# Limit cycles for a class of quintic $\mathbb{Z}_6$-equivariant systems without infinite critical points

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**Abstract**

We analyze the dynamics of a class of $\mathbb{Z}_6$-equivariant systems of the form $\dot{z} = pz^2\bar{z} + sz^3\bar{z}^2 - \bar{z}^5$, where $z$ is complex, the time $t$ is real, while $p$ and $s$ are complex parameters. This study is the natural continuation of a previous work (M.J. Álvarez, A. Gasull, R. Prohens, Proc. Am. Math. Soc. **136**, (2008), 1035–1043) on the normal form of $\mathbb{Z}_4$-equivariant systems. Our study uses the reduction of the equation to an Abel one, and provide criteria for proving in some cases uniqueness and hyperbolicity of the limit cycle surrounding either 1, 7 or 13 critical points, the origin being always one of these points.

## 1 Introduction and main results

Hilbert $XVI^{th}$ problem represents one of the open question in mathematics and it has produced an impressive amount of publications throughout the last century. The study of this problem in the context of equivariant dynamical systems is a relatively new branch of analysis and is based on the development within the last twenty years of the theory of Golubitsky, Stewart and Schaeffer in [9, 10]. Other authors [6] have specifically considered this theory when studying the limit cycles and related phenomena in systems with symmetry. Roughly speaking the

*M.J.A. was partially supported by grant MTM2008-03437. I.S.L. and A.C.M. were partially supported by the European Regional Development Fund through the programme COMPETE and through the Fundação para a Ciência e a Tecnologia (FCT) under the project PEst-C/MAT/UI0144/2013. A.C.M. was also supported by the grant SFRH/ BD/ 64374/ 2009 of FCT

Received by the editors in October 2013.

Communicated by P. De Maesschalck.

2010 *Mathematics Subject Classification* : Primary: 34C07, 34C14; Secondary: 34C23, 37C27.

*Key words and phrases* : Planar autonomous ordinary differential equations, symmetric polynomial systems, limit cycles.

presence of symmetry may complicate the bifurcation analysis because it often forces eigenvalues of high multiplicity. This is not the case of planar systems; on the contrary, it simplifies the analysis because of the reduction to isotypic components. More precisely it allows us to reduce the bifurcation analysis to a region of the complex plane.

In this paper we analyze the $\mathbb{Z}_6$-equivariant system

$$\dot{z} = \frac{dz}{dt} = (p_1 + ip_2)z^2\bar{z} + (s_1 + is_2)z^3\bar{z}^2 - \bar{z}^5 = f(z), \tag{1}$$

where $p_1, p_2, s_1, s_2 \in \mathbb{R}$.

The general form of the $\mathbb{Z}_q$-equivariant equation is

$$\dot{z} = zA(|z|^2) + B\bar{z}^{q-1} + O(|z|^{q+1}),$$

where $A$ is a polynomial of degree $[(q-1)/2]$. The study of this class of equations is developed in several books, see [3, 6], when the resonances are strong, *i.e.* $q < 4$ or weak $q > 4$. The special case $q = 4$ is also treated in several other articles, see [1, 6, 13]. In these mentioned works it is said that the weak resonances are easier to study than the other cases, as the equivariant term $\bar{z}^{q-1}$ is not dominant with respect to the function on $\bar{z}^2$. This is true if the interest lies in obtaining a bifurcation diagram near the origin, but it is no longer true if the analysis is global and involves the study of limit cycles. This is the goal of the present work: studying the global phase portrait of system (1) paying special attention to the existence, location and uniqueness of limit cycles surrounding 1, 7 or 13 critical points. As far as we know this is the first work in which the existence of limit cycles is studied for this kind of systems.

The main result of our paper is the following.

**Theorem 1.** *Consider equation* (1) *with* $p_2 \neq 0$, $|s_2| > 1$ *and define the quantities:*

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2(s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}, \qquad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2(s_1^2 + s_2^2 - 1)}}{s_2^2 - 1}.$$

*Then, the following statements are true:*

*(a) If one of the conditions*

$$(i) \quad p_1 \notin \left(\Sigma_A^-, \Sigma_A^+\right), \qquad (ii) \quad p_1 \notin \left(\frac{\Sigma_A^-}{2}, \frac{\Sigma_A^+}{2}\right)$$

*is satisfied, then equation* (1) *has at most one limit cycle surrounding the origin. Furthermore, when the limit cycle exists it is hyperbolic.*

*(b) There are equations* (1) *under condition* $(ii)$ *having exactly one limit cycle surrounding either* 1, 7 *or* 13 *critical points, and equations* (1) *under condition* $(i)$ *having exactly one hyperbolic limit cycle surrounding either* 7 *critical points if* $p_1 \neq \Sigma_A^\pm$, *or the only critical point if* $p_1 = \Sigma_A^\pm$.

