Variational inequalities in Musielak-Orlicz-Sobolev spaces

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To the memory of the Professor M. Sidi El Vally

Abstract

In this paper we prove an existence result for some class of variational boundary value problems for quasilinear elliptic equations in the Musielak-Orlicz spaces. Some results concerning the Trace mapping have also been provided, as well as existence results for some strongly nonlinear elliptic equations in Musielak-Orlicz spaces.

1 Introduction

This paper is concerned with the existence of solutions for variational boundary value problems for quasi-linear elliptic equations of the form

$$A(u) = f,$$

where the operator A is in the form:

$$A(u) \equiv \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} A_{\alpha}(x, u, \nabla u, ..., \nabla^m u) \tag{1}$$

on an open subset Ω of R^n. Existence theorems for problems of this type were first obtained by Višik [21, 22] using compactness arguments and a priori estimates on

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$(m+1)st$ derivatives. Since 1963, these problems have been extensively studied by Browder and others in the context of the theory of mappings of monotone type from a reflexive Banach space to its dual and in the case where the coefficients A_α have polynomial growth in u and its derivatives [2] , [3] , [18]. From 1970 these results have been extended by Donaldson [8], Gossez [11] , [12] and Gossez and Mustonen in [14] to the case where the coefficients A_α do not necessarily have polynomial growth in u and its derivatives. The Banach spaces in which the problems formulated (the Orlicz-Sobolev spaces) are not reflexive and the corresponding mappings of monotone type are not bounded nor everywhere defined and do not generally satisfy a global a priori bounded(and consequently are not generally coercive).

In the last decade several works have been concerned to extend the classical polynomial growth to the x-dependent polynomial growth case in the so-called variable exponent Sobolev spaces (see [16] and references within), and also [23] .

Recently Mihăilescu and Rădulescu in [19] and Fan and Guan in [10] have obtained new results which improved the already known existence results for the p(x)-Laplacian operator in the Musielak-Orlicz-Sobolev spaces $W^1L_\varphi(\Omega)$ under some assumptions such as the condition Δ_2 on φ and also the uniform convexity of φ which assure that the space $L_\varphi(\Omega)$ is reflexive.

The topic considered in this paper includes and generalizes the above settings. for example if we take $\varphi(x,t) = t^{p(x)}$, our results improve their counterparts in the statement of the variable exponent sobolev spaces $W^{m,p(x)}$ with less restrictions on the function $p(x)$, namely, we can drop the condition that $p^+ = \text{ess}\sup_{x\in\Omega} p(x)$ is finite.

The authors in [4] and [6] have studied some very important properties of the Musielak-Orlicz-Sobolev spaces $W^mL_\varphi(\Omega)$, namely some approximation theorems which assure the existence of complementary systems generate by $W^mL_\varphi(\Omega)$ and $W_0^mL_\varphi(\Omega)$. These results have been obtained under the assumption that φ satisfies the following condition of Log-Hölder type of continuity :
There exists a constant $A > 0$ such that for all $x, y \in \Omega : |x-y| \le \frac{1}{2}$ we have :

$$\frac{\varphi(x,t)}{\varphi(y,t)} \le t^{\frac{A}{\log(\frac{1}{|x-y|})}} \tag{2}$$

for all $t \ge 1$.

Note that this condition is out of the question in the classical setting because φ is independent of the first variable. For examples of Musielak-Orlicz functions satisfy the condition (2) see the Appendix.

The study of variational boundary value problems for quasi-linear elliptic equations in the general case when the Musielak-Orlicz-Sobolev spaces $W^mL_\varphi(\Omega)$ are not reflexive was initiated by the authors in [7] with the assumption that the conjugate function ψ of φ has the Δ_2 property.

In this paper we investigate these problems without any assumption of type Δ_2 on φ and its conjugate ψ we assume only that φ satisfies the condition (2). Ours results generalize those of Gossez in [11],[13], [12] and Gossez and Mustonen in [14].

This is a new research topic which is worthy of attention and opens wide doors before several works and applications. In this regard we can point out that the x-dependence of the growth and coercitivity conditions allows to consider influence of magnetic (or electric) field which provides the possibility to understand and study the problems related to non-Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field [15].

Also the result of this paper can be for example applied for finding a weak solution for the φ-Laplacian equation

$$\Delta_{\varphi} u (= div(\frac{a(x, |\nabla u|)}{|\nabla u|}.\nabla u)) + f = 0$$

where a is the derivative of φ with respect to t.

Section 2 contains some preliminaries about the Musielak-Orlicz-Sobolev spaces and some useful lemmas as well as some facts about the complementary system. In Section 3 we introduce the main results of this paper, firstly we study the trace mapping in subsection 3.1, subsection 3.2 is devoted to some imbedding results, in subsection 3.3 and 3.4 we investigate the conditions imposed in the theorem 1(below) on the mapping T and the convex K, finally in the subsection 3.5 we give an existence result for the strongly nonlinear elliptic problem. Section 4 is an appendix that contains examples of Musielak-Orlicz function satisfy the condition (2).

2 Preliminaries

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is [20]. We also include the definition of complementary system, an abstract result and some preliminaries Lemmas to be used later.

2.1 Musielak-Orlicz-Sobolev spaces

Let Ω be an open subset of R^n and let φ be a real-valued function defined in $\Omega \times R_+$ and satisfying the following conditions :
a) $\varphi(x, .)$ is an N-function i.e. convex, nondecreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$, and

$$\lim_{t \to 0} \frac{\varphi(x, t)}{t} = 0 \text{ for almost everywhere } x \in \Omega,$$

$$\lim_{t \to \infty} \frac{\varphi(x, t)}{t} = \infty \text{ for almost everywhere } x \in \Omega,$$

b) $\varphi(.,t)$ is a measurable function.
A function $\varphi(x,t)$, which satisfies the conditions a) and b) is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x,t)$ we put $\varphi_x(t) = \varphi(x,t)$ and we associate its nonnegative reciprocal function with respect to t, φ_x^{-1} i.e.

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t$$

For any two Musielak-Orlicz functions φ and γ we introduce the following ordering :
c)if there exists two positives constants c and T such that for almost everywhere $x \in \Omega$:

$$\varphi(x,t) \leq \gamma(x,ct) \text{ for } t \geq T$$

we write $\varphi \prec \gamma$ and we say that γ dominate φ globally if $T = 0$ and near infinity if $T > 0$.

d) if for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \text{ or } \lim_{t\to\infty}(\sup_{x\in\Omega} \frac{\varphi(x,ct)}{\gamma(x,t)}) = 0$$

we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near infinity respectively.

In the following the measurability of a function $u : \Omega \mapsto R$ means the Lebesgue measurability.

We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \mapsto R$ is a measurable function.
The set

$$K_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(u) < +\infty\right\}.$$

is called the Musielak-Orlicz class (the generalized Orlicz class).

The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$.
Equivalently:

$$L_\varphi(\Omega) = \left\{u : \Omega \to R \text{ mesurable } / \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \text{ for some } \lambda > 0\right\}$$

Let

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\},$$

that is, ψ is the Musielak-Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ in the sense of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\{\lambda > 0 / \int_\Omega \varphi(x, \frac{|u(x)|}{\lambda})dx, \leq 1\}.$$

which is called the Luxemburg norm and the so-called Orlicz norm by :

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1} \int_\Omega |u(x)v(x)|dx.$$

where ψ is the Musielak-Orlicz function complementary (or conjugate) to φ. These two norms are equivalent [20].

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [20].

We have $E_\varphi(\Omega) = K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega) = L_\varphi(\Omega)$ if and only if φ has the Δ_2 property for large values of t, or for all values of t, according to whether Ω has finite measure or not, i.e., there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h , integrable in Ω such that $\varphi(x, 2t) \leq k\varphi(x, t) + h(x)$ for large values of t, or for all values of t.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m \; D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ with nonnegative integers α_i $|\alpha| = |\alpha_1| + |\alpha_2| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.

Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } ||u||^m_{\varphi,\Omega} = \inf\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \leq 1\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\langle W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [20]:

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c. \tag{3}$$

$W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \prod L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed. Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W_0^m E_\varphi(\Omega)$ is the (norm) closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m}L_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega)\}$$

$$W^{-m}E_\psi(\Omega) = \{f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega)\}$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\varrho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0.$$

For two complementary Musielak-Orlicz functions φ and ψ the following inequality is called the young inequality [20]:

$$t.s \leq \varphi(x,t) + \psi(x,s) \text{ for } t,s \geq 0,\ x \in \Omega$$

This inequality implies that

$$|||u|||_{\varphi,\Omega} \leq \varrho_{\varphi,\Omega}(u) + 1.$$

We have also for two complementary Musielak-Orlicz functions φ and ψ if $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$ the Hölder inequality [20]:

$$|\int_\Omega u(x)v(x)\ dx| \leq ||u||_{\varphi,\Omega}|||v|||_{\psi,\Omega}.$$

We recall that a family $\mathfrak{R}$ of functions $u(x)$ has equi-absolutely continuous integrals if for arbitrary $\varepsilon > 0$ an $h > 0$ can be found such that for all functions in the family $\mathfrak{R}$ we have

$$\int_E u(x)dx < \varepsilon$$

provided $|E| < h$, where $|E|$ is the measure of the set E.

2.2 Preliminary lemmas

Lemma 1. *if a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \to g$ for $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$.*

Proof. Since every sequence of functions in $L_\varphi(\Omega)$ which are bounded in norm contains a $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$ convergent subsequence, It is enough to show that for any subsequence $g_{n_k}(x)$ which converges in $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$ to $g_0(x)$, we have $g_0(x) = g(x)$.

We denote by $K_m(x)$ the characteristic function of some fixed set of points on which $|g(x) - g_0(x)| \leq m$, and the function sgn $[g(x) - g_0(x)]$ by $f_0(x)$.

Suppose $\varepsilon > 0$ is prescribed. Since the functions $g_0(x)$, $g_{n_k}(x)$ have equi-absolutely continuous integrals [20], a $\delta > 0$ can be found such that

$$\int_D |g_0(x)| dx < \frac{\varepsilon}{5}, \int_D |g_{n_k}(x)| dx < \frac{\varepsilon}{5}$$

provided $|D| < \delta (D \subset \Omega)$. We shall assume that $\delta < \frac{\varepsilon}{5m}$. It follows from the convergence in measure of the subsequence $g_{n_k}(x)$ to the function $g(x)$ and the convergence of this sequence to the function $g_0(x)$ in $\sigma(L_\varphi(\Omega), E_\psi(\Omega))$ that there exists a k_0 such that , for$k > k_0$,

$$\int_\Omega [g_{n_k}(x) - g_0(x)] f_0(x) K_m(x) dx < \frac{\varepsilon}{5}$$

and mes $\Omega_k < \delta$, where

$$\Omega_k = \{|g_{n_k}(x) - g(x)| \geq \frac{\varepsilon}{5mes\ \Omega}\}.$$

Then, for $k > k_0$, we have that

$$\int_\Omega |g(x) - g_0(x)| K_m(x) dx \leq \mid \int_\Omega [g_{n_k}(x) - g_0(x)] f_0(x) K_m(x) dx \mid \tag{4}$$

$$+ \int_{\Omega \setminus \Omega_k} |g(x) - g_{n_k}(x)| dx + \int_{\Omega_k} |g_{n_k}(x)| dx \tag{5}$$

$$+ \int_{\Omega_k} |g_0(x)| dx + \int_{\Omega_k} |g(x) - g_0(x)| K_m(x) dx \tag{6}$$

$$< \frac{\varepsilon}{5} + \frac{\varepsilon}{5mes\ \Omega}\ mes\ (\Omega \setminus \Omega_k) + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + m\ mes\ \Omega_k < \varepsilon \tag{7}$$

Since ε is arbitrary, we have that

$$\int_\Omega |g(x) - g_0(x)| K_m(x) dx = 0,$$

i.e.$g_0(x) = g(x)$ almost everywhere.

Lemma 2. *[17] Let the functions A_α satisfy the conditions (A_1) and (A_3) below. if for the sequences $\eta_k \subset R^{n_1}$, $\zeta_k \subset R^{n_2}$, and $\xi_k \subset R^{n_2}$ we have $\eta_k \to \eta$, $\zeta_k \to \zeta$, and*

$$\Sigma_{|\alpha|=m} (A_\alpha(x, \eta_k, \zeta_k) - A_\alpha(x, \eta_k, \xi_k))(\zeta_{\alpha k} - \xi_{\alpha k}) \to 0$$

as $k \to \infty$, then ξ_k is bounded in R^{n_2} and $\xi_k \to \zeta$ as $k \to \infty$.

Lemma 3. *[1] Let $u \in W^{1,1}_{loc}(\Omega)$ and let f satisfy a Lipschitz condition in R. If $g(x) = f(|u(x)|)$, then $g \in W^{1,1}_{loc}(\Omega)$ and*

$$D^\alpha g(x) = f'(|u(x)|)\ sgn\ u(x).D^\alpha u(x).$$

The following lemmas are respectively a generalization of Lemma 2, Lemma 3 and Lemma 4 of [13].

Lemma 4. *Let $F : R \to R$ be uniformly Lipschitzian, with $F(0) = 0$. Let $u \in W^1_0 L_\varphi(\Omega)$. Then $f(u) \in W^1_0 L_\varphi(\Omega)$. Moreover if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e.\ in\ \{x \in \Omega; u(x) \in D^c\} \\ 0 & a.e.\ in\ \{x \in \Omega; u(x) \in D\} \end{cases} \tag{8}$$

Proof We suppose for the moment that F is also C^1. By Theorem 2.5 of [6], there exists a sequence $u_n \in D(\Omega)$ such that for $|\alpha| \le 1$ and some $\lambda > 0$, $\int_\Omega \varphi(x, (\frac{D^\alpha u_n - D^\alpha u}{\lambda})) \to 0$ as $n \to \infty$. Passing to subsequence, we can assume that $u_n \to u$ a.e. in Ω. From the relations $|F(s)| \le k|s|$, where k denote the Lipschitz constant for F, and $\frac{\partial}{\partial x_i}F(u_n) = F'(u_n)\frac{\partial u_n}{\partial x_i}$, we deduce that $F(u_n)$ remains bounded in $W^1_0 L_\varphi(\Omega)$. Thus, going to a further subsequence, we obtain $F(u_n) \to w \in W^1_0 L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and also, by a local application of the compact imbedding theorem, $F(u_n) \to w$ a.e. in Ω. Consequently $w = F(u)$, and $F(u) \in W^1_0 L_\varphi(\Omega)$. Finally, by the usual chain rule for weak derivatives,

$$\frac{\partial}{\partial x_i} F(u) = F'(u)\frac{\partial u}{\partial x_i} \tag{9}$$

a.e. in Ω. For the general case. Taking convolution with the mollifiers, we get a sequence $F_n \in C^\infty(R)$ such that $F_n \to F$ uniformly on each compact, $F_n(0) = 0$ and $|F'_n| \le k$. For each n, $F_n(u) \in W^1_0 L_\varphi(\Omega)$, and we have (9) with F replaced by F_n. similarly to the preceding arguments we conclude that $F(u) \in W^1_0 L_\varphi(\Omega)$. Finally (8) follows from the generalized chain rule for weak derivatives.

Lemma 5. *Let $u, v \in W^1_0 L_\varphi(\Omega)$ and let $w = \min\{u, v\}$. then $w \in W^1_0 L_\varphi(\Omega)$ and*

$$\frac{\partial w}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} & a.e.\ in\ \{x \in \Omega; u(x) \le v(x)\} \\ \frac{\partial v}{\partial x_i} & a.e.\ in\ \{x \in \Omega; u(x) > v(x)\} \end{cases}$$

Proof. We apply Lemma 4 with $F(s) = s^+$ using the fact that $\min\{u, v\} = u - (v - u)^+$.

