On weakly classical primary submodules

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Abstract

In this paper all rings are commutative with nonzero identity. Let M be an R-module. A proper submodule N of M is called a *classical primary submodule*, if for each $m \in M$ and elements $a, b \in R$, $abm \in N$ implies that either $am \in N$ or $b^t m \in N$ for some $t \geq 1$. We introduce the notion of "weakly classical primary submodules". A proper submodule N of M is a *weakly classical primary submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then either $am \in N$ or $b^t m \in N$ for some $t \geq 1$.

1 Introduction

Throughout this paper all rings are commutative with nonzero identity and all modules are unitary. We recall that a proper ideal P (resp. Q) of a commutative ring R is said to be *prime (resp. primary)* if whenever $ab \in P$ (resp. $ab \in Q$) for some $a, b \in R$, then $a \in P$ or $b \in P$ (resp. either $a \in Q$ or $b \in \sqrt{Q}$). Several authors have extended the notion of prime ideals to modules, see, for example [11, 16, 18]. Let M be a module over a commutative ring R. A proper submodule N of M is called *prime* if for $a \in R$ and $m \in M$, $am \in N$ implies that $m \in N$ or $a \in (N :_R M) = \{r \in R \mid rM \subseteq N\}$. Anderson and Smith [3] said that a proper ideal P of a ring R is *weakly prime* if whenever $a, b \in R$ with $0 \neq ab \in P$, then $a \in P$ or $b \in P$. Weakly prime submodules were introduced by Ebrahimi and Farzalipour in [13]. A proper submodule N of M is called *weakly prime* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in (N :_R M)$. In [12], Ebrahimi and Farzalipour said that a proper ideal Q of a commutative ring R is

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weakly primary if whenever $a, b \in R$, then $0 \neq ab \in Q$ implies that either $a \in Q$ or $b \in \sqrt{Q}$. Also, they said that a proper submodule N of M is *weakly primary* if for $a \in R$ and $m \in M$ with $0 \neq am \in N$, either $m \in N$ or $a \in \sqrt{(N :_R M)}$. A proper submodule N of M is called a *classical prime submodule*, if for each $m \in M$ and $a, b \in R$, $abm \in N$ implies that $am \in N$ or $bm \in N$. This notion of classical prime submodules has been extensively studied by Behboodi in [7, 8] (see also, [9], in which, the notion of classical prime submodules is named "weakly prime submodules"). For more information on classical prime submodules, the reader is referred to [4, 5, 10]. In [19] the authors introduced the concept of weakly classical prime submodules. A proper submodule N of an R-module M is called a *weakly classical prime submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $bm \in N$. Baziar and Behboodi [6] defined a *classical primary submodule* in M as a proper submodule N of M such that if $abm \in N$, where $a, b \in R$ and $m \in M$, then either $am \in N$ or $b^t m \in N$ for some $t \geq 1$. In this paper we introduce the concept of weakly classical primary submodules. A proper submodule N of an R-module M is called a *weakly classical primary submodule* if whenever $a, b \in R$ and $m \in M$ with $0 \neq abm \in N$, then $am \in N$ or $b^t m \in N$ for some $t \geq 1$. Clearly, every classical primary submodule is a weakly classical primary submodule.

The annihilator of M which is denoted by $\mathrm{Ann}_R(M)$ is $(0 :_R M)$. Furthermore, for every $m \in M$, $(0 :_R m)$ is denoted by $\mathrm{Ann}_R(m)$. When $\mathrm{Ann}_R(M) = 0$, M is called a *faithful R-module*. An R-module M is called a *multiplication module* if every submodule N of M has the form IM for some ideal I of R, see [14]. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$. Finitely generated faithful multiplication modules are cancellation modules [22, Corollary to Theorem 9], where an R-module M is defined to be a *cancellation module* if $IM = JM$ for ideals I and J of R implies $I = J$. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [2, Theorem 3.4], the product of N and K is independent of presentations of N and K. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [2]). Let N be a proper submodule of a nonzero R-module M. We recall from [17] that the M-radical of N, denoted by $M\text{-rad}(N)$, is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say $M\text{-rad}(N) = M$. It is shown in [14, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then $M\text{-rad}(N) = \sqrt{(N :_R M)}M$. In [20], Quartararo et al. said that a commutative ring R is a *u-ring* provided R has the property that an ideal that is contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them. They show that every Bézout ring is a u-ring. Moreover, they proved that every Prüfer domain is a u-domain. Also, any ring which contains an infinite field as a subring is a u-ring, [21, Exercise 3.63]. In [15], Gottlieb investigated submodules covered by finite unions of submodules.

Among many results in this paper, it is shown (Theorem 2.17) that N is a weakly classical primary submodule of an R-module M if and only if for every pair of ideals I, J of R and $m \in M$ with $0 \neq IJm \subseteq N$, either $Im \subseteq N$ or $J \subseteq \sqrt{(N :_R m)}$. It is proved (Theorem 2.19) that if N is a weakly classical primary submodule of an R-module M that is not classical primary, then $(N :_R M)^2 N = 0$. It is shown (Theorem 3.4) that over a *um*-ring R, N is a weakly classical primary submodule of an R-module M if and only if for every pair of ideals I, J of R and submodule L of M with $0 \neq IJL \subseteq N$, either $IL \subseteq N$ or $J \subseteq \sqrt{(N :_R L)}$. Let R be a *um*-ring, M be an R-module and F be a faithfully flat R-module. It is shown (Theorem 3.10) that N is a weakly classical primary submodule of M if and only if $F \otimes N$ is a weakly classical primary submodule of $F \otimes M$. Let $R = R_1 \times R_2 \times R_3$ be a decomposable ring and $M = M_1 \times M_2 \times M_3$ be an R-module where M_i is an R_i-module, for $i = 1, 2, 3$. In Theorem 4.8 it is proved that if N is a weakly classical primary submodule of M, then either $N = \{(0, 0, 0)\}$ or N is a classical primary submodule of M.

2 Properties of weakly classical primary submodules

Notice that for an R-module M, the zero submodule $\{0\}$ is always a weakly classical primary submodule. In the following example, we give a module in which the zero submodule is not classical primary.

Example 2.1. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \bigoplus \mathbb{Z}_q \bigoplus \mathbb{Q}$ where p, q are two distinct prime integers. Note that $pq(\bar{1}, \bar{1}, 0) = (\bar{0}, \bar{0}, 0)$, but $p(\bar{1}, \bar{1}, 0) \neq (\bar{0}, \bar{0}, 0)$ and $q^t(\bar{1}, \bar{1}, 0) \neq (\bar{0}, \bar{0}, 0)$ for every $t \geq 1$. So the zero submodule of M is not classical primary. Hence the two concepts of classical primary submodules and of weakly classical primary submodules are different in general.

For an R-module M, the set of zero-divisors of M is denoted by $Z_R(M)$.