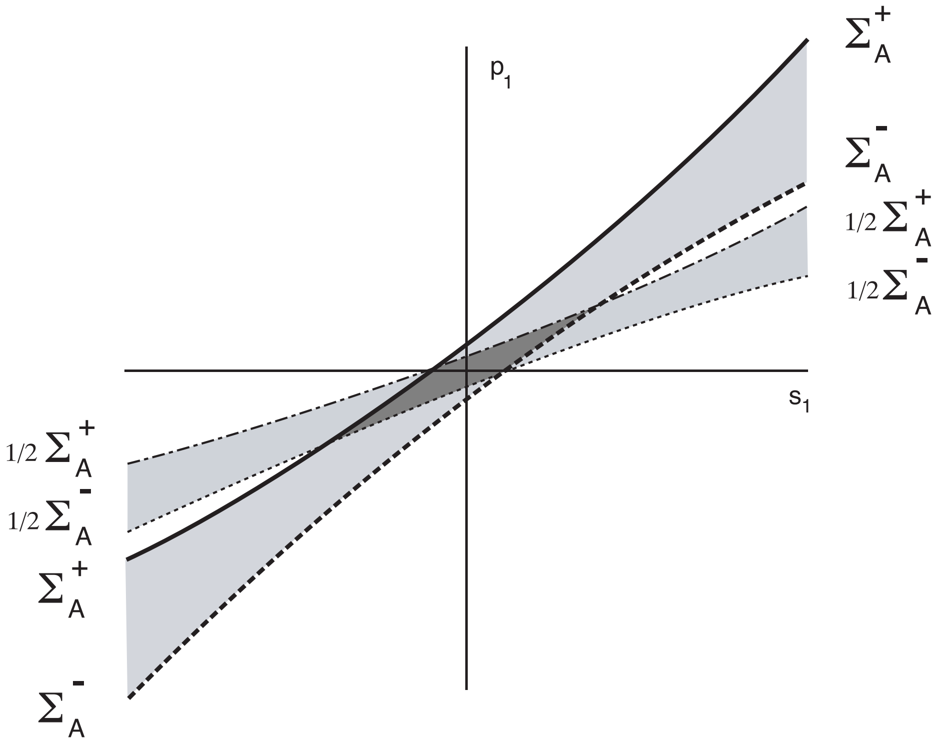


Figure 1: Equation (1) has at most one limit cycle surrounding the origin for $(s_1, p_1)$ outside the dark intersection of the two shaded areas, when $p_2 = 1$ and $s_2 = 4$. Solid line is $\Sigma_A^+$, dashed line stands for $\Sigma_A^-$, dotted line corresponds to $\Sigma_A^-/2$ and dashed-dotted to $\Sigma_A^+/2$. For $p_1$ outside the interval $\left(\Sigma_A^-, \Sigma_A^+\right)$ there are at most 7 equilibria, but when $p_1$ lies in that interval, there may be 13 equilibria surrounded by a limit cycle.

The conditions $(i)$ and $(ii)$ of Theorem 1 hold except when $p_1$ lies in $\left(\Sigma_A^-, \Sigma_A^+\right) \cap \left(\frac{\Sigma_A^-}{2}, \frac{\Sigma_A^+}{2}\right)$. The intersection of the two intervals is often empty, but this is not always the case as the example of Figure 1 shows. Examples of the relative positions of the two intervals are also given in Figure 3 below.

Our strategy for proving Theorem 1 will be to transform the system (1) into a scalar Abel equation and to study it. Conditions $(i)$ and $(ii)$ define regions where one of the functions in the Abel equation does not change sign. Since these functions correspond to derivatives of the Poincaré return map, this imposes an upper bound on the number of limit cycles. When either $p_1$ lies in the intersection of the two intervals, or $|s_2| \leq 1$, this analysis is not conclusive, and the study of the equations would require other methods. The qualitative meaning of conditions of statement (a) of the previous theorem is also briefly explained and illustrated in Remark 3 and in Figure 3 below.

The paper is organized as follows. In Section 2 we state some preliminary results while in Section 3 the study of the critical points is performed. Section 4 is entirely devoted to the proof of the main theorem of the paper.

## 2 Preliminary results

We start by obtaining the symmetries of (1). Following [9, 10], a system of differential equations $dx/dt = f(x)$ is said to be $\Gamma$-equivariant if it commutes with the group action of $\Gamma$, ie. $f(\gamma x) = \gamma f(x)$, $\forall \gamma \in \Gamma$. Here $\Gamma = \mathbb{Z}_6$ with the standard action on $\mathbb{C}$ generated by $\gamma_1 = \exp(2\pi i/6)$ acting by complex multiplication. Applying this concept to equation (1) we have the following result.

**Proposition 1.** *Equation* (1) *is* $\mathbb{Z}_6$*-equivariant.*

*Proof.* Let $\gamma_k = \exp(2\pi i k/6)$, $k = 0, \ldots, 5$. Then a simple calculation shows that $f(\gamma_k z) = \gamma_k f(z)$. This is true because the monomials in $z$, $\bar{z}$ appearing in the expression of $f$ are the following: $\bar{z}^5$, which is $\gamma_k$-equivariant, and monomials of the form $z^{\ell+1}\bar{z}^{\ell}$, that are $\mathbb{Z}_n$-equivariant for all $n$. ∎

Equation (1) represents a perturbation of a Hamiltonian one and in the following we identify conditions that some parameters have to fulfill in order to obtain this Hamiltonian. We have the following result.

**Theorem 2.** *The Hamiltonian part of* $\mathbb{Z}_6$*-equivariant equation* (1) *is*

$$\dot{z} = i(p_2 + s_2 z\bar{z})z^2\bar{z} - \bar{z}^5.$$

*Proof.* An equation $\dot{z} = F(z, \bar{z})$ is Hamiltonian if $\frac{\partial F}{\partial z} + \frac{\partial \bar{F}}{\partial \bar{z}} = 0$. For equation (1) we have

$$\frac{\partial F}{\partial z} = 2(p_1 + ip_2)z\bar{z} + 3(s_1 + is_2)z^2\bar{z}^2$$

$$\frac{\partial \bar{F}}{\partial \bar{z}} = 2(p_1 - ip_2)z\bar{z} + 3(s_1 - is_2)z^2\bar{z}^2$$

and consequently it is Hamiltonian if and only if $p_1 = s_1 = 0$. ∎

As we have briefly said in the introduction, we reduce the study of system (1) to the analysis of a scalar equation of Abel type. The first step consists in converting equation (1) from cartesian into polar coordinates.

**Lemma 1.** *The study of periodic orbits of equation* (1) *that surround the origin, for* $p_2 \neq 0$*, reduces to the study of non contractible solutions that satisfy* $x(0) = x(2\pi)$ *of the Abel equation*

$$\frac{dx}{d\theta} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x \tag{2}$$

where

$$
\begin{aligned}
A(\theta) &= \frac{2}{p_2}\Big(p_1 - p_2 s_1 s_2 + p_1 s_2^2 + (2p_1 s_2 - p_2 s_1)\sin(6\theta)\Big) + \\
&\quad + \frac{2}{p_2}\left((p_2\sin(6\theta) - p_1\cos(6\theta) + p_2 s_2)\cos(6\theta)\right), \\
B(\theta) &= \frac{2}{p_2}\left(p_2 s_1 - 2p_1 s_2 - p_2\cos(6\theta) - 2p_1\sin(6\theta)\right), \\
C(\theta) &= \frac{2p_1}{p_2}.
\end{aligned}
\tag{3}
$$

*Proof.* Using the change of variables

$$z = \sqrt{r}(\cos(\theta) + i\sin(\theta))$$

and the time rescaling, $\frac{dt}{ds} = r$, it follows that the solutions of equation (1) are equivalent to those of the polar system

$$
\begin{cases}
\dot{r} = 2rp_1 + 2r^2(s_1 - \cos(6\theta)) \\
\dot{\theta} = p_2 + r(s_2 + \sin(6\theta))
\end{cases}.
\tag{4}
$$

From equation (4) we obtain

$$\frac{dr}{d\theta} = \frac{2rp_1 + 2r^2(s_1 - \cos(6\theta))}{p_2 + r(s_2 + \sin(6\theta))}.$$

Then we apply the Cherkas transformation $x = \dfrac{r}{p_2 + r(s_2 + \sin(6\theta))}$, see [5], to get the scalar equation (2). Obviously the limit cycles that surround the origin of equation (1) are transformed into non contractible periodic orbits of equation (2), as they cannot intersect the set $\{\dot{\theta} = 0\}$. For more details see [7]. ■

As we have already mentioned in the introductory section, our goal in this work is to apply the methodology developed in [1] to study conditions for existence, location and unicity of the limit cycles surrounding 1, 7 and 13 critical points.

