Application of measure of noncompactness to ℓ_1-solvability of infinite systems of second order differential equations

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Abstract

The purpose of this work is to establish a new generalization of Darbo type fixed point theorem using the concept of the so-called α-admissibility and the Schauder fixed point theorem. We also include an example which shows that our results are applicable where the previous ones are not. Moreover, we apply our main result to the problem of existence of solutions for a class of infinite systems of second order differential equations.

1 Introduction

The theory of measures of noncompactness has been applied in the fields of topology, functional analysis and operator theory. There are various type of definitions of the notion of measures of noncompactness on metric and topological spaces in different way but it has been initially introduced by Kuratowski. In 1955, the well-known Italian mathematician G. Darbo [11] presented a fixed point theorem which guarantees the existence of a fixed point for so-called condensing operators and generalizes both the classical Schauder fixed point principle and a special type of Banach's contraction principle. He applied the notion of measure of noncompactness introduced by Kuratowski in 1930. In the last years there appeared

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so many papers concerning with the concept of measure of noncompactness. To introduce a measure of noncompactness, Kuratowski defined for the family of all bounded subsets of metric spaces (E, d) the function $\alpha(M)$ given below, which is one of the three important kinds of measure of noncompactness which arise over and over in applications:

$$\alpha(M) = \inf\{\epsilon > 0 : M \text{ may be covered by finitely many sets of diameter} \leq \epsilon\},$$

and is known by *Kuratowski measure of noncompactness* [19]. The second one is the *Istrăţescu measure of noncompactness* (*or lattis measure of noncompactness*)[16]

$$\beta(M) = \inf\{\epsilon > 0 : \text{there exists a sequence } (x_n)_n \text{ in } M \text{ with} \\ \| x_n - x_m \| \geq \epsilon \text{ for } m \neq n\},$$

and the third one is the *Hausdorff measure of noncompactness* (*or ball measure of noncompactness*)[14]

$$\gamma(M) = \inf\{\epsilon > 0 : \text{there exists a finite } \epsilon\text{-net for } M \text{ in } E\} \tag{1.1}$$

where by a finite ϵ-net for M in E we mean, as usual, a set $\{x_1, x_2, ..., x_n\} \subseteq E$ such that $\bigcup_{i=1}^{n} B_\epsilon(E; x_i)$ as finite union of open balls covers M.

As a very important generalization of Schauder's fixed point theorem, Darbo type fixed point theorem has novel applications in both linear and nonlinear models. Intuitively, such applications are characterized by some "loss of compactness" which arises in many fields: imbedding theorems between Sobolev spaces with critical exponent, imbedding over domains with irregular boundary, linear composition operators over the complex unit disc, nonlinear integral equations (also with delay), differential equations over unbounded domains, fractional differential equations, infinite systems, etc. Recently, the measure of noncompactness has been applied in several papers (see [1-4], [8-10], [13,27]).
The aim of this paper is to present a new generalization of Darbo type fixed point theorem which also improves the corresponding results given by the first author et. al [3]. Moreover, to show the applicability of our main result an example is given which shows that Darbo type fixed point theorem and some recent obtained results can not be applied. Finally, in the last section, we give an existence result for a class of infinite systems of second ordered differential equations.

2 Preliminaries

Let E be a Banach space and $\overline{\Omega}$, $Conv\Omega$ stand for the closure and closed convex hull of $\Omega \subset E$, respectively. Besides, let $\mathfrak{M}_E$ indicates the family of all nonempty bounded subsets of E and $\mathfrak{N}_E$ denotes the family of all relatively compact sets in E.

In the following we use the definition of the measure of noncompactness given in [5].

Definition 2.1 ([5]). A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a *measure of noncompactness* in E if it satisfies the following conditions

(1) The family $\ker\mu = \{\Omega \in \mathfrak{M}_E : \mu(\Omega) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$;

(2) $\Omega_1 \subset \Omega_2 \Rightarrow \mu(\Omega_1) \leq \mu(\Omega_2)$;

(3) $\mu(\overline{\Omega}) = \mu(\Omega)$;

(4) $\mu(Conv\Omega) = \mu(\Omega)$;

(5) $\mu(\lambda\Omega_1 + (1-\lambda)\Omega_2) \leq \lambda\mu(\Omega_1) + (1-\lambda)\mu(\Omega_2)$ for $\lambda \in [0,1]$;

(6) If (Ω_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $\Omega_{n+1} \subset \Omega_n$ and $\lim_{n\to\infty} \mu(\Omega_n) = 0$ then the intersection set $\Omega_\infty = \bigcap_{n=1}^{\infty} \Omega_n$ is nonempty;

We now present a fixed point theorem of Darbo type proved by Banaś and Goebel [5] which will be extended by our result.