Theorem 2.2. *Let M be an R-module, N be a submodule of M and S be a multiplicative subset of R.*

1. *If N is a weakly classical primary submodule of M such that $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a weakly classical primary submodule of $S^{-1}M$.*
2. *If $S^{-1}N$ is a weakly classical primary submodule of $S^{-1}M$ such that $S \cap Z_R(N) = \emptyset$ and $S \cap Z_R(M/N) = \emptyset$, then N is a weakly classical primary submodule of M.*

Proof. (1) Let N be a weakly classical primary submodule of M and $(N :_R M) \cap S = \emptyset$. Suppose that $\frac{0}{1} \neq \frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} \in S^{-1}N$ for some $a_1, a_2 \in R$, $s_1, s_2, s_3 \in S$ and $m \in M$. Then there exists $s \in S$ such that $sa_1a_2m \in N$. If $sa_1a_2m = 0$, then $\frac{a_1}{s_1}\frac{a_2}{s_2}\frac{m}{s_3} = \frac{sa_1a_2m}{ss_1s_2s_3} = \frac{0}{1}$, a contradiction. Since N is a weakly classical primary submodule, then we have $a_1\,(sm) \in N$ or $a_2^t(sm) \in N$ for some $t \geq 1$. Thus $\frac{a_1}{s_1}\frac{m}{s_3} = \frac{sa_1m}{ss_1s_3} \in S^{-1}N$ or $\left(\frac{a_2}{s_2}\right)^t \frac{m}{s_3} = \frac{sa_2^tm}{ss_2^ts_3} \in S^{-1}N$. Consequently $S^{-1}N$ is a weakly classical primary submodule of $S^{-1}M$.

(2) Suppose that $S^{-1}N$ is a weakly classical primary submodule of $S^{-1}M$ and $S \cap Z_R(N) = \emptyset$ and $S \cap Z_R(M/N) = \emptyset$. Let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$. Then $\frac{a}{1}\frac{b}{1}\frac{m}{1} \in S^{-1}N$. If $\frac{a}{1}\frac{b}{1}\frac{m}{1} = \frac{0}{1}$, then there exists $s \in S$ such that

$sabm = 0$ which contradicts $S \cap Z_R(N) = \emptyset$. Therefore $\frac{a}{1}\frac{b}{1}\frac{m}{1} \neq \frac{0}{1}$, and so either $\frac{a}{1}\frac{m}{1} \in S^{-1}N$ or $\left(\frac{b}{1}\right)^t \frac{m}{1} \in S^{-1}N$ for some $t \geq 1$. Assume that $\frac{a}{1}\frac{m}{1} \in S^{-1}N$. So there exists $u \in S$ such that $uam \in N$. But $S \cap Z_R(M/N) = \emptyset$, whence $am \in N$. If $\left(\frac{b}{1}\right)^t \frac{m}{1} \in S^{-1}N$ for some $t \geq 1$, then there exists $v \in S$ such that $vb^t m \in N$. Again $S \cap Z_R(M/N) = \emptyset$ implies that $b^t m \in N$. Consequently N is a weakly classical primary submodule of M. ∎

Theorem 2.3. *Let M be an R-module and N a proper submodule of M.*

1. *If N is a weakly classical primary submodule of M, then $(N :_R m)$ is a weakly primary ideal of R for every $m \in M \backslash N$ with $Ann_R(m) = 0$.*

2. *If $(N :_R m)$ is a weakly primary ideal of R for every $m \in M \backslash N$, then N is a weakly classical primary submodule of M.*

Proof. (1) Suppose that N is a weakly classical primary submodule. Let $m \in M \backslash N$ with $\mathrm{Ann}_R(m) = 0$, and $0 \neq ab \in (N :_R m)$ for some $a, b \in R$. Then $0 \neq abm \in N$. So $am \in N$ or $b^t m \in N$ for some $t \geq 1$, i.e., $a \in (N :_R m)$ or $b \in \sqrt{(N :_R m)}$. Consequently $(N :_R m)$ is a weakly primary ideal of R.

(2) Assume that $(N :_R m)$ is a weakly primary ideal of R for every $m \in M \backslash N$. Let $0 \neq abm \in N$ for some $m \in M$ and $a, b \in R$. If $m \in N$, then we are done. So we assume that $m \notin N$. Hence $0 \neq ab \in (N :_R m)$ implies that either $a \in (N :_R m)$ or $b^t \in (N :_R m)$ for some $t \geq 1$. Therefore either $am \in N$ or $b^t m \in N$, and so N is a weakly classical primary submodule of M. ∎

We recall that M is a torsion-free R-module if and only if for every $0 \neq m \in M$, $\mathrm{Ann}_R(m) = 0$. As a direct consequence of Theorem 2.3 the following result follows.

Corollary 2.4. *Let M be a torsion-free R-module and N a proper submodule of M. Then N is a weakly classical primary submodule of M if and only if $(N :_R m)$ is a weakly primary ideal of R for every $m \in M \backslash N$.*

Theorem 2.5. *Let $f : M \to M'$ be a homomorphism of R-modules.*

1. *Suppose that f is a monomorphism. If N' is a weakly classical primary submodule of M' with $f^{-1}(N') \neq M$, then $f^{-1}(N')$ is a weakly classical primary submodule of M.*

2. *Suppose that f is an epimorphism. If N is a weakly classical primary submodule of M containing $Ker(f)$, then $f(N)$ is a weakly classical primary submodule of M'.*

Proof. (1) Suppose that N' is a weakly classical primary submodule of M' with $f^{-1}(N') \neq M$. Let $0 \neq abm \in f^{-1}(N')$ for some $a, b \in R$ and $m \in M$. Since f is a monomorphism, $0 \neq f(abm) \in N'$. So we get $0 \neq abf(m) \in N'$. Hence $f(am) = af(m) \in N'$ or $f(b^t m) = b^t f(m) \in N'$ for some $t \geq 1$. Thus $am \in f^{-1}(N')$ or $b^t m \in f^{-1}(N')$. Therefore $f^{-1}(N')$ is a weakly classical primary submodule of M.

(2) Assume that N is a weakly classical primary submodule of M. Let $a, b \in R$ and $m' \in M'$ be such that $0 \neq abm' \in f(N)$. By assumption there exists $m \in M$ such that $m' = f(m)$ and so $f(abm) \in f(N)$. Since $\text{Ker}(f) \subseteq N$, we have $0 \neq abm \in N$. It implies that $am \in N$ or $b^t m \in N$ for some $t \geq 1$. Hence $am' \in f(N)$ or $b^t m' \in f(N)$. Consequently $f(N)$ is a weakly classical primary submodule of M'. ■

As an immediate consequence of Theorem 2.5(2) we have the following corollary.

Corollary 2.6. *Let M be an R-module and $L \subset N$ be submodules of M. If N is a weakly classical primary submodule of M, then N/L is a weakly classical primary submodule of M/L.*

Theorem 2.7. *Let K and N be submodules of M with $K \subset N \subset M$. If K is a weakly classical primary submodule of M and N/K is a weakly classical primary submodule of M/K, then N is a weakly classical primary submodule of M.*

Proof. Let $a, b \in R$, $m \in M$ and $0 \neq abm \in N$. If $abm \in K$, then $am \in K \subset N$ or for some $t \geq 1$, $b^t m \in K \subset N$ as it is needed. Thus, assume that $abm \notin K$. Then $0 \neq ab(m + K) \in N/K$, and so $a(m + K) \in N/K$ or $b^t(m + K) \in N/K$ for some $t \geq 1$. It means that $am \in N$ or $b^t m \in N$, which completes the proof. ■

Proposition 2.8. *Let N be a proper submodule of an R-module M. If N is a weakly primary submodule of M, then N is a weakly classical primary submodule of M.*

Proof. Assume that N is a weakly primary submodule of M. Let $a, b \in R$ and $m \in M$ such that $0 \neq abm \in N$. Therefore either $bm \in N$ or $a \in \sqrt{(N :_R M)}$. In the first case we reach the claim. In the second case there exists $t \geq 1$ such that $a^t M \subseteq N$ and so $a^t m \in N$. Consequently N is a weakly classical primary submodule. ■

Corollary 2.9. *Let R be a ring and I be a proper ideal of R.*

1. *${}_R I$ is a weakly classical primary submodule of ${}_R R$ if and only if I is a weakly primary ideal of R.*

2. *Every proper ideal of R is weakly primary if and only if for every R-module M and every proper submodule N of M, N is a weakly classical primary submodule of M.*

Proof. (1) Let ${}_R I$ be a weakly classical primary submodule of ${}_R R$. Then by Theorem 2.3(1), $(I :_R 1) = I$ is a weakly primary ideal of R. For the converse, notice that ${}_R I$ is a weakly primary submodule of ${}_R R$ if and only if I is a weakly primary ideal of R. Now, apply Proposition 2.8.