Lemma 6. *Let $u \in W^1_0 L_\varphi(\Omega)$. Then there exists a sequence u_n such that (i) $u_n \in W^1_0 L_\varphi(\Omega) \cap L^\infty$, (ii) Supp u_n is compact in Ω, (iii) $|u_n(x)| \le |u(x)|$ a.e. in Ω, (iv) $u_n(x)u(x) \ge 0$ a.e. in Ω, (v) $D^\alpha u_n \to D^\alpha u$ a.e. in Ω, for $|\alpha| \le 1$, (vi) for some $\lambda > 0$ and for $|\alpha| \le 1$,*

$$\int_\Omega \varphi(x, \frac{D^\alpha u_n - D^\alpha u}{\lambda}) \to 0$$

Proof. By Theorem 2.5 of [6], there exists a sequence $v_n \in D(\Omega)$ such that for $|\alpha| \leq 1$ and some $\lambda > 0$, $\int_\Omega \varphi(x, (\frac{D^\alpha v_n - D^\alpha u}{\lambda})) \to 0 \quad$ as $n \to \infty$. Passing to subsequence if necessary , we can assume that for $|\alpha| \leq 1$, $D^\alpha v_n \to D^\alpha u$ a.e. in Ω. Without loss of generality u and v_n can be taken ≥ 0 in Ω. We put

$$u_n = \min\{u, v_n\}.$$

Then (i)-(iv) clearly hold. By Lemma 6 we obtain

$$\frac{\partial u_n}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i} & \text{a.e. in } \Omega' = \{x \in \Omega; u(x) \leq v_n(x)\} \\ \frac{\partial v_n}{\partial x_i} & \text{a.e. in } \Omega'' = \{x \in \Omega; u(x) > v_n(x)\} \end{cases}$$

It follows that (v) holds. We write

$$\begin{aligned} \int_\Omega \varphi(x, \frac{\frac{\partial u}{\partial x_i} - \frac{\partial u_n}{\partial x_i}}{\lambda}) &= \int_{\Omega''} \varphi(x, \frac{\frac{\partial u}{\partial x_i} - \frac{\partial v_n}{\partial x_i}}{\lambda}) \\ &\leq \int_\Omega \varphi(x, \frac{\frac{\partial u}{\partial x_i} - \frac{\partial v_n}{\partial x_i}}{\lambda}). \end{aligned}$$

The right hand side goes to zero, so (vi) is true.

Lemma 7. *Let $u_n, u \in L_\varphi(\Omega)$. if $u_n \to u$ with respect to the modular convergence, then $u_n \rightharpoonup u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Proof. Let $\lambda > 0$ be such that $\int_\Omega \varphi(x, \frac{u_n - u}{\lambda})dx \to 0$. Thus, for a subsequence, $u_n \to u$ a.e. in Ω. Let $v \in L_\psi(\Omega)$. We can assume that $\lambda v \in K_\psi$ (by multiplying v by a suitable constant). Young's inequality gives

$$|(u_n - u)v| \leq \varphi(x, \frac{u_n - u}{\lambda}) + \psi(x, \lambda v),$$

so, it is enough to apply the Vitali's theorem.

2.3 Complementary system

Definition 1. *Let Y and Z be two real Banach spaces in duality with respect to a continuous pairing $<,>$ and let Y_0 and Z_0 be subspaces of Y and Z respectively. Then $(Y, Y_0; Z, Z_0)$ is called a complementary system if, by means of $<,>$, Y_0^* can be identified (i.e.,is linearly homeomorphic) to Z and Z_0^* to Y.*

Let φ and ψ be two complementary Musielak-Orlicz functions then $(L_\varphi(\Omega), E_\varphi(\Omega); L_\psi(\Omega), E_\psi(\Omega))$ is a complementary system. Other examples are $(X^{**}, X; X^*, X^*)$ and $(X^*, X^*; X^{**}, X)$ where X is Banach space. Note that in a complementary system, Y_0 is $\sigma(Y, Z)$ dense in Y. Note also that if cl Y_0 [cl Z_0] denotes the (norm) closure of Y_0 [Z_0] in Y [Z], then $(Y, clY_0; Z, clZ_0)$ is a complementary system.

The following lemma gives an important method by which from a complementary system $(Y, Y_0; Z, Z_0)$ and a closed subspace E of Y, one can construct a new complementary system $(E, E_0; F, F_0)$. Some restriction must be imposed on E. Define $E_0 = E \cap Y_0$, $F = Z/E_0^{\perp}$ and $F_0 = \{z + E_0^{\perp}; z \in Z_0\} \subset F$, where $\perp$ denotes the orthogonal in the duality (Y, Z), i.e. $E_0^{\perp} = \{z \in Z; < y, z >= 0 \text{ for all } y \in E_0\}$.

Lemma 8. *[11] The pairing $<,>$ between Y and Z induces a pairing between E and F if and only if E_0 is $\sigma(Y, Z)$ dense in E. In this case, $(E, E_0; F, F_0)$ is a complementary system if E is $\sigma(Y, Z_0)$ closed, and conversely, when Z_0 is complete, E is $\sigma(Y, Z_0)$ closed if $(E, E_0; F, F_0)$ is a complementary system.*

Remark 1. *By the statement of [6] and the above lemma, both $W_0^m L_\varphi(\Omega)$ and $W^m L_\varphi(\Omega)$ generate a complementary system in $(\Pi L_\varphi(\Omega), \Pi E_\varphi(\Omega); \Pi L_\psi(\Omega), \Pi L_\psi(\Omega))$*

2.4 An Abstract Result

Let $(Y, Y_0; Z, Z_0)$ be a complementary system and T be a mappings from the domain $D(T)$ in Y to Z which satisfy the following conditions, with respect to some element $\bar{y} \in Y_0$ and $f \in Z_0$:

(i) (finite continuity) $D(T) \supset Y_0$ and T is continuous from each finite dimensional subspaces of Y_0 to the $\sigma(Z, Y_0)$ topology of Z,

(ii) (sequential pseudo-monotonicity) for any sequence $\{y_i\}$ with $y_i \to y \in Y$ for $\sigma(Y, Z_0)$,
$T(y_i) \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup < T(y_i), y_i > \leq < z, y >$, it follows that $T(y) = z$ and $< T(y_i), y_i > \to < z, y >$,

(iii) $T(y)$ remains bounded in Z whenever $y \in D(T)$ remains bounded in Y and $< y - \bar{y}, Tu >$ remains bounded from above,

(iv) $< y - \bar{y}, t(y) - f >$ is > 0 when $y \in D(T)$ has sufficiently large norm in Y.

It is of importance to note that the condition (iii) is weaker than the condition that T transforms each bounded set of Y into a bounded set of Z. And that the condition (iv) is weaker than the assumption of coercitivity. Because in our applications, the mapping T will generally not transform a bounded set into a bounded set nor be coercive.

Given a convex set $K \subset Y$ and an element $f \in Z_0$, we are interested in finding a solution y of the variational inequality.

$$\begin{cases} y \in K \cap D(T), \\ < y - z, Ty > \leq < y - z, f > \text{ for all } z \in K. \end{cases} \tag{10}$$

Theorem 1. *[14] Let $(Y, Y_0; Z, Z_0)$ be a complementary system with Y_0 and Z_0 separable. Let $K \subset Y$ be convex, $\sigma(Y, Z_0)$ sequentially closed and such that $K \cap Y_0$ is $\sigma(Y, Z)$ dense in K. Let $f \in Z_0$ and let $T : D(T) \subset Y \to Z$ satisfy (i)...(iv) with respect to some $\overline{y} \in K \cap Y_0$ and the given f. Then the variational inequality (10) has at least one solution y.*

3 Main results

3.1 Trace

We assume that the boundary Γ of our open bounded set Ω is sufficiently good so that questions in Ω, near Γ, can be transformed, by using a partition of unity and local charts, into similar questions in R^n_+, near R^{n-1}. This will be certainly so, for our purposes below, if Γ is assumed to be C^1.

We summarize in the following theorem some properties of the trace mapping.

