On 2-pyramidal Hamiltonian cycle systems *

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Abstract

A Hamiltonian cycle system of the complete graph minus a 1–factor $K_{2v} - I$ (briefly, an HCS$(2v)$) is 2-*pyramidal* if it admits an automorphism group of order $2v - 2$ fixing two vertices. In spite of the fact that the very first example of an HCS$(2v)$ is very old and 2-pyramidal, a thorough investigation of this class of HCSs is lacking. We give first evidence that there is a strong relationship between 2-pyramidal HCS$(2v)$ and 1-*rotational* Hamiltonian cycle systems of the complete graph K_{2v-1}. Then, as main result, we determine the full automorphism group of every 2-pyramidal HCS$(2v)$. This allows us to obtain an exponential lower bound on the number of non-isomorphic 2-pyramidal HCS$(2v)$.

1 Introduction

Speaking of a *Hamiltonian cycle system* of order v, or HCS(v) for short, we mean a set of Hamiltonian cycles of K_v whose edges partition $E(K_v)$ if v is odd or $E(K_v) - I$, with I a 1-factor of K_v, if v is even. Two HCSs are isomorphic if there exists a bijection (isomorphism) between their vertex–sets mapping one into the other. An automorphism of a Hamiltonian cycle system $\mathcal{H}$ is an isomorphism of $\mathcal{H}$ with itself. The automorphisms of $\mathcal{H}$ form the *full automorphism group* of $\mathcal{H}$, denoted by $Aut(\mathcal{H})$. Speaking of *an automorphism group* of $\mathcal{H}$ one means a subgroup of $Aut(\mathcal{H})$.

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HCSs possessing a non-trivial automorphism group have attracted considerable attention (see [8] for a short recent survey on this topic). Detailed results can be found in: [9, 15] for the *cyclics*; [10] for the *dihedrals*; [4] for the *doubly transitives*; [7] for the *regulars*; [1, 6] for the *symmetrics*; [11] for those being both cyclic and symmetric. Here, we only need to recall the basic facts on the 1-*rotationals* which have been widely studied in [3, 13].

Throughout this paper every group will be denoted in multiplicative notation and its identity will be denoted by 1, except when the group is the cyclic group $\mathbb{Z}_n$. As usual, additive notation will be used in this case with identity 0.

An HCS(v) is 1-rotational under a group G (also called a *round dance neighbour design based on* G in [3]) if it admits G as an automorphism group of order $v-1$ fixing one vertex ∞. In this case the action of G on the other vertices is necessarily sharply transitive. Thus it is natural to identify the vertex-set V with $G \cup \{\infty\}$ and to see the action of G on V as the multiplication on the left with the rule that $g\infty = \infty$ for every $g \in G$.

In what follows, when speaking of the *differences* of two adjacent vertices g_1 and g_2, with $g_1, g_2 \in G$, we will mean $g_1^{-1}g_2$ and $g_2^{-1}g_1$.

We first note that a 1-rotational HCS of even order cannot exist.

There exists a 1-rotational HCS$(2n+1)$ under a group G of order $2n$ if and only if G is *symmetrically sequenceable* (see [12, Proposition 3.9]). This means that G is *binary*, namely it admits exactly one involution, and there exists a path $T = [g_1, g_2, \ldots, g_{2n}]$ (*directed terrace*) with vertex-set G satisfying the following properties:

(i) $g_{2n+1-i} = \lambda g_i$ for $1 \leq i \leq 2n$ where λ is the only involution of G, i.e., we have: $T = [g_1, g_2, \ldots, g_n, \lambda g_n, \ldots, \lambda g_2, \lambda g_1]$.

(ii) every $g \in G \setminus \{1, \lambda\}$ can be written in exactly one way as a difference of two adjacent vertices g_i and g_{i+1} of the subpath $T' = [g_1, g_2, \ldots, g_n]$.

The 1-rotational HCSs under G are precisely the G-orbits of a cycle obtainable by joining ∞ with the endpoints of a directed terrace of G.

Thus, if $T = [g_1, \ldots, g_{2n}]$ is a directed terrace of G, then the 1-rotational HCS *generated by* T is easily seen to be $\mathcal{H}(T) = \{sC \mid s \in S\}$ with $C = (\infty, g_1, \ldots, g_{2n})$, $sC = (\infty, sg_1, \ldots, sg_{2n})$ and S an arbitrary complete system of representatives for the cosets of the subgroup $\{1, \lambda\}$ of G of order 2; for instance one can take $S = \{g_1, \ldots, g_n\}$ in view of condition (i). Without loss of generality we can always assume that $g_1 = 1$, in which case T is said to be a *basic* directed terrace and C is said to be the *starter cycle* of $\mathcal{H}(T)$.