(2) Assume that every proper ideal of R is weakly primary. Let N be a proper submodule of an R-module M. Since for every $m \in M \backslash N$, $(N :_R m)$ is a proper ideal of R, then it is a weakly primary ideal of R. Hence by Theorem 2.3(2), N is a weakly classical primary submodule of M. We have the converse immediately by part (1). ■

The following example shows that the converse of Proposition 2.8 is not true.

Example 2.10. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_p \bigoplus \mathbb{Z} \bigoplus \mathbb{Z}$ where p is a prime integer. Consider the submodule $N = \{\overline{0}\} \bigoplus \{0\} \bigoplus \mathbb{Z}$ of M. Notice that $(\overline{0}, 0, 0) \neq p(\overline{1}, 0, 1) = (\overline{0}, 0, p) \in N$, but $(\overline{1}, 0, 1) \notin N$. Also $p^t(\overline{1}, 1, 1) \notin N$ for every $t \geq 1$, which shows that $p \notin (N :_{\mathbb{Z}} M)$. Therefore N is not a weakly primary submodule of M. Now, assume that $m, n, z, w \in \mathbb{Z}$ and $\overline{x} \in \mathbb{Z}_p$ be such that $(\overline{0}, 0, 0) \neq mn(\overline{x}, z, w) \in N$. Hence $\overline{mnx} = \overline{0}$ and $mnz = 0$. Therefore $p|mnx$ and $z = 0$. So $p|m$ or $p|nx$. If $p|m$, then $m(\overline{x}, z, w) = (\overline{mx}, 0, mw) = (\overline{0}, 0, mw) \in N$. Similarly, if $p|nx$, then $n(\overline{x}, z, w) = (\overline{nx}, 0, nw) = (\overline{0}, 0, nw) \in N$. Consequently N is a weakly classical prime submodule and so it is a weakly classical primary submodule.

Proposition 2.11. *Let M be a cyclic R-module. Then a proper submodule N of M is a weakly primary submodule if and only if it is a weakly classical primary submodule.*

Proof. By Proposition 2.8, the "only if" part holds. Let $M = Rm$ for some $m \in M$ and N be a weakly classical primary submodule of M. Suppose that $0 \neq rx \in N$ for some $r \in R$ and $x \in M$. Then there exists an element $s \in R$ such that $x = sm$. Therefore $0 \neq rx = srm \in N$ and since N is a weakly classical primary submodule, $x = sm \in N$ or $r^t m \in N$ for some $t \geq 1$. Hence $x \in N$ or $r^t \in (N :_R M)$. Consequently, either $x \in N$ or $r \in \sqrt{(N :_R M)}$ and so N is a weakly primary submodule of M. ∎

Definition 2.12. Let N be a proper submodule of M and $a, b \in R$, $m \in M$. If N is a weakly classical primary submodule and $abm = 0$, $am \notin N$, $b \notin \sqrt{(N :_R m)}$, then (a, b, m) is called a *classical primary triple-zero of* N.

Theorem 2.13. *Let N be a weakly classical primary submodule of a finitely generated R-module M and suppose that $abK \subseteq N$ for some $a, b \in R$ and some submodule K of M. If (a, b, k) is not a classical primary triple-zero of N for any $k \in K$, then $aK \subseteq N$ or $b^t K \subseteq N$ for some $t \geq 1$.*

Proof. Suppose that (a, b, k) is not a classical primary triple-zero of N for any $k \in K$. Assume on the contrary that $aK \not\subseteq N$ and $b \notin \sqrt{(N :_R K)}$. Then there exists $k_1 \in K$ such that $ak_1 \notin N$, and since M is finitely generated, there exists $k_2 \in K$ such that $b \notin \sqrt{(N :_R k_2)}$. If $abk_1 \neq 0$, then we have $b \in \sqrt{(N :_R k_1)}$, because $ak_1 \notin N$ and N is a weakly classical primary submodule of M. If $abk_1 = 0$, then since $ak_1 \notin N$ and (a, b, k_1) is not a classical primary triple-zero of N, we conclude once again that $b \in \sqrt{(N :_R k_1)}$. By a similar argument, since (a, b, k_2) is not a classical primary triple-zero and $b \notin \sqrt{(N :_R k_2)}$, then we deduce that $ak_2 \in N$. By our hypothesis, $ab(k_1 + k_2) \in N$ and $(a, b, k_1 + k_2)$ is not a classical primary triple-zero of N. Hence we have either $a(k_1 + k_2) \in N$ or $b \in \sqrt{(N :_R k_1 + k_2)}$. If $a(k_1 + k_2) = ak_1 + ak_2 \in N$, then since $ak_2 \in N$, we have $ak_1 \in N$, a contradiction. If $b \in \sqrt{(N :_R k_1 + k_2)}$, then since $b \in \sqrt{(N :_R k_1)}$, we have $b \in \sqrt{(N :_R k_2)}$, which again is a contradiction. Thus $aK \subseteq N$ or $b^t K \subseteq N$ for some $t \geq 1$. ∎

Definition 2.14. Let N be a weakly classical primary submodule of an R-module M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M. We say that N is a *free classical primary triple-zero with respect to IJK* if (a, b, k) is not a classical primary triple-zero of N for any $a \in I, b \in J$, and $k \in K$.

Remark 2.15. Let N be a weakly classical primary submodule of M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M such that N is a free classical primary triple-zero with respect to IJK. Then $a \in I, b \in J$, and $k \in K$ implies that either $ak \in N$ or $b^t k \in N$ for some $t \geq 1$.

Corollary 2.16. *Let N be a weakly classical primary submodule of a finitely generated R-module M and suppose that $IJK \subseteq N$ for some ideals I, J of R and some submodule K of M. If N is a free classical primary triple-zero with respect to IJK, then $IK \subseteq N$ or $J \subseteq \sqrt{(N :_R K)}$.*

Proof. Suppose that N is a free classical primary triple-zero with respect to IJK. Assume that $IK \nsubseteq N$ and $J \nsubseteq \sqrt{(N :_R K)}$. Then there exist $a \in I$ and $b \in J$ with $aK \nsubseteq N$ and $b^s K \nsubseteq N$ for every $s \geq 1$. Since $abK \subseteq N$ and N is free classical primary triple-zero with respect to IJK, then Theorem 2.13 implies that $aK \subseteq N$ or $b^t K \subseteq N$ for some $t \geq 1$, which is a contradiction. Consequently $IK \subseteq N$ or $J \subseteq \sqrt{(N :_R K)}$. ■

Let M be an R-module and N a submodule of M. For every $a \in R$, $\{m \in M \mid am \in N\}$ is denoted by $(N :_M a)$. It is easy to see that $(N :_M a)$ is a submodule of M containing N.