Theorem 2. *(a) The "restriction to Γ " mapping:*

$$\tilde{\gamma} = C^\infty(\overline{\Omega}) \to C(\Gamma) : u \mapsto u_{|\Gamma}$$

is continuous for the following topologies on $C^\infty(\overline{\Omega})$ and $C(\Gamma)$ respectively:

$$||.||^1_{\varphi,\Omega} \to ||.||_{\varphi,\Gamma} \tag{11}$$
$$\sigma(\Pi L_\varphi(\Omega), \Pi E_\psi(\Omega)) \to \sigma(L_\varphi(\Gamma), E_\psi(\Gamma)) \tag{12}$$
$$\sigma(\Pi L_\varphi(\Omega), \Pi L_\psi(\Omega)) \to \sigma(L_\varphi(\Gamma), L_\psi(\Gamma)) \tag{13}$$

(b) Green's formula holds:if $u \in W^1L_\varphi(\Omega)$ and $v \in W^1L_\psi(\Omega)$ then

$$\int_\Omega u\frac{\partial v}{\partial x}dx + \int_\Omega v\frac{\partial u}{\partial x}dx = \int_\Gamma uv\nu_i d\Gamma. \tag{14}$$

Remark 2. *Since $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_\varphi, \Pi L_\psi)$ dense in $W^1L_\varphi(\Omega)$, the condition (13) implies that $\tilde{\gamma}$ can be extended into a continuous mapping γ from $W^1L_\varphi(\Omega)$, $\sigma(\Pi L_\varphi, \Pi L_\psi)$ to $L_\varphi(\Gamma)$, $\sigma(L_\varphi, L_\psi)$. Condition (12) implies that γ is continuous from $W^1L_\varphi(\Omega)$, $\sigma(\Pi L_\varphi, \Pi E_\psi)$ to $L_\varphi(\Gamma)$, $\sigma(L_\varphi, E_\psi)$. From (11) and the fact that $C^\infty(\overline{\Omega})$ is norm dense in $W^1E_\varphi(\Omega)$, it follows that γ is continuous from $W^1E_\varphi(\Omega), ||.||$ to $E_\varphi(\Gamma), ||.||$. For u in $W^1L_\varphi(\Omega)$, γu is called the trace of u on Γ.*

Proof of Theorem. The proof is similar to that in [[13] . §3] , so we sketch it here. The conditions (11),(12)and (13) follow from a standard method of the partition of unity and the local charts. To show that (14) holds, we first use the fact that (14) is true for u and v in $C^\infty(\overline{\Omega})$, secondly we use the fact that $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_\varphi, \Pi E_\psi)$

dense in $W^1L_\varphi(\Omega)$ and that γ is continuous for(12) to conclude that (14) holds for $u \in W^1L_\varphi(\Omega)$ and $v \in C^\infty(\overline{\Omega})$, finally, since $C^\infty(\overline{\Omega})$ is $\sigma(\Pi L_\psi(\Omega), \Pi L_\varphi(\Omega))$ dense in $W^1L_\psi(\Omega)$ and since γ is continuous for (13) (with φ and ψ interchanged), we derive (14) for $u \in W^1L_\varphi(\Omega)$ and $v \in W^1L_\psi(\Omega)$.

Theorem 3. *(a) The kernel of the trace mapping $\gamma : W^1L_\varphi(\Omega) \to L\varphi(\Gamma)$ is $W_0^1L_\varphi(\Omega)$. (b) The kernel of the trace mapping $\gamma : W^1E_\varphi(\Omega) \to E_\varphi(\Gamma)$ is $W_0^1E_\varphi(\Omega)$*

Proof. We first note that the part (a) implies the part (b) because $W_0^1E_\varphi(\Omega) = W_0^1L_\varphi(\Omega) \cap W^1E_\varphi(\Omega)$. and that to prove the first assertion (a), it suffices to show that $\ker\gamma \subset W_0^1L_\varphi(\Omega)$ since the other inclusion follows from the continuity properties of γ. So we take $u \in W_0^1L_\varphi(\Omega)$ with $\gamma u = 0$ and we put

$$\tilde{u} = \begin{cases} u & \text{in } \Omega \\ 0 & \text{outside}\Omega \end{cases}$$

It is clear that $\tilde{u} \in W^1L_\varphi(R^n)$. Then the conclusion follows by using the arguments similar to those used in [6].

3.2 An imbedding results

The following three Theorems have been firstly introduced by the authors in [7]. For convincing the reader we rewrite the proof.

Theorem 4. *Let Ω have finite measure and let φ and ϕ two Msuielak-Orlicz functions such that $\phi(.,t)$ is integrable on Ω and increasing essentially more slowly than φ near infinity. If the sequence $\{u_j\}$ is bounded in $L_\varphi(\Omega)$ and convergent in measure on Ω, then it is convergent in norm in $L_\phi(\Omega)$.*

Proof. Fix ε and let $v_{j,k} = \frac{u_j(x)-u_k(x)}{\varepsilon}$. Clearly $\{v_{j,k}\}$ is bounded in $L_\varphi(\Omega)$; say $||v_{j,k}||_{\varphi,\Omega} < K$. Now there exists a positive number t_0 such that if $t > t_0$, then

$$\phi(x,t) \leq \frac{1}{4}\varphi(x, \frac{t}{K}).$$

Let $\delta > 0$ such that

$$\int_D \phi(x,t_0)dx \leq \frac{1}{4}$$

provided $|D| < \delta$.
Set

$$\Omega_{j,k} = \{x \in \Omega : |v_{j,k}(x)| \geq \phi_x^{-1}(\frac{1}{2|\Omega|})\}.$$

Since $\{u_j\}$ converges in measure, there exists an integer N such that if $j,\ k > N$, then $|\Omega_{j,k}| \leq \delta$. Set

$$\Omega'_{j,k} = \{x \in \Omega_{j,k} : |v_{j,k}(x)| \geq t_0\}, \quad \Omega''_{j,k} = \Omega_{j,k} \setminus \Omega'_{j,k}$$

For $j,k \geq N$ we have

$$\begin{aligned}\int_\Omega \phi(x,|v_{j,k}(x)|)dx &= \int_{\Omega\backslash\Omega_{j,k}} \phi(x,|v_{j,k}(x)|)dx + \int_{\Omega'_{j,k}} \phi(x,|v_{j,k}(x)|)dx \\ &\quad + \int_{\Omega''_{j,k}} \phi(x,|v_{j,k}(x)|)dx \\ &\leq \frac{|\Omega|}{2|\Omega|} + \frac{1}{4}\int_{\Omega'_{j,k}} \varphi(x,\frac{|v_{j,k}(x)|}{K})dx + \int_{\Omega_{j,k}} \phi(x,t_0)dx \leq 1.\end{aligned}$$

Hence $||u_j - u_k||_{\phi,\Omega} \leq \varepsilon$ and so $\{u_j\}$ converges in $L_\phi(\Omega)$.

Theorem 5. *Let Ω have finite measure and let φ and ϕ as in the Theorem 2. Then any bounded subset S of $L_\varphi(\Omega)$ which is precompact in $L^1(\Omega)$ is also precompact in $L_\phi(\Omega)$.*

Proof. Evidently $L_\varphi(\Omega) \hookrightarrow L^1(\Omega)$ since Ω has finite volume. If $\{u_j^*\}$ is a sequence in S , then it has a subsequence $\{u_j\}$ that converges in $L^1(\Omega)$; say $u_j \to u$ in $L^1(\Omega)$. Thus $\{u_j\}$ converges to u in measure on Ω and hence by Theorem 4 it converges also in $L_\phi(\Omega)$.

Theorem 6. *Let Ω be an open subset of R^n. Let φ a Msuielak-Orlicz function satisfies the following conditions*

$$\int_1^\infty \frac{\varphi_x^{-1}(t)}{t^{\frac{n+1}{n}}}dt = \infty, \quad \int_0^1 \frac{\varphi_x^{-1}(t)}{t^{\frac{n+1}{n}}}dt < \infty. \tag{15}$$

Let $f(x,t) = \int_0^t \frac{\varphi_x^{-1}(\tau)}{\tau^{\frac{n+1}{n}}}d\tau,\ t \geq 0$. The Sobolev conjugate φ_* of φ is the reciprocal function of f with respect to t. Then $W_0^1L_\varphi(\Omega) \hookrightarrow L_{\varphi_*}(\Omega)$. Moreover, if D is bounded subdomain of Ω, then the following imbeddings $W_0^1L_\varphi(\Omega) \hookrightarrow L_\phi(D)$ exist and are compact for any Orlicz-Msuielak function ϕ increasing essentially more slowly than φ_* near infinity such that $\phi(.,t)$ is integrable on Ω.

