonet, M. Lindström and E. Wolf, Isometric weighted composition operators on weighted Banach spaces of type H^∞, Proc. Amer. Math. Soc. **136** (2008) 4267-4273.

[7] P. S. Bourdon and S. K. Narayan, Normal weighted composition operators on the Hardy space $H^2(\mathbb{U})$, J. Math. Anal. Appl. **367** (2010) 278-286.

[8] C. C. Cowen and B. D. MacCluer, Composition Operators on Spaces of Analytic Functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA 1995.

[9] R. J. Fleming and J. E. Jamison, Isometries in Banach spaces: function spaces, CRC press 2003.

[10] R. J. Fleming and J. E. Jamison, Isometries in Banach spaces: vector-valued function spaces and operator spaces, CRC Press 2007.

[11] F. Forelli, The isometries of H^p, Canad. J. Math. **16** (1964) 721-728.

[12] L.G. Geng, Z.H. Zhou and X.T. Dong, Isometric composition operators on weighted Dirichlet-type spaces, J. Inequal. Appl. **1** (2012) 1-6.

[13] P. Greim, J. E. Jamison and A. Kaminska, Almost transitivity of some function spaces, Math. Proc. Cambridge Philos. Soc. **116** (1994) 475-488.

[14] K.-G. Grosse-Erdmann and R. Mortini, Universal functions for composition operators with non-automorphic symbol, J. Anal. Math. **107** (2009) 355-376.

[15] J. B. Guerrero and A. R. Palacios, Transitivity of the norm on Banach spaces, Extracta Math. **17** (2002) 1-58.

[16] K. Hedayatian and M. Faghih-Ahmadi, Cyclic convolution operators on the Hardy spaces, Bull. Belg. Math. Soc. Simon Stevin, 22 (2015) 291-298.

[17] W. Hornor and J. E. Jamison, Isometries of some Banach spaces of analytic functions, Integral Equations Operator Theory **41** (2001) 410-425.

[18] K. Jarosz, Any Banach space has an equivalent norm with trivial isometries, Israel J. Math. **64** (1988) 49-56.

[19] Z. Kamali, K. Hedayatian and B. Khani Robati, Non-weakly supercyclic weighted composition operators, Abstr. Appl. Anal. (2010) Art. ID 143808, 14 pp.

[20] C. Kitai, Invariant closed sets for linear operators, Ph.D. thesis. University of Toronto. Toronto 1982.

[21] M. J. Martín and D. Vukotić, Isometries of some classical function spaces among the composition operators, Contemp. Math. **393** (2006) 133-138.

[22] V.G. Miller, Remarks on finitely hypercyclic and finitely supercyclic operators, Integral Equations Operator Theory **29** (1997) 110-115.

[23] A. Montes-Rodríguez, S. Shkarin, Non-weakly supercyclic operators, J. Operator Theory 58 (2007) 39-62.

[24] A. Moradi, K. Hedayatian, B. Khani Robati and M. Ansari, Non supercyclic subsets of linear isometries on Banach spaces of analytic functions, Czechoslovak Math. J. **65** (140) (2015) 389-397.

[25] W. Novinger and D. Oberlin, Linear isometries of some normed spaces of analytic functions, Canad. J. Math. **37** (1985) 62-74.

[26] H. Rezaei, Chaotic property of weighted composition operators, Bull. Korean Math. Soc. **48** (2011) 1119-1124.

[27] S. Rolewicz, On orbits of elements, Studia Math. **32** (1969) 17-22.

[28] R. Sanders, Weakly supercyclic operators. J. Math. Anal. Appl. **292** (2004) 148-159.

[29] S. Shkarin, Non-sequential weak supercyclicity and hypercyclicity, J. Funct. Anal. 242 (2007) 37-77

[30] J. H. Shapiro, Composition Operators and Classical Function Theory, Universitext: Tracts in Mathematics, Springer, New York, NY, USA 1993.

[31] B. Yousefi and H. Rezaei, Hypercyclic property of weighted composition operators, Proc. Amer. Math. Soc. **135** (2007) 3263-3271.

[32] K. Zhu, Operator theory in function spaces, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, Inc, New York 1990.

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