Theorem 2.2 ([5]). *Let C be a nonempty, bounded, closed, and convex subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that $\mu(T(\Omega)) \leq k\mu(\Omega)$ for any nonempty subset Ω of C. Then T has a fixed point in the set C.*

Recently, the first author et. al [3] improved the theorem as above using the control function as follows.

Theorem 2.3 ([3]). *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T : C \longrightarrow C$ be a continuous function satisfying*

$$\mu(T(\Omega)) \leq \phi(\mu(\Omega)) \tag{2.1}$$

for each $\Omega \subset C$ where μ is an arbitrary measure of noncompactness and $\phi : [0,\infty) \to [0,\infty)$ is monotonic increasing (not necessarily continuous) function with $\lim_{n\to\infty} \phi^n(t) = 0$. Then T has at least one fixed point in the set C.

3 Main result

The notations and terminologies used in this section will serve to obtain the results. Throughout this section, we initially introduce the new notions of (α,ϕ,ψ)-μ-condensing and α-admissible operators. Denote with Φ the class of functions $\phi : [0,+\infty) \to [0,+\infty)$ such that

$$\liminf_{n\to\infty} \phi(a_n) = 0, \quad \text{if } \lim_{n\to\infty} a_n = 0 \tag{3.1}$$

where $\{a_n\}$ is a nonnegative sequence. For $\phi \in \Phi$, let functions $\psi : [0,+\infty) \longrightarrow [0,+\infty)$ satisfy the following conditions:

(a) ψ is a lower semi-continuous function with $\psi(t) = 0$ if and only if $t = 0$,

(b) $\liminf_{n\to\infty} \phi(a_n) < \psi(a)$ if $\lim_{n\to\infty} a_n = a > 0$.

We denote the class of all such functions by Ψ_ϕ.

In order to present our main result we give the following definition.

Definition 3.1. Let $T : W \subseteq E \to E$ be an arbitrary mapping. We say that T is *(α, ϕ, ψ)-μ-condensing* if there exist functions $\alpha : \mathfrak{M}_E \to [0, +\infty)$, $\phi \in \Phi$ and $\psi \in \Psi_\phi$ such that

$$\alpha(\Omega)\psi(\mu(T\Omega)) \leq \phi(\mu(\Omega)) \quad \text{for } \Omega \subseteq W,$$

where Ω and its image $T\Omega$ belong to $\mathfrak{M}_E$.

Notice that if a mapping $T : W \subseteq E \to E$ satisfies the Darbo condition with respect to a constant $k \in [0, 1)$ and a measure μ, that is,

$$\mu(T\Omega) \leq k\mu(\Omega), \quad \text{for } \Omega \subseteq W \text{ and } \Omega, T\Omega \in \mathfrak{M}_E,$$

then T is (α, ϕ, ψ)-μ-condensing operator, where $\alpha(\Omega) = 1$ for any set $\Omega \subseteq W$ such that $\Omega \in \mathfrak{M}_E$, ψ is the identity mapping and $\phi(t) = kt$ for all $t \geq 0$. For this case, T is called μ-contraction.

Definition 3.2. Let $T : W \subseteq E \to E$ and $\alpha : \mathfrak{M}_E \to [0, +\infty)$ be given mappings. We say that T is *α-admissible* if we have

$$\alpha(\Omega) \geq 1 \Longrightarrow \alpha(ConvT\Omega) \geq 1, \qquad \Omega \subseteq W,\ \Omega, T\Omega \in \mathfrak{M}_E.$$

The following examples show that there exist such mappings with α-admissibility.

Example 3.3. Let $E = [0, \infty)$. Define $T : E \to E$ and $\alpha : \mathfrak{M}_E \to [0, +\infty)$ by

$$T(x) = e^x - 1 \quad \text{for all } x \in E$$

and $\alpha(\Omega) = \text{diam}(\Omega)$ which may be written as $\alpha(\Omega) = \sup \Omega - \inf \Omega$ for any $\Omega \in \mathfrak{M}_E$. Obviously, if $\alpha(\Omega) \geq 1$, then $M_\Omega - m_\Omega \geq 1$ where M_Ω and m_Ω are the supremum and the infimum of Ω, respectively. Hence, we get

$$\alpha(ConvT\Omega) = \alpha(T\Omega) = (e^{M_\Omega} - e^{m_\Omega}) \geq e^{m_\Omega}(e - 1) \geq e - 1 > 1$$

which implies that T is an α-admissible mapping.