Proof. Evidently the function $s = \varphi_*(x,t)$ as defined above is an Orlicz-Msuielak function and satisfies the differential equation

$$\varphi_x^{-1}(s)\frac{ds}{dt} = s^{\frac{n+1}{n}}, \tag{16}$$

and hence, since $s < \varphi_x^{-1}(s)\psi_x^{-1}(s)$,

$$\frac{ds}{dt} \leq s^{\frac{1}{n}}\psi_x^{-1}(s).$$

Therefore $\nu(t) = [\varphi_*(x,t)]^{\frac{n-1}{n}}$ satisfies the differential inequality

$$\frac{d\nu}{dt} \leq \frac{n-1}{n}\psi_x^{-1}((\nu(t))^{\frac{n-1}{n}}). \tag{17}$$

Let $u \in W_0^1 L_\varphi(\Omega)$ and suppose, for the moment, that u is bounded on Ω and is not zero in $L_\varphi(\Omega)$. Then $\int_\Omega \varphi_*(x, \frac{|u(x)|}{\lambda})dx$ decreases continuously from infinity to zero as λ increases from zero to infinity, and accordingly assumes the value unity for some positive value of λ. Thus

$$\int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx = 1, \quad K = ||u||_{\varphi_*}. \tag{18}$$

Let $f(x) = \nu(\frac{|u(x)|}{K})$. Evidently $u \in W_0^{1,1}(\Omega)$ and ν is Lipschitz on the range of $\frac{|u(x)|}{K}$ so that, by Lemma 3, $f \in W_0^{1,1}(\Omega)$. By Sobolev inequality we have

$$||f||_{0,\frac{n}{n-1}} \leq K_1 \sum_1^n ||D^j f||_{0,1} = K_1 \sum_1^n \frac{1}{K} \int_\Omega \nu'(\frac{|u(x)|}{K})|D^j u(x)|dx. \tag{19}$$

By (18) and Hölder's inequality, we obtain

$$1 = \{\int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx\}^{\frac{n-1}{n}} = ||f||_{0,\frac{n}{n-1}} \leq \frac{cK_1}{K} \sum_1^n ||\nu'(\frac{|u|}{K})||_\psi ||D^j u||_\varphi. \tag{20}$$

Making use of (17), we have

$$\begin{aligned} ||\nu'(\frac{|u|}{K})||_\psi &\leq \frac{n-1}{n} ||\psi_x^{-1}((\nu(\frac{|u|}{K}))^{\frac{n-1}{n}})||_\psi \\ &= \frac{n-1}{n} \inf\{\lambda > 0 : \int_\Omega \psi(x, \frac{\psi_x^{-1}(\varphi_*(x, \frac{|u(x)|}{K}))}{\lambda})dx \leq 1\}. \end{aligned}$$

Suppose $\lambda > 1$. Then

$$\int_\Omega \psi(x, \frac{\psi_x^{-1}(\varphi_*(x, \frac{|u(x)|}{K}))}{\lambda})dx \leq \frac{1}{\lambda} \int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx = \frac{1}{\lambda} < 1.$$

Thus

$$||\nu'(\frac{|u|}{K})||_\psi \leq \frac{n-1}{n}. \tag{21}$$

Hence,

$$1 \leq \frac{K_3}{K} ||u||_\varphi^1$$

so that

$$||u||_{\varphi_*} = K \leq K_3 ||u||_\varphi^1 \tag{22}$$

To extend (22) to arbitrary $u \in W_0^1 L_\varphi(\Omega)$ let

$$u_k(x) = \begin{cases} |u(x)| & \text{if } |u(x)| \leq k \\ k \operatorname{sgn} u(x) & \text{if } |u(x)| > k \end{cases}$$

Clearly u_k is bounded and it belongs to $W_0^1L_\varphi(\Omega)$ by Lemma 3. Moreover, $||u_k||_{\varphi_*}$ increases with k but is bounded by $K_4||u||_\varphi$. Therefore, $\lim_{k\to\infty}||u_k||_{\varphi_*} = K$ exists and $K \leq K_4||u||_\varphi^1$. By Fatou's Lemma

$$\int_\Omega \varphi_*(x, \frac{|u(x)|}{K})dx \leq \lim_{k\to\infty}\int_\Omega \varphi_*(x, \frac{|u_k(x)|}{K})dx \leq 1$$

whence $u \in L_{\varphi_*}(\Omega)$ and (22) holds.

If D is a bounded subdomain of Ω, we have

$$W_0^1L_\varphi(\Omega) \hookrightarrow W_0^{1,1}(\Omega) \hookrightarrow L^1(\Omega),$$

the latter imbedding being compact a bounded subset of $W_0^1L_\varphi(D)$ is bounded in $L_{\varphi_*}(D)$ and precompact in $L^1(D)$, and hence precompact in $L_\phi(D)$ by Theorem 5 whenever ϕ increases essentially more slowly than φ_* near infinity. ∎

In the following two sections we study the condition introduced in the Theorem 1.

3.3 Conditions on the mapping T

Let φ and ψ be two complementary Musielak-Orlicz functions. We assume that $\varphi(.;t)$ is locally integrable. We are interested here in the Dirichlet problem for the operator

$$A(u) \equiv \sum_{|\alpha|\leq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x, u, \nabla u, ..., \nabla^m u) \tag{23}$$

on Ω.

The following notations will be used. If $\xi = \{\xi_\alpha; |\alpha| \leq m\} \in R^n$ is an m-jet, with $\alpha = (\alpha_1, ..., \alpha_n)$ a multi-index of integers and $|\alpha| = \alpha_l + ... + \alpha_n$, then $\zeta = \{\xi_\alpha; |\alpha| = m\} \in R^{n_2}$ denotes its top order part and $\eta = \{\xi_\alpha; |\alpha| < m\} \in R^{n_1}$ its lower order part. For u a derivable function, $\xi(u)$ denotes $\{D^\alpha u; |\alpha| \leq m\} \in R^n$.

The basic conditions imposed on the coefficients A_α of (23) are the followings:

(A_1) Each $A_\alpha(x, \xi)$ is a real valued function defined on $\Omega \times R^{n_0}$ is measurable in x for fixed ξ and continuous inξ for fixed x.

(A_2) There exist two Musielak-Orlicz functions φ and γ with $\gamma \prec\prec \varphi$, functions a_α in $E_\psi(\Omega)$, constants c_1 and c_2 such that for all x in Ω and ξ in R^{n_0},if

$$|\alpha| = m : |A_\alpha(x, \xi)| \leq a_\alpha(x) + c_1\sum_{|\beta|=m}\psi_x^{-1}(\varphi(x, c_2\xi_\beta)) + c_1\sum_{|\beta|<m}\phi_x^{-1}(\varphi(x, c_2\xi_\beta)),$$

if

$$|\alpha| < m : |A_\alpha(x,\xi)| \le a_\alpha(x) + c_1 \sum_{|\beta|=m} \psi_x^{-1}(\gamma(x, c_2\xi_\beta)) + c_1 \sum_{|\beta|<m} \psi_x^{-1}(\varphi(x, c_2\xi_\beta)).$$

Where ψ and ϕ are the complementary functions of φ and γ respectively.

(A_3) For each $x \in \Omega$, $\eta \in R^{n_1}$, ξ, and ξ' in R^{n_2} with $\xi \neq \xi'$,

$$\sum_{|\alpha|=m} (A_\alpha(x,\xi,\eta) - A_\alpha(x,\xi',\eta))(\xi_\alpha - \xi'_\alpha) > 0.$$

(A_4) There exist functions $b_\alpha(x)$ in $E_\psi(\Omega)$, $b(x)$ in $L^1(\Omega)$, positive constants d_1 and d_2 such that, for some fixed element v in $W_0^m E_\varphi(\Omega)$,

$$\sum_{|\alpha|\le m} A_\alpha(x,\xi)(\xi_\alpha - D^\alpha v) \ge d_1 \sum_{|\alpha|\le m} \varphi(x, d_2\xi_\alpha) - \sum_{|\alpha|\le m} b_\alpha(x)\xi_\alpha - b(x)$$

for all x in Ω and ξ in R^{n_0}.