In the next theorem we characterize weakly classical primary submodules.

Theorem 2.17. *Let M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is weakly classical primary;*
2. *For every $a, b \in R$, $(N :_M ab) \subseteq (0 :_M ab) \cup (N :_M a) \cup \big(\cup_{t \geq 1}(N :_M b^t)\big)$;*
3. *For every $a \in R$ and $m \in M$ with $am \notin N$, $(N :_R am) \subseteq (0 :_R am) \cup \sqrt{(N :_R m)}$;*
4. *For every $a \in R$ and $m \in M$ with $am \notin N$, $(N :_R am) = (0 :_R am)$ or $(N :_R am) \subseteq \sqrt{(N :_R m)}$;*
5. *For every $a \in R$ and every ideal I of R and $m \in M$ with $0 \neq aIm \subseteq N$, either $am \in N$ or $I \subseteq \sqrt{(N :_R m)}$;*
6. *For every ideal I of R and $m \in M$ with $I \nsubseteq \sqrt{(N :_R m)}$, $(N :_R Im) = (0 :_R Im)$ or $(N :_R Im) = (N :_R m)$;*
7. *For every pair of ideals I, J of R and $m \in M$ with $0 \neq IJm \subseteq N$, either $Im \subseteq N$ or $J \subseteq \sqrt{(N :_R m)}$.*

Proof. (1)⇒(2) Suppose that N is a weakly classical primary submodule of M. Let $m \in (N :_M ab)$. Then $abm \in N$. If $abm = 0$, then $m \in (0 :_M ab)$. Assume that $abm \neq 0$. Hence $am \in N$ or $b^t m \in N$ for some $t \geq 1$. Therefore $m \in (N :_M a)$ or $m \in \cup_{t \geq 1}(N :_M b^t)$. Consequently, $(N :_M ab) \subseteq (0 :_M ab) \cup (N :_M a) \cup \big(\cup_{t \geq 1}(N :_M b^t)\big)$.
(2)⇒(3) Let $am \notin N$ for some $a \in R$ and $m \in M$. Assume that $x \in (N :_R am)$.

Then $axm \in N$, and so $m \in (N :_M ax)$. Since $am \notin N$, then $m \notin (N :_M a)$. Thus by part (2), $m \in (0 :_M ax)$ or $m \in \cup_{t\geq 1}(N :_M x^t)$, whence $x \in (0 :_R am)$ or $x \in \sqrt{(N :_R m)}$. Therefore $(N :_R am) \subseteq (0 :_R am) \cup \sqrt{(N :_R m)}$.
(3)⇒(4) By the fact that if an ideal (a subgroup) is the union of two ideals (two subgroups), then it is equal to one of them.
(4)⇒(5) Suppose that for some $a \in R$, an ideal I of R and $m \in M$, $0 \neq aIm \subseteq N$. Hence $I \subseteq (N :_R am)$ and $I \nsubseteq (0 :_R am)$. If $am \in N$, then we are done. So, assume that $am \notin N$. Therefore by part (4) we have that $I \subseteq \sqrt{(N :_R m)}$.
(5)⇒(6) Assume that I is an ideal of R and $m \in M$ such that $I \nsubseteq \sqrt{(N :_R m)}$. Let $x \in (N :_R Im)$. Thus $xIm \subseteq N$. If $xIm = 0$, then $x \in (0 :_R Im)$. If $xIm \neq 0$, then by part (5) we have $xm \in N$ and so $x \in (N :_R m)$. Hence $(N :_R Im) = (0 :_R Im) \cup (N :_R m)$. Consequently $(N :_R Im) = (0 :_R Im)$ or $(N :_R Im) = (N :_R m)$.
(6)⇒(7) Let $0 \neq IJm \subseteq N$ for some ideals I, J of R and $m \in M$ with $J \nsubseteq \sqrt{(N :_R m)}$. Therefore $I \subseteq (N :_R Jm)$. On the other hand part (6) implies that either $(N :_R Jm) = (0 :_R Jm)$ or $(N :_R Jm) = (N :_R m)$. The former cannot hold, because $IJm \neq 0$. Hence the second case implies that $Im \subseteq N$.
(7)⇒(1) Is trivial. ■

Theorem 2.18. *Let N be a weakly classical primary submodule of M and suppose that (a, b, m) is a classical primary triple-zero of N for some $a, b \in R$ and $m \in M$. Then the following conditions hold:*

1. $abN = 0$.
2. $a(N :_R M)m = 0$.
3. $b(N :_R M)m = 0$.
4. $(N :_R M)^2 m = 0$.
5. $a(N :_R M)N = 0$.
6. $b(N :_R M)N = 0$.

Proof. (1) Suppose that $abN \neq 0$. Then there exists $n \in N$ with $abn \neq 0$. Hence $0 \neq ab(m + n) = abn \in N$, so we conclude that $a(m + n) \in N$ or $b^t(m + n) \in N$ for some $t \geq 1$. Thus $am \in N$ or $b^t m \in N$, which contradicts the assumption that (a, b, m) is classical primary triple-zero. Thus $abN = 0$.

(2) Let $axm \neq 0$ for some $x \in (N :_R M)$. Then $a(b + x)m \neq 0$, because $abm = 0$. Since $xm \in N$, $a(b + x)m \in N$. Then $am \in N$ or $(b + x)^t m \in N$ for some $t \geq 1$. Hence $am \in N$ or $b^t m \in N$, which contradicts our hypothesis.

(3) The proof is similar to part (2).

(4) Assume that $x_1x_2m \neq 0$ for some $x_1, x_2 \in (N :_R M)$. Then by parts (2) and (3), $(a + x_1)(b + x_2)m = x_1x_2m \neq 0$. Clearly $(a + x_1)(b + x_2)m \in N$. Then $(a + x_1)m \in N$ or $(b + x_2)^t m \in N$ for some $t \geq 1$. Therefore $am \in N$ or $b^t m \in N$ which is a contradiction. Consequently $(N :_R M)^2 m = 0$.

(5) Let $axn \neq 0$ for some $x \in (N :_R M)$ and $n \in N$. Therefore by parts (1) and (2) we conclude that $0 \neq a(b + x)(m + n) = axn \in N$. So $a(m + n) \in N$

or $(b+x)^t(m+n) \in N$ for some $t \geq 1$. Hence $am \in N$ or $b^t m \in N$. This contradiction shows that $a(N :_R M)N = 0$.

(6) Similart to part (5). ■

A submodule N of an R-module M is called a nilpotent submodule if $(N :_R M)^k N = 0$ for some positive integer k (see [1]), and we say that $m \in M$ is nilpotent if Rm is a nilpotent submodule of M.

Theorem 2.19. *If N is a weakly classical primary submodule of an R-module M that is not classical primary, then $(N :_R M)^2 N = 0$ and so N is nilpotent.*

Proof. Suppose that N is a weakly classical primary submodule of M that is not classical primary. Then there exists a classical primary triple-zero (a, b, m) of N for some $a, b \in R$ and $m \in M$. Assume that $(N :_R M)^2 N \neq 0$. Hence there are $x_1, x_2 \in (N :_R M)$ and $n \in N$ such that $x_1 x_2 n \neq 0$. By Theorem 2.18, $0 \neq (a+x_1)(b+x_2)(m+n) = x_1 x_2 n \in N$. So $(a+x_1)(m+n) \in N$ or $(b+x_1)^t(m+n) \in N$ for some $t \geq 1$. Therefore $am \in N$ or $b^t m \in N$, a contradiction. ■

Remark 2.20. Let M be a multiplication R-module and K, L be submodules of M. Then there are ideals I, J of R such that $K = IM$ and $L = JM$. Thus $KL = IJM = IL$. In particular $KM = IM = K$. Also, for any $m \in M$ we define $Km := KRm$. Hence $Km = IRm = Im$.