Example 3.4. Let $E = [0, \frac{\pi}{2}]$. Define $T : E \to E$ and $\alpha : \mathfrak{M}_E \to [0, +\infty)$ by

$$T(x) = \sqrt{\sin x} \quad \text{for all } x \in E$$

and $\alpha(\Omega) = \sup_{x,y\in\Omega} f(x, y)$ where $\Omega \in \mathfrak{M}_E$ and

$$f(x, y) = \begin{cases} 2 - \cos(x - y) & x > y, \\ \frac{1}{3} & \text{o.w.} \end{cases}$$

for all $x, y \in \Omega \subset E$. Notice that $\alpha(\Omega) < 1$ if and only if the nonempty bounded set Ω is singleton. Similar to the previous example, suppose that $\Omega \in \mathfrak{M}_E$ and

$\alpha(\Omega) \geq 1$. Then following the definition of α there exist $x, y \in \Omega$ so that $x > y$ and $f(x,y) = 2 - \cos(x-y)$. Clearly, we have

$$T(x) = \sqrt{\sin x} > \sqrt{\sin y} = T(y)$$

which implies that

$$\alpha(ConvT\Omega) \geq \alpha(T\Omega) = \sup_{a,b \in T\Omega} f(a,b) \geq f(Tx, Ty) = 2 - \cos(Tx - Ty) \geq 1.$$

This means that T is α-admissible.

Now, we are ready to present our main result of this section.

Theorem 3.5. *Let $C \in \mathfrak{M}_E$ be a closed and convex subset of a Banach space E and $T : C \to C$ be a continuous (α, ϕ, ψ)-μ-condensing operator, where μ is an arbitrary measure of noncompactness. Suppose that T is α-admissible and $\alpha(C) \geq 1$. Then T has at least one fixed point which belongs to $\ker\mu$.*

Proof. Consider C_n as a sequence of convex sets in $\mathfrak{M}_E$ which is recursively defined by

$$C_{n+1} = ConvTC_n, \;\; C_0 = C.$$

Clearly, applying mathematical induction and using the fact that C is convex and $TC \subset C$ one can obtain $C_{n+1} \subseteq C_n$. On the other hand, α-admissibility T together with the assumption $\alpha(C) \geq 1$ implies that

$$\alpha(C_1) = \alpha(ConvTC_0) \geq 1.$$

Again using the mathematical induction, we obtain that $\alpha(C_n) \geq 1$ for $n \geq 1$. Since T is (α, ϕ, ψ)-μ-condensing we have

$$\psi(\mu(C_{n+1})) = \psi(\mu(ConvTC_n)) = \psi(\mu(TC_n)) \leq \alpha(C_n)\psi(\mu(TC_n)) \leq \phi(\mu(C_n)) \quad \text{for } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

Suppose that for some $n_0 \in \mathbb{N} \cup \{0\}$, $\mu(C_{n_0}) = 0$. Then the monotonicity of μ implies that $\mu(C_n) = 0$ for all $n \geq n_0$ and so $\mu(C_\infty) = 0$ where $C_\infty = \bigcap_{n=0}^{\infty} C_n$. Clearly, C_∞ is a bounded, closed and convex set which belongs to $\ker\mu$ and moreover C_∞ is invariant under the continuous mapping T. Now, the classical Schauder fixed point theorem guarantees the existence of fixed point for the mapping T and this completes the proof for this case.

Now consider $\mu(C_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mu(C_{n+1}) \leq \mu(C_n)$ for all $n \in \mathbb{N} \cup \{0\}$, consequently, there exists $\theta \in \mathbb{R}_+$ such that

$$\mu(C_n) \to \theta \quad \text{as } n \to \infty.$$

If $\theta = 0$, then following the discussion as above one can conclude that T has a fixed point in $C_\infty = \bigcap_{n=0}^{\infty} C_n$ with $\mu(C_\infty) = 0$. Otherwise, in view of (3.2) and the properties (a) and (b) of ψ we have

$$\psi(\theta) \leq \liminf_{n\to\infty} \psi(\mu(C_n)) \leq \liminf_{n\to\infty} \phi(\mu(C_n)) < \psi(\theta)$$

which is a contradiction. ∎

Here, some immediate remarks are provided as follows.