A natural way for proving the existence of a limit cycle is to show that, in the Poincaré compactification, infinity has no critical points and both infinity and the origin have the same stability. Therefore, we would like to find the sets of parameters for which these conditions are satisfied. In the following lemma we determine the stability of infinity.

**Lemma 2.** *Consider equation* (1) *in the Poincaré compactification of the plane. Then:*

*(i) There are no critical points at infinity if and only if* $|s_2| > 1$;

*(ii) When* $s_2 > 1$, *infinity is an attractor (resp. a repellor) when* $s_1 > 0$ *(resp.* $s_1 < 0$*) and the opposite when* $s_2 < -1$.

*Proof.* The proof follows the same steps as the Lemma 2.2 in [1]. After the change of variable $R = 1/r$ in system (4) and reparametrization $\dfrac{dt}{ds} = R$, we get the system

$$\begin{cases} R' = \frac{dR}{ds} = -2R(s_1 - \cos(6\theta)) - 2p_1R^2, \\ \\ \theta' = \frac{d\theta}{ds} = s_2 + \sin(6\theta) + p_2R, \end{cases}$$

giving the invariant set $\{R = 0\}$, that corresponds to the infinity of system (4). Consequently, it has no critical points at infinity if and only if $|s_2| > 1$. To compute the stability of infinity in this case, we follow [11] and study the stability of $\{R = 0\}$ in the system above. This stability is given by the sign of

$$\int_0^{2\pi} \frac{-2\,(s_1 - \cos(6\theta))}{s_2 + \sin(6\theta)} d\theta = \frac{-\mathrm{sgn}(s_2) 4\pi s_1}{\sqrt{s_2^2 - 1}},$$

and the result follows. ∎

## 3 Analysis of the critical points

In this section we are going to analyze which conditions must be satisfied to ensure that equation (4) has one, seven or thirteen critical points. Obviously, the origin of the system is always a critical point. We are going to prove that it is *monodromic*: there is no trajectory of the differential equations that approaches the critical point with a definite limit direction.

For this purpose, let us define the generalized Lyapunov constants. Consider the solution of the following scalar equation

$$\frac{dr}{d\theta} = \sum_{i=1}^{\infty} R_i\,(\theta)\, r^i, \tag{5}$$

where $R_i(\theta)$, $i \geqslant 1$ are $T-$periodic functions. To define the generalized Lyapunov constants, consider the solution of (5) that for $\theta = 0$ passes through $\rho$. It may be written as

$$r(\theta, \rho) = \sum_{i=1}^{\infty} u_i(\theta)\rho^i$$

with $u_1(0) = 1$, $u_k(0) = 0, \forall\, k \geqslant 2$. Hence, the return map of this solution is given by the series

$$\Pi(\rho) = \sum_{i=1}^{\infty} u_i(T)\rho^i.$$

For a given system, in order to determine the stability of a solution, the only significant term in the return map is the first nonvanishing term that makes it differ from the identity map. Moreover, this term will determine the stability of this solution. On the other hand, if we consider a family of systems depending on parameters, each of the $u_i(T)$ depends on these parameters. We will call $k^{th}$ generalized Lyapunov constant $V_k = u_k(T)$ the value of this expression assuming $u_1(T) = 1, u_2(T) = \ldots, = u_{k-1}(T) = 0$.

**Lemma 3.** *The origin of system* (4) *is monodromic, if $p_2 \neq 0$. Moreover, its stability if given by the sign of $p_1$, if it is not zero, and by the sign of $s_1$ if $p_1 = 0$.*

*Proof.* To prove that the origin is monodromic we calculate the arriving directions of the flow to the origin, see [2, Chapter IX] for more details. Concretely if we write system (4) in cartesian coordinates we get

$$\begin{aligned} \dot{x} = P(x,y) = &\left(p_1x^3 - p_2x^2y + p_1xy^2 - p_2y^3\right) + (s_1-1)x^5 - s_2x^4y \\ &+(2s_1+10)x^3y^2 - 2s_2x^2y^3 + (s_1-5)xy^4 - s_2y^5 \\ \dot{y} = Q(x,y) = &\left(p_2x^3 + p_1x^2y + p_2xy^2 + p_1y^3\right) + s_2x^5 + (s_1+5)x^4y \\ &+2s_2x^3y^2 + (2s_1-10)x^2y^3 + s_2xy^4 + (s_1+1)y^5, \end{aligned} \tag{6}$$

Then any solutions arriving at the origin will be tangent to the directions $\theta$ that are the zeros of $r\dot{\theta} = R(x,y) = xQ(x,y) - yP(x,y)$. Since the lowest order term of $R(x,y)$ is $p_2(x^2+y^2)^2$, which is always different from zero away from the origin, it follows that the origin is either a focus or a center.

To know the stability of the origin, we compute the two first Lyapunov constants. As it is well-known, the two first Lyapunov constants of an Abel equation are given by

$$V_1 = \exp\left(\int_0^{2\pi} C(\theta)d\theta\right) - 1, \qquad V_2 = \int_0^{2\pi} B(\theta)d\theta.$$

Applying this to equation (2) with the expressions given in (3) we get the following result:

$$V_1 = \exp\left(4\pi\frac{p_1}{p_2}\right) - 1,$$

and if $V_1 = 0$, then $V_2 = 4\pi s_1$, and we get the result. ∎

**Remark 1.** *It is clear that $(V_1, V_2) = (0,0)$ if and only if $(p_1, s_1) = (0,0)$ i.e. equation* (1) *is hamiltonian (see Theorem 2). Consequently, as the origin remains being monodromic, in this case it is a center.*

In the next result we study the equilibria of equation (4) with $r \neq 0$. They will be the non-zero critical points of the system.