We refer to [20] for a survey on sequenceable groups. Here we recall that every binary solvable group except Q_8 (the group of quaternions) has been proved to be symmetrically sequenceable in [2].

We say that an HCS(v) is 2-*pyramidal* if it admits an automorphism group G of order $v-2$ fixing 2 vertices ∞ and $\overline{\infty}$. First observe that such an HCS(v) has v even apart from the trivial case of $v = 3$. In fact, for v odd, the edge connecting the two vertices fixed by G should be covered by a cycle C of the HCS and then we see that every $g \in G$ would have to fix C pointwise. This is possible only in the case that G is the trivial group.

Reasoning as above one can also see that in every 2-pyramidal HCS of even order, the edge $[\infty, \overline{\infty}]$ is always in the removed 1-factor I.

It is not difficult to see that for any given 2-pyramidal HCS under G, the action of G on the non-fixed vertices is sharply transitive. Hence the vertex-set V can be identified with $G \cup \{\infty, \overline{\infty}\}$, $v = 2n+2$, $|G| = 2n$, and the action of G on V is the multiplication on the left with the rule that $g\infty = \infty$ and $g\overline{\infty} = \overline{\infty}$ for every $g \in G$.

Given $g \in G$, we will denote by τ_g the bijection on G defined by $\tau_g(x) = gx$ for every $x \in G$. By abuse of notation, for $\infty, \overline{\infty} \notin G$, the bijections on $G \cup \{\infty\}$ or $G \cup \{\infty, \overline{\infty}\}$ acting as τ_g on G and fixing the *infinities* will be also denoted by τ_g.

In what follows we will denote by $\widehat{G}$ the group $\{\tau_g \mid g \in G\}$. The action of $\widehat{G}$ on the vertex-set naturally extends to edges and cycles and thus $\widehat{G}$ is an automorphism group of the HCS. Moreover each automorphism τ_g preserves differences between adjacent vertices and the map $g \mapsto \tau_g$ is an isomorphism between G and $\widehat{G}$.

In the next section we will see that every 2-pyramidal HCS$(2n+2)$ is *generated* by a suitable 1-rotational HCS$(2n+1)$.

In the third section we will prove that the full automorphism group of a 2-pyramidal HCS$(2n+2)$ under G always is isomorphic with G itself for $n \geq 3$. As a consequence, there exists an HCS$(2n+2)$ with full automorphism group G for any symmetrically sequenceable group G.

Finally, in the last section we show that for $n \geq 3$, up to isomorphism, every 2-pyramidal HCS$(2n+2)$ is generated by exactly two 1-rotational HCS$(2n+1)$ so that the number of non-isomorphic 2-pyramidal HCS$(2n+2)$ is exactly half the number of non–isomorphic 1-rotational HCS$(2n+1)$. This fact, using the enumerative results on 1-rotational HCS$(2n+1)$ obtained in [13], allows us to claim that there are at least $2^{\lceil 3n/4 \rceil - 1}$ pairwise non-isomorphic 2-pyramidal HCS$(2n+2)$ for every $n \geq 6$.

2 On the structure of 2-pyramidal HCSs

The famous HCS$(2n+1)$ by Walecki (see [18]) is 1-rotational under $\mathbb{Z}_{2n}$. We denote this $W(2n+1)$. It is well known that inserting in every cycle of it a new vertex $\overline{\infty}$ between the two vertices at distance n from ∞ one obtains a 2-pyramidal HCS$(2n+2)$ under $\mathbb{Z}_{2n}$ that will be denoted by $W(2n+1)_+$. (see Figure 1).

Indeed we are going to show that the 2-pyramidal HCSs are precisely those obtainable in this way starting from any 1-rotational HCS.

From now on, if C is a $(2n+1)$-cycle with a vertex denoted by ∞, then C_+ will denote the $(2n+2)$-cycle obtainable from C by inserting a new vertex $\overline{\infty}$ between the two vertices of C at distance n from ∞. If $\mathcal{H}$ is any collection of $(2n+1)$-cycles passing through ∞, then we set $\mathcal{H}_+ = \{C_+ \mid C \in \mathcal{H}\}$.