Remark 3.6. Theorem 2.2 is easily implied by the recent result using the identity mapping instead of ψ, $\phi(t) = kt$ for $t \geq 0$ and considering the constant function $\alpha(\Omega) = 1$ for $\Omega \in \mathfrak{M}_E$.

Remark 3.7. In Theorem 3.5, it is easy to see that substituting the identity mapping for ψ and considering the constant function $\alpha(\Omega) = 1$ for $\Omega \in \mathfrak{M}_E$ we infer the relation (2.1). Also we note that the properties of function ϕ in Theorem 2.3 can imply (3.1).

The effectiveness of the generalization with respect to our main result may be seen from the following example, while both Theorems 2.2 and 2.3 are not applicable.

Example 3.8. Let $E = [-3, 2]$, and $T : E \to E$ be a mapping given by

$$Tx = \begin{cases} e^{-0.1x} - 2 & x \in [-3, 1], \\ -\frac{5}{4}x + e^{-0.1} - \frac{3}{4} & x \in (1, 2]. \end{cases}$$

Suppose that $\alpha : \mathfrak{M}_E \to [0, +\infty)$ where

$$\alpha(\Omega) = \begin{cases} 2 & \Omega \subseteq [-3, 1], \\ \frac{1}{3} & \Omega \subseteq (1, 2], \\ 0 & \text{o.w.} \end{cases}$$

Clearly, T is a continuous α-admissible mapping and one can check this out using the fact that $T([-3, 1]) \subseteq [-3, 1]$. We claim that

$$\alpha(\Omega)\mu(T\Omega) \leq \phi(\mu(\Omega)),$$

for any set $\Omega \in \mathfrak{M}_E$. To prove this we have the following possible cases:

Case 1. Consider $\Omega \subseteq [-3, 1]$ such that

$$\max \Omega = M, \ \ \min \Omega = m, \ \ m \neq M \tag{3.3}$$

then defining the measure of noncompactness μ by $\mu(\Omega) = \operatorname{diam}(\Omega)$, we get

$$\alpha(\Omega)\mu(T\Omega) = 2 \operatorname{diam}(T\Omega) = 2(e^{-0.1m} - e^{-0.1M}) \leq \frac{1}{2}(M - m). \tag{3.4}$$

To prove the recent inequality, note that the mean value theorem states there exists a point c in (m, M) such that

$$\frac{1}{10}e^{-0.1c} = |\frac{e^{-0.1m} - e^{-0.1M}}{M - m}|.$$

On the other hand, since $c \in (-3, 1)$ we obtain

$$\frac{1}{10}e^{-0.1c} < \frac{1}{10}e^{0.3} \cong 0.1349,$$

which shows that the inequality in (3.4) is true. This implies that

$$\alpha(\Omega)\mu(T\Omega) \leq \phi(\mu(\Omega)),$$

with $\Omega \subseteq [-3, 1]$ and $\phi(t) = \frac{1}{2}t$.

Case 2. Let $\Omega \subseteq (1, 2]$ and (3.3) still be assumed. Then we have

$$\alpha(\Omega)\mu(T\Omega) = \frac{1}{3}(\frac{5}{4}(M - m)) < \frac{3}{4}(M - m) = \phi(\mu(\Omega)),$$

where $\phi(t) = \frac{3}{4}t$.

Case 3. The case that Ω neither containing in $[-3, 1]$ nor $(1, 2]$ is trivial.

Therefore, we conclude that

$$\alpha(\Omega)\psi(\mu(T\Omega)) \leq \phi(\mu(\Omega)),$$

for any set $\Omega \in \mathfrak{M}_E$ so that ψ is the identity mapping and $\phi(t) = \frac{3}{4}t$. Now, all conditions of Theorem 3.5 are fulfilled and this implies that T has a fixed point containing in a set $\Omega_0 \subseteq [-3, 2]$ with $\mu(\Omega_0) = 0$, i.e., $\operatorname{diam}(\Omega_0) = 0$. This shows that Ω_0 is a singleton and so the fixed point of T is unique.