Corollary 2.21. *If N is a weakly classical primary submodule of a multiplication R-module M that is not classical primary, then $N^3 = 0$.*

Proof. Since M is multiplication, then $N = (N :_R M)M$. Therefore by Theorem 2.19 and Remark 2.20, $N^3 = (N :_R M)^2 N = 0$. ■

Definition 2.22. ([17]) Let N be a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted by M-rad(N), is defined to be the intersection of all prime submodules of M containing N. If M has no prime submodule containing N, then we say M-rad$(N) = M$.

Let M be an R-module. Assume that Nil(M) is the set of all nilpotent elements of M. If M is faithful, then Nil(M) is a submodule of M and if M is faithful multiplication, then $\mathrm{Nil}(M) = \mathrm{Nil}(R)M = \bigcap Q$ $(= M\text{-rad}(\{0\}))$, where the intersection runs over all prime submodules of M, [1, Theorem 6].

We recall from [14, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then $M\text{-rad}(N) = \sqrt{(N :_R M)}M$.

Theorem 2.23. *Let N be a weakly classical primary submodule of M. If N is not classical primary, then*

1. $\sqrt{(N :_R M)} = \sqrt{Ann_R(M)}$.
2. *If M is multiplication, then* M-rad(N)=M-rad$(\{0\})$. *If in addition M is faithful, then* M-rad$(N) = \mathrm{Nil}(M)$.

Proof. (1) Assume that N is not classical primary. By Theorem 2.19, $(N :_R M)^2N = 0$. Then

$$\begin{aligned}(N :_R M)^3 &= (N :_R M)^2(N :_R M)\\ &\subseteq ((N :_R M)^2N :_R M)\\ &= (0 :_R M),\end{aligned}$$

and so $(N :_R M) \subseteq \sqrt{(0 :_R M)}$. Hence, we have $\sqrt{(N :_R M)} = \sqrt{(0 :_R M)} = \sqrt{\mathrm{Ann}_R(M)}$.

(2) Suppose that M is multiplication. Then, by part (1) we have that

$$M\text{-rad}(N) = \sqrt{(N :_R M)}M = \sqrt{(0 :_R M)}M = M\text{-rad}(\{0\}).$$

Now, if in addition M is faithful, then $M\text{-rad}(N) = M\text{-rad}(\{0\}) = \mathrm{Nil}(M)$. ■

Regarding Remark 2.20 we have the next proposition.

Proposition 2.24. *Let R be a Noetherian ring, M a multiplication R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is a weakly classical primary submodule of M;*
2. *If $0 \neq N_1N_2m \subseteq N$ for some submodules N_1, N_2 of M and $m \in M$, then either $N_1m \subseteq N$ or $N_2^tm \subseteq N$ for some $t \geq 1$.*

Proof. (1)⇒(2) Let $0 \neq N_1N_2m \subseteq N$ for some submodules N_1, N_2 of M and $m \in M$. Since M is multiplication, there are ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Therefore $0 \neq N_1N_2m = I_1I_2m \subseteq N$, and so by Theorem 2.17 either $I_1m \subseteq N$ or $I_2 \subseteq \sqrt{(N :_R m)}$. In the first case we have $N_1m = I_1m \subseteq N$. Notice the fact that every ideal of a Noetherian ring contains a power of its radical. So, in the second case, there exists some $t \geq 1$ such that $I_2^t \subseteq \left(\sqrt{(N :_R m)}\right)^t \subseteq (N :_R m)$. Therefore $N_2^tm = I_2^tm \subseteq N$.
(2)⇒(1) Suppose that $0 \neq I_1I_2m \subseteq N$ for some ideals I_1, I_2 of R and some $m \in M$. In part (2) set $N_1 := I_1M$ and $N_2 := I_2M$. Therefore $N_1m = I_1m \subseteq N$ or $N_2^tm = I_2^tm \subseteq N$ for some $t \geq 1$. Consequently N is a weakly classical primary submodule of M. ■

3 Weakly classical primary submodules of modules over specific rings

First, we recall the two concepts of u-rings and um-rings and then investigate weakly classical primary submodules over these rings.

Definition 3.1. ([20]) A commutative ring R is a *u-ring* provided R has the property that an ideal that is contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Proposition 3.2. *Let M be an R-module and N be a weakly classical primary submodule of M.Then*

1. *For every $a, b \in R$ and $m \in M$,*

$$(N :_R abm) = (0 :_R abm) \cup (N :_R am) \cup \left(\cup_{t\geq 1}(N :_R b^t m)\right);$$

2. *If R is a u-ring, then for every $a, b \in R$ and $m \in M$, $(N :_R abm) = (0 :_R abm)$ or $(N :_R abm) = (N :_R am)$ or $(N :_R abm) = (N :_R b^t m)$ for some $t \geq 1$.*

Proof. (1) Let $a, b \in R$ and $m \in M$. Suppose that $r \in (N :_R abm)$. Then $ab(rm) \in N$. If $ab(rm) = 0$, then $r \in (0 :_R abm)$. Therefore we assume that $ab(rm) \neq 0$. So, either $a(rm) \in N$ or $b^t(rm) \in N$ for some $t \geq 1$. Thus, either $r \in (N :_R am)$ or $r \in (N :_R b^t m)$ for some $t \geq 1$. Consequently $(N :_R abm) = (0 :_R abm) \cup (N :_R am) \cup \left(\cup_{t\geq 1}(N :_R b^t m)\right)$.

(2) Apply part (1). ∎

Lemma 3.3. *A ring R is a um-ring if and only if $M \subseteq \bigcup_{i=1}^{n} M_i$, where M_i's are some R-modules and n is a positive integer implies that $M \subseteq M_i$ for some $1 \leq i \leq n$.*

Proof. ($\Leftarrow$) It is clear.

($\Rightarrow$) Suppose that R is a um-ring. Let $M \subseteq \bigcup_{i=1}^{n} M_i$ for some R-modules $M_1, M_2, \ldots, M_n$. Then $M = \bigcup_{i=1}^{n} (M_i \cap M)$ and so $M = M_i \cap M$ for some $1 \leq i \leq n$. Therefore $M \subseteq M_i$ for some $1 \leq i \leq n$. ∎

Theorem 3.4. *Let R be a um-ring, M be an R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is weakly classical primary;*

2. *For every $a, b \in R$, $(N :_M ab) = (0 :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = (N :_M b^t)$ for some $t \geq 1$;*

3. *For every $a, b \in R$ and every submodule L of M, $0 \neq abL \subseteq N$ implies that $aL \subseteq N$ or $b^t L \subseteq N$ for some $t \geq 1$;*

4. *For every $a \in R$ and every submodule L of M with $aL \nsubseteq N$, $(N :_R aL) = (0 :_R aL)$ or $(N :_R aL) \subseteq \sqrt{(N :_R L)}$;*

5. *For every $a \in R$, every ideal I of R and every submodule L of M, $0 \neq aIL \subseteq N$ implies that $aL \subseteq N$ or $I \subseteq \sqrt{(N :_R L)}$;*