**Lemma 4.** *Let $-\pi/6 < \theta < \pi/6$. Then the equilibria of system* (4) *with $r \neq 0$ are given by:*

$$r = \frac{-p_2}{s_2 + \sin(2\theta_\pm)}, \qquad \theta_\pm = \frac{1}{3}\arctan(\Delta_\pm), \tag{7}$$

*where* $\Delta_\pm = \dfrac{p_1 \pm u}{p_2 - p_1s_2 + p_2s_1}$ *and* $u = \sqrt{p_1^2 + p_2^2 - (p_1s_2 - p_2s_1)^2}$.

*Proof.* Let $-\pi/6 < \theta < \pi/6$. To compute the critical points of system (4), we have to solve the following nonlinear system:

$$\begin{aligned} 0 &= 2rp_1 + 2r^2\left(s_1 - \cos(6\theta)\right) \\ 0 &= p_2 + r\left(s_2 + \sin(6\theta)\right). \end{aligned} \tag{8}$$

Let $x = 6\theta$ and $t = \tan\frac{x}{2}$, so $t = \tan 3\theta$. Then, doing some simple computations one gets

$$\sin x = \frac{2t}{1+t^2} \qquad \cos x = \frac{1-t^2}{1+t^2}. \tag{9}$$

Eliminating $r$ from equations (8) and using the previous formulas we get

$$(-p_2 + p_1 s_2 - p_2 s_1)t^2 + 2p_1 t + p_2 + p_1 s_2 - p_2 s_1 = 0.$$

Solving the previous equation for $t$, yields the result.

Finally, consider the interval $-\pi/6 \leqslant \theta < \pi/6$, let $x = 6\theta$ and $\tau = \cot\frac{x}{2}$, so $\tau = \cot 3\theta$. The same expressions (9) for $\sin x$ and $\cos x$ hold if $t$ is replaced by $\tau = 1/t$, hence the rest of the proof is applicable. ∎

We will prove now that simultaneous equilibria of the type $(r,\theta) = (r,0)$, $(r,\theta) = (\tilde{r},\pi/6)$ are not possible.

**Lemma 5.** *If $|s_2| > 1$, then there are no parameter values for which there are simultaneous equilibria of system* (4) *for $\theta = 0$ and $\theta = \pi/6$ different from the origin.*

*Proof.* If we solve

$$\begin{cases} 0 = p_1 + r(s_1 - \cos 6\theta) \\ 0 = p_2 + r(s_2 + \sin 6\theta) \end{cases}$$

for $\theta = 0$, we get $p_1 s_2 = p_2(s_1 - 1)$ with the restriction $\operatorname{sign} p_2 = -\operatorname{sign} s_2$, to have $r$ well defined in the second equation. On the other hand, solving the same system for $\theta = \pi/6$, we get $p_1 s_2 = p_2(s_1 + 1)$ with the same restriction $\operatorname{sign} p_2 = -\operatorname{sign} s_2$. This means $p_2 = 0$ that implies $r = 0$ and the result follows. ∎

In the following we summarize the conditions that parameters have to fulfill in order that system (4) has exactly one, seven or thirteen critical points (see figure 2).

**Lemma 6.** *Consider system* (4) *with $|s_2| > 1$. If $s_2 p_2 \geq 0$, then the only equilibrium is the origin. If $s_2 p_2 < 0$ then the number of equilibria of system* (4) *is determined by the quadratic form:*

$$\mathcal{Q}(p_1,p_2) = p_1^2 + p_2^2 - (p_1 s_2 - p_2 s_1)^2 = (1-s_2^2)p_1^2 + (1-s_1^2)p_2^2 + 2s_1 s_2 p_1 p_2 \tag{10}$$

*Concretely:*

1. *exactly one equilibrium (the origin) if $\mathcal{Q}(p_1,p_2) < 0$;*
2. *exactly seven equilibria (the origin and one non-degenerate saddle-node per sextant) if $\mathcal{Q}(p_1,p_2) = 0$;*
3. *exactly thirteen equilibria (the origin and two per sextant, a saddle and a node) if $\mathcal{Q}(p_1,p_2) > 0$.*

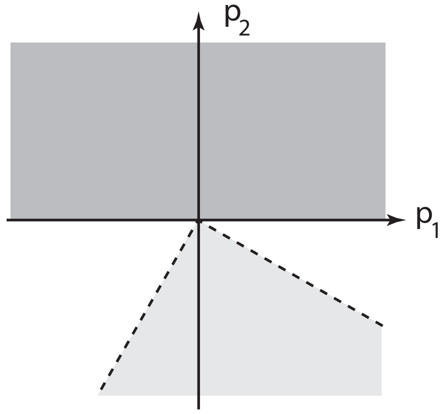


Figure 2: Bifurcation diagram for equilibria of equation (4) with $s_2 > 1$ on the $(p_1, p_2)$ plane. For $(p_1, p_2)$ in the shaded regions, the only equilibrium is the origin. In the white region there are two other equilibria, a saddle and a node, in each sextant. On the dotted line, the saddle and the node come together into a saddle-node in each sextant. When $p_2$ tends to 0 in the white region, the two equilibria tend to the origin. For $s_2 < -1$, the diagram is obtained by reflecting on the $p_1$ axis.

*Proof.* We will do the proof for the case $s_2 > 1$, the other case being analogous.

There will exist more critical points different from the origin if and only if the formulas (7) given in Lemma 4 are realizable with $r > 0$. This will not occur either when the expression for $r$ is negative (that corresponds to the condition $p_2 \geq 0$) or when the discriminant in $\Delta_\pm$ is negative (that corresponds to $\mathcal{Q}(p_1, p_2) < 0$).

In order to have exactly six more critical points, two conditions have to be satisfied: $p_2 < 0$ to ensure positive values of $r$, as $s_2 > 1$, and the quantities $\Delta_\pm$ have to coincide, *i.e.* the discriminant $u$ in Lemma 4 has to be zero, that is the condition (2) given in the statement of the lemma. Hence, $r_+ = r_-$ and $\theta_+ = \theta_-$.