Remark 3.9. In the previous example, we note that T is not a μ-contraction and thus Theorem 2.2 can not be applied. To clarify this fact let $\Omega = [\frac{3}{2}, 2] \in \mathfrak{M}_E$ then we get

$$\mu(T\Omega) = \frac{5}{4}(2 - \frac{3}{2}) > \frac{1}{2} = \mu(\Omega).$$

Remark 3.10. Considering Example 3.8, we notice that T does not hold in (2.1) and hence Theorem 2.3 is not usable. Indeed, by taking $\Omega = [\frac{6}{5}, 2] \in \mathfrak{M}_E$ and applying (2.1) we should have

$$\mu(T\Omega) = \frac{5}{4}(2 - \frac{6}{5}) = 1 \leq \phi(\mu(\Omega)) = \phi(\frac{4}{5}).$$

where ϕ is as defined in Theorem 2.3. This implies that

$$\phi^n(\frac{4}{5}) \nrightarrow 0 \quad \text{as } n \to \infty$$

which contradicts to

$$\lim_{n\to\infty} \phi^n(t) = 0 \ \text{ for } t \geq 0$$

as a property of ϕ in Theorem 2.3.

4 Infinite systems of second order differential equations

The existence of solutions of infinite systems of ordinary differential equations have been investigated in several papers using various types of fixed point theorems (see [6], [7], [22], [24] and the references therein). Here, in this section, we study the solvability of a class of infinite systems of second order differential equations in the Banach space $(\ell_1, \|\cdot\|_1)$. It is well known that in the space

ℓ_1 the Hausdorff measure of noncompactness γ which is defined in (1.1) can be formulated by the following formula ([5]):

$$\gamma(B) = \lim_{n\to\infty}\left(\sup_{u\in B}\sum_{k\geq n}|u_k|\right) \tag{4.1}$$

where $u = (u_i) \in \ell_1$ and $B \in \mathfrak{M}_{\ell_1}$. Using the technique of measure of noncompactness, we are going to show how the abstract result obtained in this paper can be applied to the infinite system of second order differential equations as form of

$$-\frac{d^2u_i}{dt^2} = f_i(t, u_0, u_1, u_2, ...) \tag{4.2}$$

with the initial conditions $u_i(0) = u_i(T) = 0$ for $i = 0, 1, 2, ...$ and $t \in I = [0, T]$.

From now on, for the simplicity, we will write $f_i(t, u)$ instead of $f_i(t, u_0, u_1, u_2, ...)$.

Let $C(I, \mathbb{R})$ denote the space of all continuous real functions and $C^2(I, \mathbb{R})$ be the class of all functions with two continuous derivatives on I. It is clear that $u \in C^2(I, \mathbb{R})$ is a solution of Eq. (4.2) if and only if $u \in C(I, \mathbb{R})$ is a solution of the system of integral equations

$$u_i(t) = \int_0^T G(t,s)f_i(s, u(s))ds, \quad \text{for } t \in I \tag{4.3}$$

where $f_i(t, u) \in C(I, \mathbb{R})$, $i = 0, 1, 2, ...$ and the green function associated to (4.2) is given by

$$G(t,s) = \begin{cases} \frac{t}{T}(T-s), & 0 \leq t \leq s \leq T, \\ \frac{s}{T}(T-t), & 0 \leq s \leq t \leq T. \end{cases} \tag{4.4}$$

Indeed, by (4.4) we have

$$u_i(t) = \int_0^t \frac{s}{T}(T-t)f_i(s, u(s))ds + \int_t^T \frac{t}{T}(T-s)f_i(s, u(s))ds,$$

now compute

$$\frac{du_i(t)}{dt} = -\frac{1}{T}\int_0^t sf_i(s, u(s))ds + \frac{1}{T}\int_t^T (T-s)f_i(s, u(s))ds.$$

Again using the differentiation from the both sides of recent equality we obtain

$$\frac{d^2u_i(t)}{dt^2} = -\frac{1}{T}(tf_i(t, u(t))) + \frac{1}{T}((t-T)f_i(t, u(t))) = -f_i(t, u(t)).$$

To get more details about green functions we refer the reader to see [12].

System (4.2) will be investigated under the following hypotheses:

(H1) the functions f_i are defined on $I \times \mathbb{R}^\infty$ and take real values ($i = 0, 1, 2, ...$). In addition, the operator f is given on the space $I \times \ell_1$ as follows

$$(t, u) \mapsto (fu)(t) = (f_1(t, u), f_2(t, u), ...)$$

which maps the space $I \times \ell_1$ into ℓ_1 and is such that the class of all functions $\{(fu)(t)\}_{t\in I}$ is equicontinuous at every point of the space ℓ_1.