Associated to the differential operator (23) we define a mapping T from

$$D(T) = \{u \in W_0^m L_\varphi(\Omega); A_\alpha(\xi(u)) \in L_\psi(\Omega) \text{ for all } |\alpha| \le m\} \subset W_0^m L_\varphi(\Omega)$$

into $W^{-m}L_\psi(\Omega)$ by the formula

$$< v, Tu >= \int_\Omega \sum_{|\alpha|\le m} A_\alpha(\xi(u)) D^\alpha v dx$$

for $v \in W_0^m L_\varphi(\Omega)$.

Theorem 7. *Let Ω be an open subset of R^n. Assume that the coefficients of (23) satisfy $(A_1),...,(A_4)$. Then the corresponding mapping T in $(W_0^m L_\varphi(\Omega), W_0^m E_\varphi(\Omega), W^{-m}L_\psi(\Omega), W^{-m}E_\psi(\Omega))$ satisfies the conditions (i), . . . , (iv) of Theorem 1 with respect to $\overline{u} = v$ and any $f \in W^{-m}E_\psi(\Omega)$.*

Proof. The proof is generally similar to that of Theorem 5 of [7], so, we sketch it here.

For the complementary system $(W_0^m L_\varphi(\Omega), W_0^m E_\varphi(\Omega), W^{-m}L_\psi(\Omega), W^{-m}E_\psi(\Omega))$ we use the notation (Y, Y_0, Z, Z_0).

The property (i) follows immediately from the following Lemma :

Lemma 9. *[7, Lemma 5]. Suppose that A_1 and A_2 hold (with $a(x) \in L_\psi(\Omega)$). Then the mapping $\omega = (\omega_\beta)_{|\beta|\le m} \mapsto (A_\alpha(\omega))_{|\alpha|\le m}$ sends $\Pi E_\varphi(\Omega)$ into $\Pi L_\psi(\Omega)$ and is finitely continuous from $\Pi E_\varphi(\Omega)$ to the $\sigma(\Pi L_\psi(\Omega), \Pi E_\varphi(\Omega))$ topology of $\Pi L_\psi(\Omega)$.*

Now, we show that (ii) holds true :

It follows by the fact that $A_\alpha(,\xi(y_i))$ remains bounded in $L_\psi(\Omega)$ for all $|\alpha| \leq m$ that there exists $h_\alpha \in L_\psi(\Omega)$ such that $A_\alpha(,\xi(y_i)) \to h_\alpha$ for $\sigma(L_\psi(\Omega), E_\varphi(\Omega))$ for each $|\alpha| \leq m$. Hence the linear form $z \in Z = Y_0^*$ can be identified to $(h_\alpha) \in \Pi L_\psi(\Omega)$, i.e.,

$$(z, v) = \int_\Omega \sum_{|\alpha|\leq m} h_\alpha D^\alpha v dx \tag{24}$$

holds for all v in Y.

Therefor, by Theorem 6, we may assume that $D^\alpha y_i(x) \to D^\alpha y$ a.e. in Ω for all $|\alpha| \leq m-1$. We show that

$$\Sigma_{|\alpha|=m}(A_\alpha(x,\eta(y_i),\xi(y_i)) - A_\alpha(x,\eta(y_i),\zeta(y)))(D^\alpha(y_i) - D^\alpha(y)) \to 0.$$

Then, using lemma 2 with the specialization $\eta_k = \eta(y_i)$, $\xi_k = \xi(y_i)$ and $\zeta_k = \zeta(y)$ for each $x \in \Omega$, it follows that $D^\alpha y_i(x) \to D^\alpha y$ a.e. in Ω for all $|\alpha| = 1$.

We have shown that $D^\alpha(y_i)(x) \to D^\alpha(y)(x)$ a.e. in Ω for all $|\alpha| \leq m$, at least for a subsequence. By (A_1) we can conclude that $A_\alpha(x,\zeta(y_i)) \to A_\alpha(x,\zeta(y))$ a.e. in Ω for all $|\alpha| \leq m$. On the other hand, $A_\alpha(x,\zeta(y_i)) \to h_\alpha$ for $\sigma(L_\psi(\Omega), E_\varphi(\Omega))$, so that by Lemma 1 $A_\alpha(x,\zeta(y)) = h_\alpha$ for each $|\alpha| \leq m$. Hence $y \in D(T)$ and $T(y) = z$.

Using (A_3) and (A_4) we can conclude that

$$\liminf \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y_i))D^\alpha(y_i) \geq \int_\Omega \sum_{|\alpha|\leq m} A_\alpha(x,\zeta(y))D^\alpha(y). \tag{25}$$

Thus, bearing in mind the assumption that $\limsup < T(y_i), y_i > \leq < z, y >$, we obtain

$$(T(y_i), y_i) \to (z, y) = (T(y), y). \tag{26}$$

Concerning the condition (iii) we sue the fact that $A_\alpha(x,\xi(y))$ remains bounded in $L_\psi(\Omega)$ for all $|\alpha| \leq m$, which clearly implies that $T(y)$ remains bounded in $W^{-m}L_\psi(\Omega)$.

Finally, we show, using (A_4), that the set

$$\{y \in D(T); < y - v, T(y) - f > \leq 0\} \tag{27}$$

is bounded in $W_0^m L_\varphi(\Omega)$, which clearly yields the condition (iv). ∎

3.4 Conditions on the convex set K

We turn now to the study of the conditions imposed in Theorem 1 on the convex set K. The main key is the verification of the $\sigma(Y,Z)$ density of $K \cap Y_0$ in K. This is a question of approximation within a convex set. Here we limit ourselves to second order operators, i.e $m = 1$, and the obstacle problem.

Given an obstacle function $\Lambda : \Omega \to R$, we consider

$$K = \{y \in W_0^1 L_\varphi(\Omega); y \geq \Lambda \ \ a.e. \in \Omega\}. \tag{28}$$

This convex set is sequentially $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed in $W_0^1 L_\varphi(\Omega)$. Indeed, let $y_n \in K$ converge to $y \in W_0^m L_\varphi(\Omega)$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$; it follows that, for a subsequence, $y_n \to y$ a.e. in Ω, which gives $u \in K$.

Theorem 8. *Let $\Omega \subset R^n$ be a bounded Lipschitz domain. Assume that there exists $\overline{\Lambda} \in K \cap W_0^1 E_\varphi(\Omega)$ such that $\Lambda - \overline{\Lambda}$ is continuous on Ω. Then, for each $y \in K$, there exists a sequence $y_n \in K \cap W_0^1 E_\varphi(\Omega)$ such that, for some $\lambda > 0$ and all $|\alpha| \leq 1$,*

$$\int_\Omega \varphi(x, \frac{D^\alpha y - D^\alpha y_n}{\lambda})dx \to 0$$

as $n \to \infty$.

Proof As in [14], we first observe that it suffices to prove the theorem in the case of an obstacle function which is continuous and ≤ 0 on Ω. Indeed, assuming the theorem in that case and denoting the set (28) by K_Λ, we see that if $y \in K_\Lambda$, then $y - \overline{\Lambda} \in K_{\Lambda - \overline{\Lambda}}$, and so there exists $\Theta_n \in K_{\Lambda-\overline{\Lambda}} \cap W_0^1 E_\varphi(\Omega)$ and $\Theta_n + \overline{\Lambda} \to y$ for the modular convergence. So, from now on, we suppose that Λ itself is continuous and ≤ 0 on Ω. Let $y \in K$. By Lemma 6 there exists $\lambda_1 > 0$ and a sequence s_n, such that (i) $s_n \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$, (ii) the support of s_n is compact in Ω. (iii) $|s_n(x)| \leq |y(x)|$ a.e in Ω, (iv) $s_n(x)y(x) > 0$ a.e. in Ω, (v) for all $|\alpha| \leq 1$,

$$\int_\Omega \varphi(x, \frac{D^\alpha y - D^\alpha s_n}{\lambda_1})dx \to 0 \ \ as \ \ n \to \infty.$$