We prove now that these additional critical points are saddle-nodes. By symmetry, we only need to prove it for one of them. The Jacobian matrix of system (4) evaluated at $(r_+, \theta_+)$ is

$$J_{(r_+,\theta_+)} = \begin{pmatrix} 2p_1 + 4r_+(s_1 - \cos(6\theta_+) & 12r_+^2 \sin(6\theta_+) \\ s_2 + \sin(6\theta_+) & 6r_+ \cos(6\theta_+) \end{pmatrix}. \tag{11}$$

Evaluating the Jacobian matrix (11) at the concrete expression (7) of the critical point and taking into account the condition $p_1^2 + p_2^2 = (p_1 s_2 - p_2 s_1)^2$, the eigenvalues of the matrix are

$$\lambda_1 = 0 \qquad\qquad \lambda_2 = 2p_1 - \frac{6p_2(p_2 + p_2 s_1 - p_1 s_2)}{(p_2 s_2 + p_1)(1 + s_1) - p_1 s_2^2}.$$

Therefore, $(r_+, \theta_+)$ has a zero eigenvalue. To show that these critical points are saddle-nodes we use the following reasoning. It is well–known, see for instance [2], that the sum of the indices of all critical points contained in the interior of a limit cycle of a planar system is $+1$. As under our hypothesis the infinity does not have critical points on it, it is a limit cycle of the system and it has seven singularities in its interior: the origin, that is a focus and hence has index +1, and 6 more critical points, all of the same type because of the symmetry. Consequently, the index of these critical points must be 0. As we have proved that they are semi-hyperbolic critical points then they must be saddle-nodes.

In order that equation (4) has exactly thirteen critical points it is enough that $r > 0$ (*i.e.* $p_2 < 0$) and the discriminant in $\Delta_\pm$ of Lemma 4 is positive, that is the condition (3) of the statement.

To get the stability of the twelve critical points we evaluate the Jacobian matrix (11) at these critical points, and taking into account the condition $p_1^2 + p_2^2 > (p_1 s_2 - p_2 s_1)^2$, the eigenvalues of the critical points $(r_+, \theta_+)$ are $\lambda_{1,2} = R_+ \pm S_+$, while the eigenvalues of $(r_-, \theta_-)$ are $\alpha_{1,2} = R_- \pm S_-$, where

$$R_\pm = \frac{p_1^2 s_1 \pm 3p_2^2 s_1 \mp 2p_1 p_2 s_2 + 2p_1 u}{\mp p_1 s_1 \mp p_2 s_2 + u},$$

$$S_\pm = \frac{\sqrt{48(p_1^2 + p_2^2)u(u \mp p_1 s_1 \mp p_2 s_2) + (2p_1^2 s_1 + 6p_2^2 s_1 - 4p_1 p_2 s_1 2 \pm 4p_1 u)^2}}{p_1 \mp p_1 s_1 \mp p_2 s_2}.$$

In the following we will show that one of the critical points has index $+1$, while the other is a saddle.

Doing some computations one gets that the product of the eigenvalues of $(r_+, \theta_+)$ and $(r_-, \theta_-)$ are, respectively

$$R_+^2 - S_+^2 = \frac{-12(p_1^2 + p_2^2)u}{u - p_1 s_1 - p_2 s_2}$$

$$R_-^2 - S_-^2 = \frac{-12(p_1^2 + p_2^2)u}{u + p_1 s_1 + p_2 s_2}.$$

The numerator of both expressions is negative.

If $p_1 s_1 + p_2 s_2 > 0$ then $S_-^2 > 0$ and $R_-^2 - S_-^2 < 0$. Consequently the critical point $(r_-, \theta_-)$ is a saddle. On the other hand, the denominator of $R_+^2 - S_+^2$ is negative. This is true as

$$\begin{aligned}
&u - p_1 s_1 - p_2 s_2 < 0 \iff u^2 < (p_1 s_1 + p_2 s_2)^2 \iff \\
&\iff p_1^2 + p_2^2 - p_2^2 s_1^2 - p_1^2 s_2^2 + 2p_1 p_2 s_1 s_2 < p_1^2 s_1^2 + p_2^2 s_2^2 + 2p_1 p_2 s_1 s_2 \\
&\iff 0 < (p_1^2 + p_2^2)(s_1^2 + s_2^2 - 1),
\end{aligned}$$

that is always true. Hence, $R_+^2 - S_+^2 > 0$ and $(r_+, \theta_+)$ has index $+1$.

If $p_1 s_1 + p_2 s_2 < 0$, doing similar reasonings one gets that the critical point $(r_+, \theta_+)$ is a saddle while $(r_-, \theta_-)$ has index $+1$. ∎

Note that $\mathcal{Q}$ is a quadratic form on $p_1, p_2$, and its determinant $1 - s_1^2 - s_2^2$, is negative if $s_2 > 1$. Hence, for each choice of $s_1, s_2$ with $s_2 > 1$, the points where $\mathcal{Q}(p_1, p_2)$ is negative lie on two sectors, delimited by the two perpendicular lines where $\mathcal{Q}(p_1, p_2) = 0$. Since $\mathcal{Q}(0, p_2) = (1 - s_2^2)p_2^2 < 0$ for $s_2 > 1$, then the sectors where there are two equilibria in each sextant do not include the $p_2$ axis, as in Figure 2.

**Lemma 7.** *Consider $|s_2| > 1$ and define the following two numbers:*

$$\Sigma_A^- = \frac{p_2 s_1 s_2 - \sqrt{p_2^2\left(s_1^2 + s_2^2 - 1\right)}}{s_2^2 - 1}, \quad \Sigma_A^+ = \frac{p_2 s_1 s_2 + \sqrt{p_2^2\left(s_1^2 + s_2^2 - 1\right)}}{s_2^2 - 1}.$$

*Let $A(\theta)$ be the function given in Lemma 1. Then the function $A(\theta)$ changes sign if and only if $p_1 \in \left(\Sigma_A^-, \Sigma_A^+\right)$.*

*Proof.* Writing $x = \sin(6\theta)$, $y = \cos(6\theta)$, the function $A(\theta)$ in (3) becomes

$$A(x,y) = \frac{2}{p_2}\Big(p_1 - p_2 s_1 s_2 + p_1 s_2^2 + (2p_1 s_2 - p_2 s_1)x + (p_2 x - p_1 y + p_2 s_2)y\Big).$$

Next we solve the set of equations

$$\begin{aligned} A(x,y) &= 0, \\ x^2 + y^2 &= 1. \end{aligned}$$

to get the solutions

$$\begin{aligned} x_1 &= -s_2,\ y_1 = \sqrt{1 - s_2^2} \\ x_2 &= -s_2,\ y_2 = -\sqrt{1 - s_2^2} \end{aligned}$$

$$x_\pm = \frac{p_1 p_2 s_1 - p_1^2 s_2 \pm p_2\sqrt{p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2}}{p_1^2 + p_2^2}$$

$$y_\pm = \frac{p_2^2 s_1 - p_1 p_2 s_2 \mp p_1\sqrt{p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2}}{p_1^2 + p_2^2}$$

Observe now the the two first pairs of solutions $(x_1, y_1), (x_2, y_2)$ cannot be solutions of our equation $A(\theta) = 0$ since $x = \sin(6\theta) = -s_2 < -1$.