(H2) there exist a nonnegative mapping $g : I \to \mathbb{R}_+$, function $\Lambda : I \times \ell_1 \to \mathbb{R}$ and a superadditive mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$, i.e., $\phi(s) + \phi(t) \leq \phi(s+t)$ for $s, t \geq 0$, such that

$$\Lambda(t, u) \geq 0 \Longrightarrow |f_i(t, u)| \leq g_i(t)\phi(|u_i|) \tag{4.5}$$

where $t \in I$, $u = (u_i) \in \ell_1$ and $i \geq j$ for some $j \in \mathbb{N} \cup \{0\}$;

(H3) for any fixed element $t \in I$ function $G(t,s)g(s)$ is integrable on I such that

$$g(s) = \limsup_{i \to \infty} g_i(s).$$

Moreover, if a nonnegative sequence $\{a_n\}$ converges to a positive number a then

$$\liminf_{n \to \infty} \phi(a_n) < \frac{a}{M} \tag{4.6}$$

where $M \in \mathbb{R}^+$ such that

$$\sup_{t \in I} \int_o^T G(t,s)g(s)ds \leq M;$$

(H4) there is a function u such that

$$\Lambda(t, u(t)) \geq 0 \quad \text{for all } t \in I, \tag{4.7}$$

besides, for $t \in I$ we have the following implication

$$\Lambda(t, v(t)) \geq 0 \Longrightarrow \Lambda\left(t, \left(\int_0^T G(t,s) f_i(s, v(s))ds\right)\right) \geq 0 \tag{4.8}$$

for all $v(t) \in \ell_1$.

Now we are ready to formulate the following result.

Theorem 4.1. *Under the hypotheses (H1)-(H4), system (4.2) has at least one solution $u(t) = (u_i(t))$ such that $u(t) \in \ell_1$ for $t \in I$.*

Proof. Consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on $C(I, \ell_1)$ in the following way, for $t \in I$

$$(\mathcal{F}u)(t) = ((\mathcal{F}_i u)(t)) = \left(\int_0^T G(t,s) f_i(s, u(s))ds\right),$$
$$u(t) = (u_i(t)) \in \ell_1, \ u_i \in C(I, \mathbb{R}).$$

Clearly, the operator $\mathcal{F}$ is continuous on $C(I, \ell_1)$ using condition (H1). The continuity of function $\mathcal{F}u$ is also obvious and $(\mathcal{F}u)(t) \in \ell_1$ if $u(t) = (u_i(t)) \in \ell_1$, followed from the fact that ϕ is superadditive together with (4.5) and (H3):

$$\begin{aligned} \|\mathcal{F}u)(t)\|_1 &\leq \sum_{i=0}^{\infty} \int_0^T G(t,s)|f_i(s, u(s))|ds \\ &\leq \frac{T}{4} \int_0^T \|(fu)(s))\|_1 ds < \infty. \end{aligned}$$

Now, consider the operator $\mathcal{F} = (\mathcal{F}_i)$ on a nonempty bounded set $B \in \mathfrak{M}_{\ell_1}$ including the functions $u(t) = (u_i(t)) \in \ell_1$ with $\Lambda(t, u(t)) \geq 0$ for arbitrary fixed element $t \in I$. Then using (4.1) we get

$$\begin{aligned}
\gamma(\mathcal{F}B) &= \lim_{n\to\infty} \left(\sup_{u(t)\in B} \sum_{k\geq n} \left| \int_0^T G(t,s) f_k(s,u(s))ds \right| \right) \\
&\leq \lim_{j\to\infty} \left(\sup_{u(t)\in B} \sum_{k\geq j} \int_0^T G(t,s) g_k(s)\phi(|u_k(s)|)ds \right) \\
&\leq M \lim_{j\to\infty} \left(\sup_{u\in B} \phi(\sum_{k\geq j} \|u_k\|) \right) \\
&= M \lim_{j\to\infty} \phi\left(\sup_{u\in B} \sum_{k\geq j} \|u_k\| \right) \\
&\leq M\phi(\gamma(B)).
\end{aligned}$$

Hence we conclude that

$$\alpha(B)\psi(\mathcal{F}B) \leq \phi(\gamma(B))$$

for any nonempty bounded set $B \in \mathfrak{M}_{\ell_1}$ where $\alpha : \mathfrak{M}_{\ell_1} \to [0, \infty)$ is given by

$$\alpha(B) = \begin{cases} 1 & \Lambda(t, u(t)) \geq 0 \text{ for some } u \in B \text{ and all } t \in I; \\ 0 & \text{o.w.} \end{cases}$$

and $\psi(s) = \frac{s}{M}$ for $s \in \mathbb{R}_+$. Obviously, $\psi \in \Psi_\phi$ followed by (4.6). On the other hand, using condition (H4) we infer that the operator $\mathcal{F}$ is α-admissible and satisfies all the conditions of Theorem 3.5 and hence $\mathcal{F}$ has at least one fixed point $u = u(t)$ such that $u(t) \in \ell_1$ for all $t \in I$. Consequently, the function $u = u(t)$ is a solution of infinite system (4.2) and this completes the proof. ∎

Now, in order to illustrate the result as above we provide the following example.