By the convexity of φ we have

$$\int_\Omega \varphi(x, \frac{D^\alpha s_n}{\lambda_1}) \leq \frac{1}{2}\int_\Omega \varphi(x, \frac{2(D^\alpha s_n - D^\alpha y)}{\lambda_1}) + \frac{1}{2}\int_\Omega \varphi(x, \frac{2D^\alpha y}{\lambda_1})$$

Then, taking λ_1 large if necessary, we can also assume

$$\int_\Omega \varphi(x, \frac{D^\alpha s_n}{\lambda_1}) < \infty \tag{29}$$

for all $|\alpha| \leq 1$ and all n. Since $\Lambda \leq 0$ on Ω, it follows from (iii) and (iv) that $s_n \in K$. For each n, take Ω'_n, Ω_n'' open with supp $s_n \subset\subset \Omega_n'' \subset\subset \Omega'_n \subset\subset \Omega$ and $h_n \in \mathcal{D}(\Omega)$ with $0 \leq h_n(x) \leq 1$ on Ω and $h_n(x) = 1$ on Ω'_n. Define

$$y_n = (s_n + \nu_n h_n) * \rho_{\delta_n}$$

where $\nu_n > 0$ and ρ_{δ_n} is a mollification kernel. Clearly $y_n \in \mathfrak{D}(\Omega)$ if δ_n is taken sufficiently small. Moreover, for a given $\nu_n > 0$, $y_n \in K$ if δ_n is sufficiently small. Indeed

$$s_n + \nu_n h_n \begin{cases} \geq \Lambda + \nu_n & \text{on } \Omega'_n \\ = \nu_n h_n & \text{on } \Omega \setminus \text{ supp } s_n \end{cases}$$

and so, for δ_n sufficiently small and by the continuity of Λ we have

$$(s_n + \nu_n h_n) * \rho_{\delta_n} \geq (\Lambda + \nu_n) * \rho_{\delta_n} \geq \Lambda \text{ on } \Omega_n'',$$

on $\Omega \setminus \Omega_n''$ we have, for δ_n sufficiently small,

$$(s_n + \nu_n h_n) * \rho_{\delta_n} = \nu_n h_n * \rho_{\delta_n} \geq 0 \geq \Lambda;$$

consequently $y_n \geq \Lambda$ on all Ω, i.e. $y_n \in K$.

Let $\varepsilon > 0$ be given. We will show that n, ν_n and δ_n can be chosen such that $y_n \in K \cap D(\Omega)$ and

$$\int_\Omega \varphi(x, \frac{(D^\alpha y - D^\alpha y_n)}{6\lambda_1} \leq \varepsilon. \tag{30}$$

for $|\alpha| \leq 1$, which will complete the proof. By the preceding discussion, only (30) remains to be verified. For that purpose we write

$$\int_\Omega \varphi(x, \frac{D^\alpha y - D^\alpha y_n}{6\lambda_1})dx \leq \frac{1}{3}\int_\Omega \varphi(x, \frac{D^\alpha y - D^\alpha s_n}{2\lambda_1})dx$$
$$+ \frac{1}{3}\int_\Omega \varphi(x, \frac{D^\alpha s_n - D^\alpha s_n * \rho_{\delta_n}}{2\lambda_1})dx + \frac{1}{3}\int_\Omega \varphi(x, \frac{\nu_n D^\alpha h_n * \rho_{\delta_n}}{2\lambda_1})dx$$

First we choose n such that the first term is $\leq \frac{\varepsilon}{3}$ for $|\alpha| \leq 1$, which is possible by (v) above. Then we choose ν_n such that for all $\delta_n > 0$, the third term is $\leq \frac{\varepsilon}{3}$ for $|\alpha| \leq 1$, which is possible since this third term is

$$\leq \frac{1}{3}||\varphi(x, \frac{\nu_n D^\alpha h_n}{2\lambda_1}||_{L^1(\Omega)}.$$

Finally we use (29) together with Theorem 2.3 of [6], to choose δ_n, such that the second term is $\leq \frac{\varepsilon}{3}$ for $|\alpha| \leq 1$. ■

In the following theorem we construct an approximation within K, by assuming other condition on the obstacle function.

Theorem 9. *Let $\Omega \subset R^n$ be a bounded Lipschitz domain. Assume that the obstacle function Λ belongs to $W^1E_\varphi(\Omega)$. Then the conclusion of Theorem 8 holds.*

Proof. Let $y \in K$, by [6, Theorem 2.5], there exists a sequence $z_n \in \mathcal{D}(\Omega)$ and $\lambda > 0$ such that, for $|\alpha| \leq 1$,

$$\int_\Omega \varphi(x, \frac{D^\alpha y - D^\alpha z_n}{\lambda})dx \to 0 \tag{31}$$

as $n \to \infty$. Put

$$y_n = \sup\{z_n, \Lambda\}.$$

Clearly $y_n \geq \Lambda$ a.e. in Ω. We also have $y_n \in E_\varphi(\Omega)$ since it is the supremum of two functions in that space; moreover, by the chain rule for weak derivatives, we have

$$\frac{\partial y_n}{\partial x_i} = \begin{cases} \frac{\partial z_n}{\partial x_i} & \text{a.e. in } \Omega'_n = \{x \in \Omega; z_n(x) \geq p(x)\} \\ \frac{\partial p}{\partial x_i} & \text{a.e. in } \Omega''_n = \{x \in \Omega; z_n(x) < p(x)\} \end{cases}$$

which shows that $y_n \in W^1E_\varphi(\Omega)$. To show that y_n belongs to $W_0^1E_\varphi(\Omega)$, take $p \in K$ and write

$$z_n \leq y_n \leq \sup\{z_n, p\} \tag{32}$$

By Lemma 5 we know that $\sup\{z_n, \Lambda\} \in W_0^1L_\varphi(\Omega)$; taking the trace on $\partial\Omega$ of the functions in (32) and using the fact that this is an order preserving operation, we obtain

$$\gamma z_n \leq \gamma y_n \leq \gamma \sup\{z_n, p\} = 0,$$

so that $\gamma y_n = 0$, which implies, by Theorem 3 that $y_n \in W_0^1E_\varphi(\Omega)$. It remains to prove that $y_n \to y$ for the modular convergence in $W_0^1L_\varphi(\Omega)$(for a subsequence if necessary). Since $y \geq \Lambda$, we have $|y_n - y| \leq |z_n - y|$, and consequently, by (31)

$$\int_\Omega \varphi(x, \frac{y_n - y}{\lambda})dx \to 0 \tag{33}$$

To deal with the derivatives, we first take a subsequence such that $z_n \to y$ a.e. in Ω and replace λ by a larger number, if necessary, so that $(\frac{\partial p}{\partial x_i} - \frac{\partial y}{\partial x_i})/\lambda \in K_\varphi(\Omega)$; we then have

$$\int_\Omega \varphi(x, \frac{\frac{\partial y_n}{\partial x_i} - \frac{\partial y}{\partial x_i}}{\lambda})dx = \int_{\Omega'_n} \varphi(x, \frac{\frac{\partial z_n}{\partial x_i} - \frac{\partial y}{\partial x_i}}{\lambda})dx + \int_\Omega \varphi(x, \frac{\frac{\partial p}{\partial x_i} - \frac{\partial y}{\partial x_i}}{\lambda})\chi_{\Omega''_n}\, dx$$

The first integral in the right hand side goes to zero by (31) and the second integral goes to zero since $\chi_{\Omega''_n} \to 0$ a.e. in Ω.

Remark 3. The arguments of the previews theorem can easily be adapted to deal with the double obstacle problem. Here one considers the convex set

$$K = \{y \in W_0^1 L_\varphi(\Omega); \Lambda(x) \leq y(x) \leq \Theta(x) \text{ a.e. in } \Omega\}$$

where $\Lambda, \Theta : \Omega \to R$. If Λ and Θ belong to $W^1 E_\varphi(\Omega)$ and if K is nonempty, then for each $y \in K$ there exists a sequence $y_n \in K \cap W_0^1 E_\varphi(\Omega)$ which converges to y for the modular convergence in $W_0^1 L_\varphi(\Omega)$. Starting with a sequence z_n as in the proof of Theorem 9, it suffices to take

$$y_n = \inf\{\sup\{z_n, \Lambda\}, \Theta\}.$$

Remark 4. The arguments of Theorem 9 can be adapted to deal with the obstacle problem without Dirichlet boundary conditions. Here one considers the convex set

$$K = \{y \in W^1 L_\varphi(\Omega); y \geq \Lambda \text{ a.e. in } \Omega\}$$

If Ω is a bounded Lipschitz domain, and if $\Lambda \in W^1 E_\varphi(\Omega)$, then, for each $y \in K$, there exists a sequence $u_n \in K \cap W^1 E_\varphi(\Omega)$ which converges to y for the modular convergence in $W^1 L_\varphi(\Omega)$. It suffices in the above proof to apply Theorem 2.6 of [6] instead of Theorem 2.5.