On the other hand, if we look for the intervals where the function $A(\theta)$ does not change sign we have two possibilities: either $|x_\pm| > 1$ (and consequently $|y_\pm| > 1$) or the discriminant of $x_\pm$, $\Delta = p_1^2 + p_2^2 - (p_2 s_1 - p_1 s_2)^2$ is negative or zero. In the case $\Delta < 0$, the solutions will be complex non-real, and in the second case $\Delta = 0$, the function will have a zero but a double one, and it will not change its sign.

The first possibility turns out to be impossible in our region of parameters. The second possibility leads to the region $p_1 \in \mathcal{R} \setminus (\Sigma_A^+, \Sigma_A^-)$. ■

**Lemma 8.** *Consider $|s_2| > 1$ and define the following two numbers:*

$$\Sigma_B^\pm = \frac{\Sigma_A^\pm}{2}.$$

*Let $B(\theta)$ be the function given in Lemma 1. Then the function $B(\theta)$ changes sign if and only if $p_1 \in \left(\Sigma_B^-, \Sigma_B^+\right)$.*

*Proof.* By direct computations with the same substitution as the one in the proof of the previous lemma, $x = \sin(6\theta), y = \cos(6\theta)$, we get that the zeroes of the system

$$\begin{aligned} B(x,y) &= 0, \\ x^2 + y^2 &= 1, \end{aligned}$$

are

$$\begin{aligned} x_{\pm} &= \frac{2p_1p_2s_1 - 4p_1^2s_2 \pm p_2\sqrt{4p_1^2 + p_2^2 - (p_2s_1 - 2p_1s_2)^2}}{4p_1^2 + p_2^2}, \\ y_{\pm} &= \frac{p_2^2s_1 - 2p_1p_2s_2 \mp 2p_1\sqrt{4p_1^2 + p_2^2 - (p_2s_1 - 2p_1s_2)^2}}{4p_1^2 + p_2^2}. \end{aligned}$$

Applying arguments similar to those in the previous proof, we get that the function $B(\theta)$ will not change sign if and only if $p_1 \notin (\Sigma_B^+, \Sigma_B^-)$. ∎

**Remark 2.**

- *In general systems there are examples for which the function $A(\theta)$ changes sign but $B(\theta)$ does not and vice-versa.*
- *The condition given in Lemma 7 under which the function $A(\theta)$ does not change sign is closely related to the number of critical points of system* (4)*. More precisely, the condition $\Delta < 0$ in the proof of Lemma 7 is the same as the condition for existence of a unique critical point in system* (4)*, while the condition $\Delta = 0$, together with $p_2 < 0$, is equivalent to the existence of exactly 7 critical points.*

In the following we will need some results on Abel equations proved in [12] and [8]. We summarize them in a theorem.

**Theorem 3.** *Consider the Abel equation* (2) *and assume that either $A(\theta) \not\equiv 0$ or $B(\theta) \not\equiv 0$ does not change sign. Then it has at most three solutions satisfying $x(0) = x(2\pi)$, taking into account their multiplicities.*

**Remark 3.** *Condition (a) of Theorem 1 implies that one of the functions $A(\theta), B(\theta)$ of the Abel equation does not change sign. If condition (i) is satisfied then $A(\theta)$ does not change sign and, as the third derivative of the Poincaré return map of the Abel equation is this function $A(\theta)$ then, the Abel equation can have at most 3 limit cycles. In our case, one of them is the origin and another one is infinity. Consequently only one non-trivial limit cycle can exist, both for the scalar equation and for the planar system.*

*If condition (ii) is verified then it is the function $B(\theta)$ that does not change sign and, by a change of coordinates, one can get another scalar equation for which the third derivative of the Poincaré map is the function $B(\theta)$, getting the same conclusions as before.*

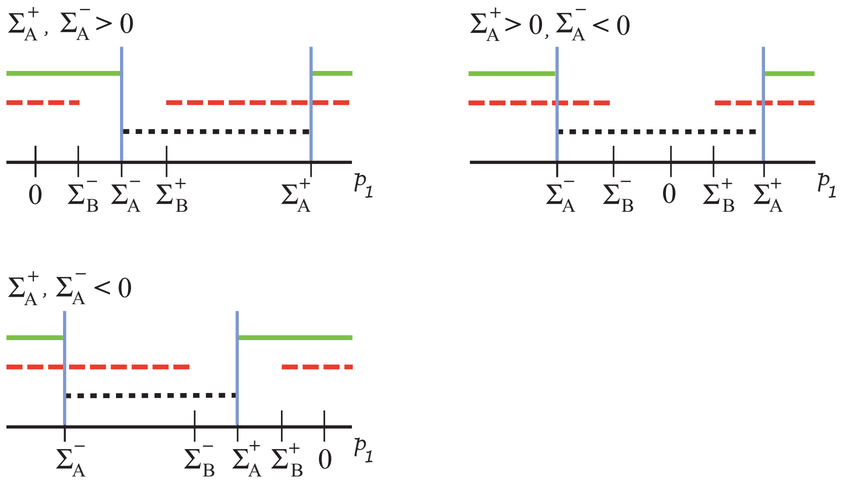


Figure 3: A sketch of intervals where the conditions of Theorem 1 are satisfied, with $s_2 > 1, p_2 < 0$. The result follows from the relationship between the function $A(\theta)$ and $B(\theta)$ changing sign and the number of critical points of (4). The green solid line indicates where the function $A(\theta)$ does not change sign, the red dashed one where $B(\theta)$ does not change sign, while the dotted line represents the interval where 13 critical points exist. The blue vertical lines (that correspond to $\Sigma_A^{\pm}$) stand for 7 critical points. There is another possibility, not shown here, that the two intervals are disjoint, see Figure 1.