Example 4.2. Let us consider the system of differential equations as form of

$$-\frac{d^2u_n}{dt^2} = \frac{t(T-t)\exp(-nt)}{(n+1)^4} + \sum_{m=n}^{\infty} \frac{u_m(t)\sqrt{t}}{(1+n^2)(m+1)^2}, \quad n \in \mathbb{N}\cup\{0\},\ t \in I = [0,T],\ 0 < T < 2\sqrt{2}. \tag{4.9}$$

Clearly, $a_{nm}(t) := \sqrt{t}(1+n^2)^{-1}(m+1)^{-2}$ is continuous and $\sum_{m=n}^{\infty} a_{nm}(t)$ is absolutely uniformly continuous on I where $m, n \in \mathbb{N}\cup\{0\}$. Since $a_n(t) := \sum_{m=n}^{\infty} |a_{nm}(t)|$ is uniformly bounded on I we can consider the following notation

$$A = \sup\{a_n(t), t \in I, n \in \mathbb{N}\cup\{0\}\} < \infty. \tag{4.10}$$

Remark that

$$(fu)(t) = (f_n(t,u)) := \left(\frac{t(T-t)\exp(-nt)}{(n+1)^4} + \sum_{m=n}^{\infty} \frac{u_m(t)\sqrt{t}}{(1+n^2)(m+1)^2} \right) \in \ell_1 \quad \text{if } u(t) = \{u_n(t)\} \in \ell_1,$$

because we have

$$\|(fu)(t)\|_1 \leq \sum_{n=0}^{\infty} \frac{t(T-t)\exp(-nt)}{(n+1)^4} + \sum_{n=0}^{\infty}\sum_{m=n}^{\infty} \left|\frac{u_m(t)\sqrt{t}}{(1+n^2)(m+1)^2}\right| \\ \leq \frac{\pi^4 T^2}{360} + \left(1+\frac{\pi^2}{6}\right)\sqrt{T}\|u(t)\|_1 < \infty$$

To show that the operator $(fu)(t) = ((f_n u)(t))$ is uniformly continuous on ℓ_1 related to $t \in I$ we need to prove that the sequence $(f_n(u))$ is equicontinuous. Let us fix $\epsilon > 0$ arbitrarily and $u(t) = (u_n(t)) \in \ell_1$. Then using (4.10) and taking $v(t) = (v_n(t)) \in \ell_1$ with $\|u(t) - v(t)\|_1 \leq \delta(\epsilon) := \epsilon A^{-1}$ we obtain

$$|(f_n u)(t) - (f_n v)(t)| \leq \sum_{m=n}^{\infty} \frac{|u_m(t) - v_m(t)|\sqrt{t}}{(1+n^2)(m+1)^2} \\ \leq A\|u(t) - v(t)\|_1 \leq \epsilon$$

for any fixed n, which yields the continuity as desire and hence condition (H1) is satisfied. To prove that conditions (H2)-(H4) hold consider a function $\Lambda : I \times \ell_1 \to \mathbb{R}$ which takes nonnegative values if and only if $u(t) = (u_n(t)) \in \ell_1$ where $\{u_n(t)\}$ is a nonincreasing sequence in $\mathbb{R}_+$, $u(0) = u(T) = 0$ and we have

$$\frac{t(T-t)\exp(-nt)}{(n+1)^4} = o(u_n(t)), \tag{4.11}$$

uniformly with respect to $t \in (0, T)$. Recall that the notion of little-o is used for comparison of growth of two arbitrary sequences a_n and b_n and is defined by

$$a_n = o(b_n) \iff \lim_{n\to\infty} \frac{a_n}{b_n} = 0, \quad b_n \not\equiv 0.$$