Combining Theorem 1, Theorem 7 and Theorem 8 or Theorem 9 we obtain the following existence result for the obstacle problem associated to a second order differential operator of the form (23).

Theorem 10. *Let $\Omega \subset R^n$ be a bounded Lipschitz domain. Assume that the coefficients of (23) satisfy $(A_1), ..., (A_4)$ with $m = 1$ and let T the corresponding mapping in the complementary system $(W_0^1 L_\varphi(\Omega), W_0^1 E_\varphi(\Omega), W^{-1} L_\psi(\Omega), W^{-1} E_\psi(\Omega))$. Let K as in Theorem 8 or Theorem 9. Then for any $f \in W^{-1} E_\psi(\Omega)$, the following variational inequality*

$$\begin{cases} y \in K \cap D(T), \\ < y - z, Ty > \leq < y - z, f > \text{ for all } z \in K. \end{cases}$$

has at least one solution.

Remark 5. *By the bipolar Theorem in any complementary system, Y_0 is $\sigma(Y, Z)$ dense in Y. Then the above result can be applied in particular if $K = Y$, i.e. for the equation $Tu = f$ with f given in Z_0.*

3.5 Strongly nonlinear elliptic problems

Now we are interested in the study of the so-called ”strongly nonlinear” inequalities, i.e. problem of the forme

$$A(u) + g(x,u) \leq f \tag{34}$$

where A is given by (23) with $m = 1$.

Consider the complementary system $(W_0^1 L_\varphi(\Omega), W_0^1 E_\varphi(\Omega), W^{-1}L_\psi(\Omega), W^{-1}E_\psi(\Omega))$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain. It will be denoted below by (Y, Y_0, Z, Z_0).

For the mapping $T : D(T) \subset Y \to Z$, we assume the properties (i), (ii), (iii) and the following slightly stronger form of (iv):

$$(iv)* \quad \langle u - \overline{u}, tu - f\rangle \to +\infty \text{ as } ||u||_Y \to +\infty \text{ in } D(T).$$

For the perturbing function $g : \Omega \times R \to R$, we assume the usual conditions.

(G_1) $g(x,u)$ is a Caratheodory function and satisfies the sign condition

$$g(x,u)u \geq 0$$

for a.e. x in Ω and all u in R.

(G_2) For each $r \geq 0$, there exists $h_r \in L^1(\Omega)$ such that

$$|g(x,u)| \leq h_r(x)$$

for a.e. x in Ω and all u in R with $|u| \leq r$.

For the convex set $K \subset Y$ we need the following two approximation properties (which, together, imply that $K \cap Y_0$ is $\sigma(Y,Z)$ dense in K).

(K_1) For each $u \in K \cap L^\infty(\Omega)$ there exists a sequence $u_n \in K \cap L^\infty(\Omega) \cap Y_0$ such that $u_n \to u$ for $\sigma(Y,Z)$ with $||u_n||_\infty$ bounded.

(K_2) For each $u \in K$ there exists a sequence $u_n \in K \cap L^\infty(\Omega)$ and a constant c such that $u_n \to u$ for $\sigma(Y,Z)$ and $|u_n(x)| \leq c|u(x)|$ for a.e. x in Ω and all u in R.

The following abstract result for the strongly nonlinear problems is due to J.P. Gossez in [14]

Theorem 11. *[14, Proposition 13] Let $K \subset Y$ be convex, $\sigma(Y,Z_0)$ sequentially closed, and satisfy (K_1) and (K_2), Let $f \in Z_0$. Let $T : D(T) \subset Y \to Z$ satisfy (i), (ii), (iii) and (iv)* with respect to some $\overline{u} \in K \cap Y_0 \cap L^\infty(\Omega)$ and the given f. Let $g(x,u)$ satisfy (G_1) and (G_2). Then there exists $u \in K \cap D(T)$ such that $g(x,u) \in L^1(\Omega)$, $g(x,u)u \in L^1(\Omega)$ and*

$$\langle u - v, Tu\rangle + \int_\Omega g(x,u)(u-v)dx \leq \langle u - v, f\rangle \tag{35}$$

for all $v \in K \cap L^\infty(\Omega)$.

Remark 6. *Let K be given by(28). It follows from the Proof of Theorem 8 that if there exists* $\overline{\Lambda}$ *in* $K \cap Y_0 \cap L^\infty\Omega$ *such that* $\Lambda - \overline{\Lambda}$ *is continuous on* Ω*, then condition* (K_1) *holds. If* $\Lambda \in W^1E_\varphi(\Omega)$*, the proof of Theorem 9 shows that* (K_1) *holds. For* (K_2) *it suffices to assume the obstacle function* Λ *bounded from above. Indeed, for any* $u \in K$*, the truncated function*

$$u_n(x) = \begin{cases} |u(x)| & \text{if } |u(x)| \leq n \\ n \text{ sgn } u(x) & \text{if } |u(x)| > n \end{cases}$$

lies above Λ *as soon as* $n >$ *ess sup* Λ*, and the other requirements in* (K_2) *are obvious.*

Theorem 12. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$*. Assume that the coefficients of (23) satisfy the conditions* $(A_1), ..., (A_4)$*. Then the corresponding mapping T in the complementary system* $(W_0^1L_\varphi(\Omega), W_0^1E_\varphi(\Omega), W^{-1}L_\psi(\Omega), W^{-1}E_\psi(\Omega))$ *satisfies the conditions (i), (ii), (iii), (iv)* with respect to* $\overline{u} = v$ *and any* $f \in W^{-1}E_\psi(\Omega)$*.*

Proof. Similar argument as in the proof of property (vi) of Theorem 7 shows that for any c, the set

$$\{u \in D(T);\ \langle u - \omega,\ Tu - f\rangle\ \leq c\}$$

is bounded in Y, which yields (iv)*.

In the particular case when $K = W^mL_\varphi(\Omega)$ we conclude the following theorem :

Theorem 13. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^n$*. Assume that the coefficients of (23) satisfy the conditions* $(A_1), ..., (A_4)$ *with* $m = 1$*. Let T the corresponding mapping in the complementary system* $(W_0^1L_\varphi(\Omega), W_0^1E_\varphi(\Omega), W^{-1}L_\psi(\Omega), W^{-1}E_\psi(\Omega))$ *and let* $g(x,u)$ *satisfy* (G_1) *and* (G_2)*. Then there exists* $u \in D(T)$ *such that* $g(x,u) \in L^1(\Omega)$*,* $g(x,u)u \in L^1(\Omega)$ *and*

$$\langle u - v, Tu\rangle + \int_\Omega g(x,u)(u - v)dx = \langle u - v, f\rangle \tag{36}$$

for all $v \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$*.*

4 Appendix

In this section we give examples of Musielak-Orlicz function that satisfy the condition (2).

Let $p : \Omega \mapsto [1, \infty)$ be a measurable function such that there exist a constant $c > 0$ such that for all points $x, y \in \Omega$ with $|x - y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \leq \frac{c}{\log(\frac{1}{|x-y|})}.$$

Then the following Musielak-Orlicz functions satisfy the condition (2)

1. $\varphi(x,t) = t^{p(x)}$,
2. $\varphi(x,t) = t^{p(x)} \log(1+t)$,
3. $\varphi(x,t) = t(\log(t+1))^{p(x)}$,
4. $\varphi(x,t) = (e^t)^{p(x)} - 1$.

The function (4) do not satisfy the Δ_2 condition and the conjugate functions of (2) and (3) also do not satisfy the Δ_2 condition.

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