6. *For every ideal I of R and every submodule L of M with $I \nsubseteq \sqrt{(N :_R L)}$, $(N :_R IL) = (0 :_R IL)$ or $(N :_R IL) = (N :_R L)$;*

7. *For every pair of ideals I, J of R and every submodule L of M, $0 \neq IJL \subseteq N$ implies that $IL \subseteq N$ or $J \subseteq \sqrt{(N :_R L)}$.*

Proof. Similar to that of Theorem 2.17. ∎

Remark 3.5. The zero submodule of the $\mathbb{Z}$-module $\mathbb{Z}_6$, is a weakly classical primary submodule (weakly primary ideal) of $\mathbb{Z}_6$. Notice that $2 \cdot 3 \in 6\mathbb{Z}$, but neither $2 \in 6\mathbb{Z}$ nor $3 \in \sqrt{6\mathbb{Z}} = 2\mathbb{Z} \cap 3\mathbb{Z}$. Therefore $(0 :_{\mathbb{Z}} \mathbb{Z}_6) = 6\mathbb{Z}$ is not a weakly primary ideal of $\mathbb{Z}$.

Proposition 3.6. *Let R be a um-ring, M be an R-module and N be a proper submodule of M. If N is a weakly classical primary submodule of M, then $(N :_R L)$ is a weakly primary ideal of R for every faithful submodule L of M that is not contained in N.*

Proof. Assume that N is a weakly classical primary submodule of M and L is a faithful submodule of M such that $L \nsubseteq N$. Let $0 \neq ab \in (N :_R L)$ for some $a, b \in R$. Then $0 \neq abL \subseteq N$, because L is faithful. Hence Theorem 3.4 implies that $aL \subseteq N$ or $b^t L \subseteq N$ for some $t \geq 1$, i.e., $a \in (N :_R L)$ or $b \in \sqrt{(N :_R L)}$. Consequently $(N :_R L)$ is a weakly primary ideal of R. ∎

Lemma 3.7. *Let R be a ring and Q be a proper ideal of R. The following conditions are equivalent:*

1. *Q is a weakly primary ideal of R;*
2. *For every element $a \in R \backslash Q$, either $(Q :_R a) = (0 :_R a)$ or $(Q :_R a) \subseteq \sqrt{Q}$;*
3. *For every $a \in R$ and every ideal I of R, $0 \neq aI \subseteq Q$ implies that either $a \in Q$ or $I \subseteq \sqrt{Q}$;*
4. *For every ideal I of R with $I \nsubseteq \sqrt{Q}$, either $(Q :_R I) = (0 :_R I)$ or $(Q :_R I) = Q$;*
5. *For every pair of ideals I, J of R, $0 \neq IJ \subseteq Q$ implies that either $I \subseteq Q$ or $J \subseteq \sqrt{Q}$.*

Proof. (1)⇒(2) Assume that Q is a weakly primary ideal of R. Let $a \in R \backslash Q$ and $x \in (Q :_R a)$. Then $ax \in Q$. If $ax = 0$, then $x \in (0 :_R a)$. Suppose that $ax \neq 0$. So $x \in \sqrt{Q}$. Hence $(Q :_R a) \subseteq (0 :_R a) \cup \sqrt{Q}$. Therefore either $(Q :_R a) = (0 :_R a)$ or $(Q :_R a) \subseteq \sqrt{Q}$.
(2)⇒(3) Suppose that for some $a \in R$ and ideal I of R, $0 \neq aI \subseteq Q$. Thus $I \subseteq (Q :_R a)$. Since $aI \neq 0$, then $(Q :_R a) \neq (0 :_R a)$. Then, part (2) implies that $I \subseteq (Q :_R a) \subseteq \sqrt{Q}$.
(3)⇒(4) Suppose that $I \nsubseteq \sqrt{Q}$ for some ideal I of R. Let $x \in (Q :_R I)$. Then $xI \subseteq Q$. If $xI = 0$, then $x \in (0 :_R I)$. If $xI \neq 0$, then by part (3) we have that $x \in Q$. Hence $(Q :_R I) = (0 :_R I) \cup Q$. Consequently $(Q :_R I) = (0 :_R I)$ or $(Q :_R I) = Q$.
(4)⇒(5) Assume that I, J are ideals of R such that $0 \neq IJ \subseteq Q$. Then $I \subseteq (Q :_R J)$. Suppose that $J \nsubseteq \sqrt{Q}$. Thus part (4) implies that $(Q :_R J) = (0 :_R J)$ or $(Q :_R J) = Q$. Since $IJ \neq 0$, then we have only $(Q :_R J) = Q$, and so $I \subseteq Q$.
(5)⇒(1) is straightforward. ∎

Theorem 3.8. *Let R be a Noetherian um-ring, M be a faithful multiplication R-module and N be a proper submodule of M. The following conditions are equivalent:*

1. *N is a weakly classical primary submodule of M;*

2. *If $0 \neq N_1N_2N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M, then either $N_1N_3 \subseteq N$ or $N_2^tN_3 \subseteq N$ for some $t \geq 1$;*

3. *If $0 \neq N_1N_2 \subseteq N$ for some submodules N_1, N_2 of M, then either $N_1 \subseteq N$ or $N_2^t \subseteq N$ for some $t \geq 1$;*

4. *N is a weakly primary submodule of M;*

5. *$(N :_R M)$ is a weakly primary ideal of R.*

Proof. (1)⇒(2) Let $0 \neq N_1N_2N_3 \subseteq N$ for some submodules N_1, N_2, N_3 of M. Since M is multiplication, there exist ideals I_1, I_2 of R such that $N_1 = I_1M$ and $N_2 = I_2M$. Therefore $0 \neq I_1I_2N_3 \subseteq N$. Since R is Noetherian, Theorem 2.24 implies that $I_1N_3 \subseteq N$ or $I_2^tN_3 \subseteq N$ for some $t \geq 1$. Thus, either $N_1N_3 \subseteq N$ or $N_2^tN_3 \subseteq N$.
(2)⇒(3) is easy.
(3)⇒(4) Suppose that $0 \neq IK \subseteq N$ for some ideal I of R and some submodule K of M. It is sufficient to set $N_1 := K$ and $N_2 := IM$ in part (3).
(4)⇒(1) By Proposition 2.8.
(1)⇒(5) By Proposition 3.6.
(5)⇒(4) Let $0 \neq IK \subseteq N$ for some ideal I of R and some submodule K of M. Since M is multiplication, then there is an ideal J of R such that $K = JM$. Hence $0 \neq JI \subseteq (N :_R M)$ which by Lemma 3.7 implies that either $J \subseteq (N :_R M)$ or $I \subseteq \sqrt{(N :_R M)}$. If $I \subseteq \sqrt{(N :_R M)}$, the we are done. If $J \subseteq (N :_R M)$, then $K = JM \subseteq N$. ■

Proposition 3.9. *Let R be a Noetherian um-ring. Let M be a faithful multiplication R-module and N a submodule of M. Then the following conditions are equivalent:*

1. *N is a weakly classical primary submodule;*

2. *$(N :_R M)$ is a weakly primary ideal of R;*

3. *$N = IM$ for some weakly primary ideal I of R.*

Proof. (1) ⇔ (2) . By Theorem 3.8.
(2) ⇒ (3) Since $(N :_R M)$ is a weakly primary ideal and $N = (N :_R M) M$, then condition (3) holds.
(3) ⇒ (2) By the fact that every multiplication module over a Noetherian ring is a Noetherian module, M is Noetherian and so finitely generated. Suppose that $N = IM$ for some weakly primary ideal I of R. Since M is a multiplication module, we have $N = (N : M) M$. Therefore $N = IM = (N : M) M$ and so $I = (N : M)$, because by [22, Corollary to Theorem 9] M is cancellation. ■

Theorem 3.10. *Let R be a um-ring and M be an R-module.*

1. If F is a flat R-module and N is a weakly classical primary submodule of M such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly classical primary submodule of $F \otimes M$.