## 4 Limit cycles

*Proof of Theorem 1.* We first define the function $c(\theta) = s_2 + \sin(6\theta)$ and the set $\Theta := \{(r, \theta) : \dot{\theta} = p_2 + (s_2 + \sin(6\theta))r = 0\}$. Since $|s_2| > 1$, we have $c(\theta) \neq 0, \ \forall \theta \in [0, 2\pi]$.

$(a)$ Let's assume first that condition $(i)$ is satisfied. By Lemma 1, we reduce the study of the periodic orbits of equation (1) to the analysis of the non contractible periodic orbits of the Abel equation (2). Since $p_1 \notin \left(\Sigma_A^-, \Sigma_A^+\right)$, by condition $(i)$ of Lemma 7, we know that function $A(\theta)$ in the Abel equation does not change sign. Hence, from Theorem 3, the maximum number of solutions satisfying $x(0) = x(2\pi)$ in system (2) taking into account their multiplicities, is three. One of them is trivially $x = 0$. Since $c(\theta) \neq 0$, by simple calculations we can prove that the curve $x = 1/c(\theta)$ is a second solution satisfying this condition. As shown in [1], undoing the Cherkas transformation we get that $x = 1/c(\theta)$ is mapped into infinity of the differential equation. Then, by Lemma 1, the maximum number of limit cycles of equation (1) is one. Moreover, from the same lemma it follows that the limit cycle is hyperbolic. From the symmetry, it follows that a unique limit cycle must surround the origin.

$(b)$ We follow the same analysis method as in [1]. When $p_1 > \Sigma_A^+$, by Lemma 3, both the origin and infinity in the Poincaré compactification are repelors. In particular the origin is an unstable focus. On the other hand, from Lemma 7, $A(\theta)$ does not vanish when $p_1 > \Sigma_A^+$, and the origin is the unique critical point. It is easy to see that, since $p_2 A(\theta) > 0$, then the exterior of the closed curve $\Theta$ is pos-

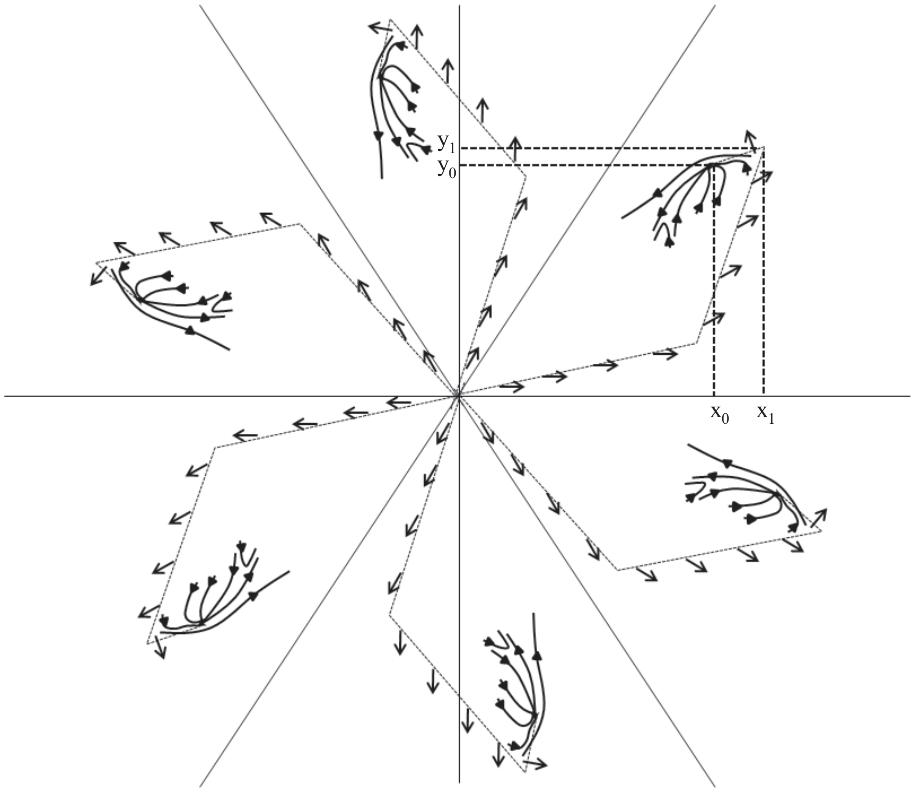


Figure 4: The polygonal curve with no contact with the flow of the differential equation and the separatrices of the saddle-nodes of system (8).

itively invariant and therefore, by applying the Poincaré-Bendixson Theorem and part $(a)$ of this theorem, there is exactly one hyperbolic limit cycle surrounding the curve $\Theta$. Moreover, this limit cycle is stable.

When $p_1 = \Sigma_A^+$, six more semi-elementary critical points appear and they are located on $\Theta$. They are saddle-nodes as shown in Lemma 6. We will show that at this value of $p_1$ the periodic orbit still exists and it surrounds the seven critical points. We will prove this by constructing a polygonal line with no contact with the flow of the differential equation. On the polygonal, the vector field points outside, and consequently, as the infinity is a repelor, the $\omega$-limit set of the unstable separatrices of the saddle-nodes must be a limit cycle surrounding $\Theta$, see Figure 4 and Example 1 below.

The $\mathbb{Z}_6-$equivariance of system (1) allows us to study the flow in only one sextant of the phase space in cartesian coordinates, the behaviour in the rest of the phase space being identical. The polygonal line will join the origin to one of the saddle-nodes as in Figure 4.

We explain the construction of the polygonal line when $0 < s_1 \leq 1, s_2 > 1$ and $p_2 < 0$. In this case, $p_1 = \Sigma_A^+$ implies that $p_1 > 0$ and, in the notation of Lemma 4, we have $u = 0$. Hence

$$\frac{1}{\Delta_\pm} = \frac{(s_2^2-1)(1+s_1)}{s_1 s_2 - \sqrt{s_1^2+s_2^2-1}} - s_2 < -1$$

and using the expression in Lemma 4, we find that the angular coordinate $\theta_0$ of the saddle-node satisfies $-1 < \tan 3\theta_0 < 0$, and therefore $\pi/4 < \theta_0 < \pi/3$.