It is easy to see that $\{u \in \ell_1 : \Lambda(t, u(t)) \geq 0, t \in I\} \neq \emptyset$. Let $\Lambda(t, u(t)) \geq 0$, then using the fact that $u(0) = u(T) = 0$ and (4.11) we get

$$\frac{t(T-t)\exp(-nt)}{(n+1)^4} \leq u_n(t), \text{ for } n > m \text{ and some } m \in \mathbb{N} \cup \{0\}, \tag{4.12}$$

for all $t \in I$. Hence, we have

$$|(f_n u)(t)| \leq \frac{t(T-t)\exp(-nt)}{(n+1)^4} + \sum_{k=n}^{\infty} \frac{|u_k(t)|\sqrt{t}}{(1+n^2)(k+1)^2} \\ \leq g_n(t)|u_n(t)|$$

where

$$g_n(t) = 1 + \frac{\pi^2\sqrt{t}}{6(1+n^2)} \text{ for all } t \in I,\ n > m.$$

Since $g(t) := \limsup_{n\to\infty} g_n(t) = 1$ so we get

$$M := \sup_{t\in I} \int_0^T G(t,s)g(s)ds = \sup_{t\in I} \left(\int_0^t \frac{s(T-t)}{T} ds + \int_t^T \frac{t(T-s)}{T} ds \right) = \frac{T^2}{8}.$$

Next, by taking $\phi(t)$ as the identity mapping we conclude that conditions (H2), (H3) and (4.7) are satisfied. It only remains to show that (4.8) holds. To do this, let $\Lambda(t,u(t)) \geq 0$ such that $t \in I$ and $u(t) = (u_n(t)) \in \ell_1$. Then from the definition of Λ, $(u_n(t))$ is a nonincreasing sequence in $\mathbb{R}_+$ and we have

$$\begin{aligned} f_{n+1}(t,u(t)) &= \frac{t(T-t)\exp(-(n+1)t)}{(n+2)^4} + \sum_{m=n+1}^{\infty} \frac{u_m(t)\sqrt{t}}{(n^2+2n+2)(m+1)^2} \\ &\leq \frac{t(T-t)\exp(-nt)}{(n+1)^4} + \sum_{m=n+1}^{\infty} \frac{u_m(t)\sqrt{t}}{(n^2+1)(m+1)^2}, \end{aligned}$$

for $t \in I$ and $n = 0,1,2,....$ This implies that

$$0 \leq f_{n+1}(t,u(t)) \leq f_n(t,u(t)), \quad t \in I,\ n = 0,1,2,....$$

Therefore,

$$0 \leq \int_0^T G(t,s) f_{n+1}(s,u(s))ds \leq \int_0^T G(t,s) f_n(s,u(s))ds, \quad t \in I,\ n = 0,1,2,....$$

We only need to show that

$$\frac{t(T-t)\exp(-nt)}{(n+1)^4} = o\Big(\int_0^T G(t,s) f_n(s,u(s))ds\Big),$$

uniformly with respect to $t \in (0,T)$. To do this, we should prove that

$$\frac{(n+1)^4}{t(T-t)} \int_0^T G(t,s)\exp(nt) f_n(s,u(s))ds \to \infty \text{ as } n \to \infty \tag{4.13}$$

uniformly on $(0,T)$. By a direct computation we see that

$$\begin{aligned} &\frac{(n+1)^4}{t(T-t)} \int_0^T G(t,s)\exp(nt) f_n(s,u(s))ds \\ &\quad \geq \frac{(n+1)^4}{tT} \int_0^{\frac{t}{2}} s^2(T-s)\exp(n(t-s))ds \\ &\quad \geq \frac{(n+1)^4 \exp(\frac{nt}{2})}{T^2} \bigg(\frac{2(-\frac{3}{n}+T)\exp(\frac{nt}{2})}{n^3} - \frac{t^2(-\frac{3}{n}+T)}{4n} \\ &\qquad - \frac{t(-\frac{3}{n}+T)}{n^2} - \frac{2(-\frac{3}{n}+T)}{n^3} \bigg) \\ &\quad \geq \frac{(n+1)^4}{T^2} \bigg(\frac{2(-\frac{3}{n}+T)}{n^3} - \frac{T^2(-\frac{3}{n}+T)}{4n} \\ &\qquad - \frac{T(-\frac{3}{n}+T)}{n^2} - \frac{2(-\frac{3}{n}+T)}{n^3} \bigg), \end{aligned}$$

for $n > \frac{3}{T}$, which converges uniformly to zero as $n \to \infty$ and this verifies (4.13). Therefore, all conditions (H1)-(H4) are fulfilled and applying Theorem 4.1 we conclude that system (4.9) has a solution in ℓ_1.

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