2. Suppose that F is a faithfully flat R-module. Then N is a weakly classical primary submodule of M if and only if $F \otimes N$ is a weakly classical primary submodule of $F \otimes M$.

Proof. (1) Let $a, b \in R$. Then by Theorem 3.4, either $(N :_M ab) = (0 :_M ab)$ or $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = \left(N :_M b^t\right)$ for some $t \geq 1$. Assume that $(N :_M ab) = (0 :_M ab)$. Then by [5, Lemma 3.2],

$$(F \otimes N :_{F\otimes M} ab) = F \otimes (N :_M ab) = F \otimes (0 :_M ab)$$

$$= (F \otimes 0 :_{F\otimes M} ab) = (0 :_{F\otimes M} ab).$$

Now, suppose that $(N :_M ab) = (N :_M a)$. Again by [5, Lemma 3.2],

$$(F \otimes N :_{F\otimes M} ab) = F \otimes (N :_M ab) = F \otimes (N :_M a)$$

$$= (F \otimes N :_{F\otimes M} a).$$

With a similar argument we can show that if $(N :_M ab) = \left(N :_M b^t\right)$ for some $t \geq 1$, then $(F \otimes N :_{F\otimes M} ab) = \left(F \otimes N :_{F\otimes M} b^t\right)$. Consequently by Theorem 3.4 we deduce that $F \otimes N$ is a weakly classical primary submodule of $F \otimes M$.

(2) Let N be a weakly classical primary submodule of M and assume that $F \otimes N = F \otimes M$. Then $0 \to F \otimes N \overset{\subseteq}{\to} F \otimes M \to 0$ is an exact sequence. Since F is a faithfully flat module, $0 \to N \overset{\subseteq}{\to} M \to 0$ is an exact sequence. So $N = M$, which is a contradiction. So $F \otimes N \neq F \otimes M$. Then $F \otimes N$ is a weakly classical primary submodule by (1). Now for the converse, let $F \otimes N$ be a weakly classical primary submodule of $F \otimes M$. We have $F \otimes N \neq F \otimes M$ and so $N \neq M$. Let $a, b \in R$. Then by Theorem 3.4, $(F \otimes N :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)$ or $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} a)$ or $(F \otimes N :_{F\otimes M} ab) = \left(F \otimes N :_{F\otimes M} b^t\right)$ for some $t \geq 1$. Suppose that $(F \otimes N :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)$. Hence

$$F \otimes (N :_M ab) = (F \otimes N :_{F\otimes M} ab) = (0 :_{F\otimes M} ab)$$

$$= (F \otimes 0 :_{F\otimes M} ab) = F \otimes (0 :_M ab).$$

Thus $0 \to F \otimes (0 :_M ab) \overset{\subseteq}{\to} F \otimes (N :_M ab) \to 0$ is an exact sequence. Since F is a faithfully flat module, $0 \to (0 :_M ab) \overset{\subseteq}{\to} (N :_M ab) \to 0$ is an exact sequence which implies that $(N :_M ab) = (0 :_M ab)$. With a similar argument we can deduce that if $(F \otimes N :_{F\otimes M} ab) = (F \otimes N :_{F\otimes M} a)$ or $(F \otimes N :_{F\otimes M} ab) = \left(F \otimes N :_{F\otimes M} b^t\right)$ for some $t \geq 1$, then $(N :_M ab) = (N :_M a)$ or $(N :_M ab) = \left(N :_M b^t\right)$. Consequently N is a weakly classical primary submodule of M by Theorem 3.4. ■

Corollary 3.11. *Let R be a um-ring, M be an R-module and X be an indeterminate. If N is a weakly classical primary submodule of M, then $N[X]$ is a weakly classical primary submodule of $M[X]$.*

Proof. Assume that N is a weakly classical primary submodule of M. Notice that $R[X]$ is a flat R-module. Then by Theorem 3.10, $R[X] \otimes N \simeq N[X]$ is a weakly classical primary submodule of $R[X] \otimes M \simeq M[X]$. ■

4 Weakly classical primary submodules in direct products of modules

Let R be a ring and M_1, M_2 be two R-modules. Then $M = M_1 \times M_2$ is an R-module, and for R-submodules N_1 of M_1 and N_2 of M_2, $N = N_1 \times N_2$ is an R-submodule of M.

Theorem 4.1. *Let M_1, M_2 be R-modules and N_1 be a proper submodule of M_1. Then the following conditions are equivalent:*

1. *$N = N_1 \times M_2$ is a weakly classical primary submodule of $M = M_1 \times M_2$;*

2. *N_1 is a weakly classical primary submodule of M_1 and for each $r, s \in R$ and $m_1 \in M_1$ we have*

$$rsm_1 = 0,\ rm_1 \notin N_1,\ s \notin \sqrt{(N_1 : m_1)} \Rightarrow rs \in Ann_R(M_2).$$

Proof. (1)⇒(2) Suppose that $N = N_1 \times M_2$ is a weakly classical primary submodule of $M = M_1 \times M_2$. Let $r, s \in R$ and $m_1 \in M_1$ be such that $0 \neq rsm_1 \in N_1$. Then $(0,0) \neq rs(m_1, 0) \in N$. Thus $r(m_1, 0) \in N$ or $s^t(m_1, 0) \in N$ for some $t \geq 1$, and so $rm_1 \in N_1$ or $s^t m_1 \in N_1$ for some $t \geq 1$. Consequently N_1 is a weakly classical primary submodule of M_1. Now, assume that $rsm_1 = 0$ for some $r, s \in R$ and $m_1 \in M_1$ such that $rm_1 \notin N_1$ and $s \notin \sqrt{(N_1 : m_1)}$. Suppose that $rs \notin \mathrm{Ann}_R(M_2)$. Therefore there exists $m_2 \in M_2$ such that $rsm_2 \neq 0$. Hence $(0,0) \neq rs(m_1, m_2) \in N$, and so $r(m_1, m_2) \in N$ or $s^t(m_1, m_2) \in N$ for some $t \geq 1$. Thus $rm_1 \in N_1$ or $s^t m_1 \in N_1$ for some $t \geq 1$, which is a contradiction. Consequently $rs \in \mathrm{Ann}_R(M_2)$.
(2)⇒(1) Let $r, s \in R$ and $(m_1, m_2) \in M = M_1 \times M_2$ be such that $(0,0) \neq rs(m_1, m_2) \in N = N_1 \times M_2$. First assume that $rsm_1 \neq 0$. Then by part (2), $rm_1 \in N_1$ or $s^t m_1 \in N_1$ for some $t \geq 1$. So $r(m_1, m_2) \in N$ or $s^t(m_1, m_2) \in N$, and thus we are done. If $rsm_1 = 0$, then $rsm_2 \neq 0$. Therefore $rs \notin \mathrm{Ann}_R(M_2)$, and so part (2) implies that either $rm_1 \in N_1$ or $s^t m_1 \in N_1$ for some $t \geq 1$. Again we have that $r(m_1, m_2) \in N$ or $s^t(m_1, m_2) \in N$ which shows N is a weakly classical primary submodule of M. ■

The following two propositions have easy verifications.