The first segment in the polygonal is the line $\theta = \pi/4$. Using the expression (4) we obtain that on this line $\dot{\theta} = p_2 + r(s_2 - 1)$ which is negative for $r$ between 0 and $r_1 = -p2/(s_2 - 1) > 0$. Thus, the vector field is transverse to this segment and points away from the saddle-node on it.

Another segment in the polygonal is obtained using the eigenvector $v$ corresponding to the non-zero eigenvalue of the saddle-node $z_0$. The line $z_0 + tv$ is the tangent to the separatrix of the hyperbolic region of the saddle-node. Let $t_0$ be the first positive value of $t$ for which the vector field fails to be transverse to this line.

If the lines $z_0 + tv$ and $\theta = \pi/4$ cross at a point with $0 < t < t_0$ and with $0 < r < r_1$, then the polygonal consists of the two segments. If this is not the case, then the segment joining $z_0 + t_0 v$ to $r_1 \left(\sqrt{2}/2, \sqrt{2}/2\right)$ will also be transverse to the vector field, and the three segments will form the desired polygonal.

By using the same arguments presented above and the fact that infinity is a repelor, it follows by the Poincaré-Bendixson Theorem that the only possible $\omega$-limit for the unstable separatrix of the saddle-node is a periodic orbit which has to surround the six saddle-nodes, see again Figure 4.

If $p_1 = \Sigma_A^+$ then the function $A(\theta)$ does not change sign; therefore the arguments presented in part $(a)$ of the proof of this theorem assure the hyperbolicity of the limit cycle.

When we move $p_1$ towards zero but still very close to $\Sigma_A^+$, then $B(\theta)$ is strictly positive because $\Sigma_A^+ > \Sigma_B^+$ and there are 13 critical points as shown in Lemma 6: the origin (which is a focus), six saddles and six critical points of index $+1$ on $\Theta$. Applying one more time part $(a)$ of the proof of this theorem, we know that the maximum number of limit cycles surrounding the origin is one. If $p_1 = \Sigma_A^+$ the limit cycle is hyperbolic and it still exists for the mentioned value of $p_1$. Then we have the vector field with $B(\theta)$ not changing sign, 12 non-zero critical points and a limit cycle which surrounds them together with the origin. ∎

**Example 1.** *As an example of the construction of the polygonal line in the proof of Theorem 1 we present a particular case, done with Maple. Let's fix parameters* $p_2 = -1$, $s_1 = -0.5$, $s_2 = 1.2$. *With these values of the parameters, we have* $\Sigma_A^- = -0.52423$, $\Sigma_A^+ = 3.25151$.

*To construct the polygonal line we work in cartesian coordinates. A key point in the process of identifying the three segments of the polygonal line is knowing explicitly the eigenvector corresponding to the non-zero eigenvalue of the saddle-node* $(x_0, y_0) = (1.358, 1.5)$. *This eigenvector is* $v = (-0.8594, -0.5114)$ *so the slope of the tangent to the hyperbolic direction of the saddle-node is* $0.5114/0.8594$ *and the straight line of this slope passing through the saddle-node is* $R \equiv \{y = 1.5 + \dfrac{0.5114}{0.8594}(x - 1.358)\}$. *The tangent to the separatrix of the hyperbolic region of the saddle-node is locally transverse to this line. The scalar product of the vector field associated to equation* (8) *with the normal vector to* $R$, $(0.5114, -0.8594)$, *is given by the equation*

$$-2.39191647949065x^5 + 2.34410741916533x^4 + 4.86235167862649x^3 - 2.71272659052423x^2 - 2.33924612305747x - 0.92289951077311$$

*when evaluated on the straight line R; its unique real root is* $x \simeq -1.1737$ *and the flow is transversal from inside out through R for any* $x > -1.1737$.
*Let us now define the polygonal line as*

$$(x(t), y(t)) = \begin{cases} (t, t) & \text{if } 0 \leqslant t < 2\sqrt{\frac{1}{2.8}}, \\ \\ (t, 1.5(t - 2\sqrt{\frac{1}{3}}) + 2\sqrt{\frac{1}{2.8}}) & \text{if } 2\sqrt{\frac{1}{2.8}} \leqslant t < 1.425, \\ \\ ((t-2)\dfrac{x_1 - x_0}{x_1 - 2} + x_0, (t-2)\dfrac{y_1 - y_0}{y_1 - 2} + y_0) & \text{if } 1.425 \leqslant t < 2. \end{cases}$$

*Let's now consider the point* $(x_1, y_1) = (1.4250, 1.5399)$, *which stands at the intersection between the second and third segments of the polygonal line. The scalar product between the normal to each segment and the flow of the differential equation is negative, when evaluated on the corresponding segments. To show the calculation, we will exemplify it for the first segment, the remaining cases being treated similarly.*

*We elected the segment of the line* $L \equiv \{y = x\}$ *and when substituting it into the system in cartesian coordinates* (6) *we obtain*

$$\begin{cases} \dot{x} = 4x^5(s_1 - s_2 + 1) + 2x^3(p_1 - p_2) \\ \dot{y} = 4x^5(4s_1 + 5s_2 - 4) + 2x^3(p_1 + p_2). \end{cases}$$

*A normal vector to the line L is* $(-1, 1)$ *and the scalar product of* $(\dot{x}, \dot{y})$ *with* $(-1, 1)$ *yields* $f(x) = x^3(4p_2 + x^2(9s_2 - 8))$. *Solving this last equation leads to* $f(x) < 0$ *for* $-2\sqrt{\frac{-p_2}{9s_2-8}} \leqslant x \leqslant 2\sqrt{\frac{-p_2}{9s_2-8}}$. *So we choose* $0 \leqslant x \leqslant 2\sqrt{\frac{-p_2}{9s_2-8}}$, *and we get that this scalar product is negative in the region where the polygonal line is defined as* $(t, t)$.

*By using the same arguments presented above it follows that the only possible* $\omega$*-limit for the unstable separatrix of the saddle-node is a periodic orbit which has to surround the six saddle-nodes, see again Figure 4.*

**Acknowledgements** An important part of the work was carried out during a FCT-supported visit of A.C. Murza to the Departament Ciències Matemàtiques i Informàtica de la Universitat de les Illes Balears, under the supervision of Professors María J. Álvarez and Rafel Prohens, to whom A.C. Murza is deeply thankful for their kind guidance.

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