Proposition 4.2. *Let M_1, M_2 be R-modules and N_1 be a proper submodule of M_1. Then $N = N_1 \times M_2$ is a classical primary submodule of $M = M_1 \times M_2$ if and only if N_1 is a classical primary submodule of M_1.*

Proposition 4.3. *Let M_1, M_2 be R-modules and N_1, N_2 be proper submodules of M_1, M_2, respectively. If $N = N_1 \times N_2$ is a weakly classical primary (resp. classical primary) submodule of $M = M_1 \times M_2$, then N_1 is a weakly classical primary (resp. classical primary) submodule of M_1 and N_2 is a weakly classical primary (resp. classical primary) submodule of M_2.*

Example 4.4. Let $R = \mathbb{Z}$, $M = \mathbb{Z} \times \mathbb{Z}$ and $N = p\mathbb{Z} \times q\mathbb{Z}$ where p, q are two distinct prime integers. Since $p\mathbb{Z}$, $q\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $p\mathbb{Z}$, $q\mathbb{Z}$ are weakly classical primary $\mathbb{Z}$-submodules of $\mathbb{Z}$. Notice that $(0,0) \neq pq(1,1) = (pq,pq) \in N$, but $p(1,1) \notin N$ and $q^t(1,1) \notin N$ for every $t \geq 1$. So N is not a weakly classical primary submodule of M. This example shows that the converse of Proposition 4.3 is not true.

Let R_i be a commutative ring with identity and M_i be an R_i-module, for $i = 1,2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R-module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2.

Theorem 4.5. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that $N = N_1 \times M_2$ is a proper submodule of M. Then the following conditions are equivalent:*

1. *N_1 is a classical primary submodule of M_1;*
2. *N is a classical primary submodule of M;*
3. *N is a weakly classical primary submodule of M.*

Proof. (1)⇒(2) Let $(a_1,a_2)(b_1,b_2)(m_1,m_2) \in N$ for some $(a_1,a_2),(b_1,b_2) \in R$ and $(m_1,m_2) \in M$. Then $a_1b_1m_1 \in N_1$ so either $a_1m_1 \in N_1$ or $b_1^tm_1 \in N_1$ for some $t \geq 1$, which shows that either $(a_1,a_2)(m_1,m_2) \in N$ or $(b_1,b_2)^t(m_1,m_2) \in N$. Consequently N is a classical primary submodule of M.
(2)⇒(3) It is clear that every classical primary submodule is a weakly classical primary submodule.
(3)⇒(1) Let $abm \in N_1$ for some $a,b \in R_1$ and $m \in M_1$. We may assume that $0 \neq m' \in M_2$. Therefore $0 \neq (a,1)(b,1)(m,m') \in N$. So either $(a,1)(m,m') \in N$ or $(b,1)^t(m,m') \in N$ for some $t \geq 1$. Therefore $am \in N_1$ or $b^tm \in N_1$. Hence N_1 is a classical primary submodule of M_1. ∎

Proposition 4.6. *Let $R = R_1 \times R_2$ be a decomposable ring and $M = M_1 \times M_2$ be an R-module where M_1 is an R_1-module and M_2 is an R_2-module. Suppose that N_1, N_2 are proper submodules of M_1, M_2, respectively. If $N = N_1 \times N_2$ is a weakly classical primary submodule of M, then N_1 is a weakly prime submodule of M_1 and N_2 is a weakly prime submodule of M_2.*

Proof. Suppose that $N = N_1 \times N_2$ is a weakly classical primary submodule of M. By hypothesis, there exist $x \in M_1 \backslash N_1$ and $y \in M_2 \backslash N_2$. First, we show that N_1 is a weakly prime submodule of M_1. Let $0 \neq am_1 \in N_1$ for some $a \in R_1$ and $m_1 \in M_1$. Then $0 \neq (1,0)(a,1)(m_1,y) \in N_1 \times N_2 = N$. Notice that if $(a,1)(m_1,y) \in N_1 \times N_2 = N$, then $y \in N_2$ which is a contradiction. So we get $(1,0)^t(m_1,y) \in N_1 \times N_2 = N$ for some $t \geq 1$. Thus $m_1 \in N_1$. Hence N_1 is a weakly prime submodule of M_1. A similar argument shows that N_2 is a weakly prime submodule of M_2. ∎

The following example shows that the converse of Proposition 4.6 is not true in general.

Example 4.7. Let $R = M = \mathbb{Z} \times \mathbb{Z}$ and $N = p\mathbb{Z} \times q\mathbb{Z}$ where p, q are two distinct prime integers. Since $p\mathbb{Z}$, $q\mathbb{Z}$ are prime ideals of $\mathbb{Z}$, then $p\mathbb{Z}$, $q\mathbb{Z}$ are weakly primary (weakly classical primary) $\mathbb{Z}$-submodules of $\mathbb{Z}$. Notice that $(0,0) \neq (p,1)(1,q)(1,1) = (p,q) \in N$, but $(p,1)(1,1) \notin N$ and $(1,q)^t(1,1) \notin N$ for every $t \geq 1$. So N is not a weakly classical primary submodule of M.

Theorem 4.8. *Let $R = R_1 \times R_2 \times R_3$ be a decomposable ring and $M = M_1 \times M_2 \times M_3$ be an R-module where M_i is an R_i-module, for $i = 1,2,3$. If N is a weakly classical primary submodule of M, then either $N = \{(0,0,0)\}$ or N is a classical primary submodule of M.*

Proof. Since $\{(0,0,0)\}$ is a weakly classical primary submodule in any module, we may assume that $N = N_1 \times N_2 \times N_3 \neq \{(0,0,0)\}$. We assume that N is not a classical primary submodule of M and reach a contradiction. Without loss of generality we may assume that $N_1 \neq 0$ and so there is $0 \neq n \in N_1$. We claim that $N_2 = M_2$ or $N_3 = M_3$. Suppose that there are $m_2 \in M_2 \backslash N_2$ and $m_3 \in M_3 \backslash N_3$. Get $r \in (N_2 :_{R_2} M_2)$ and $s \in (N_3 :_{R_3} M_3)$. Since

$$(0,0,0) \neq (1,r,1)(1,1,s)(n,m_2,m_3) = (n,rm_2,sm_3) \in N,$$

then $(1,r,1)(n,m_2,m_3) = (n,rm_2,m_3) \in N$ or $(1,1,s)^t(n,m_2,m_3) = (n,m_2,s^tm_3) \in N$ for some $t \geq 1$. Therefore either $m_3 \in N_3$ or $m_2 \in N_2$, a contradiction. Hence $N = N_1 \times M_2 \times N_3$ or $N = N_1 \times N_2 \times M_3$. Let $N = N_1 \times M_2 \times N_3$. Then $(0,1,0) \in (N :_R M)$. Clearly $(0,1,0)^2N \neq \{(0,0,0)\}$. So $(N :_R M)^2N \neq \{(0,0,0)\}$ which is a contradiction, by Theorem 2.19. In the case when $N = N_1 \times N_2 \times M_3$ we have that $(0,0,1) \in (N :_R M)$ and similar to the previous case we reach a contradiction. ■

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