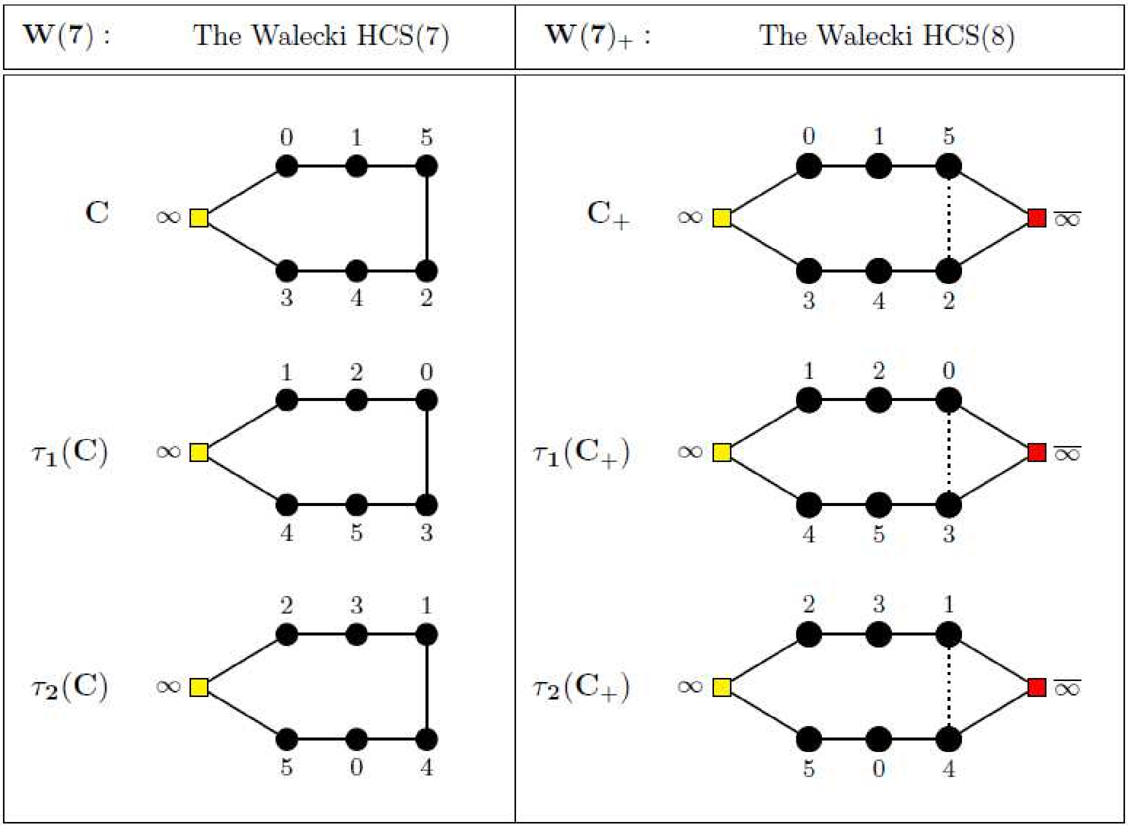

Figure 1:

Proposition 2.1. *If $\mathcal{H}$ is a* 1*-rotational HCS*$(2n+1)$ *under G, then $\mathcal{H}_+$ is a* 2*-pyramidal HCS*$(2n+2)$ *under G.*

Conversely, every 2*-pyramidal HCS*$(2n+2)$ *under G has the form $\mathcal{H}_+$ with $\mathcal{H}$ a suitable* 1*-rotational HCS*$(2n+1)$ *under G.*

Proof. Let $\mathcal{H}$ be a 1-rotational HCS$(2n+1)$ under G and let $C = (\infty, g_1, \ldots, g_{2n})$ be its starter cycle. Thus $T = [g_1, \ldots, g_{2n}]$ is a basic directed terrace of G and we have $\mathcal{H} = \{g_iC \mid 1 \leq i \leq n\}$, with $g_iC = (\infty, g_ig_1, \ldots, g_ig_n)$. For $i = 1, 2, \ldots, n$, the two vertices at distance n from ∞ in the cycle g_iC are the endpoints of the edge $[g_ig_n, g_ig_{n+1}]$. All these edges form the cosets of $\{1, \lambda\}$ in G by condition (i) on directed terraces and hence they form, together with $[\infty, \overline{\infty}]$, a 1-factor I of the complete graph K_{2n+2} with vertex-set $G \cup \{\infty, \overline{\infty}\}$. Thus, we easily see that $\mathcal{H}_+$ is a decomposition of $K_{2n+2} - I$, i.e., an HCS$(2n+2)$. We also see that the cycles of $\mathcal{H}_+$ are those of the $\widehat{G}$-orbit of the cycle C_+. Therefore $\mathcal{H}_+$ is 2-pyramidal under G.

Now assume that $\mathcal{H}'$ is a 2-pyramidal HCS$(2n+2)$ under G. Let I be the 1-factor not covered by the cycles of $\mathcal{H}'$ and recall that $[\infty, \overline{\infty}]$ is in I.

Let λ be any involution of G and suppose that there exists an edge e of a cycle C of $\mathcal{H}'$ which is a right coset of $\{1, \lambda\}$ in G. Of course τ_λ switches the endpoints of e and hence it acts on C as a reflection in the *axis* of e. This is absurd since the reflection in the axis of an edge of an even-cycle has no fixed vertex while we know that τ_λ fixes both ∞ and $\overline{\infty}$. We conclude that no right coset of $\{1, \lambda\}$ is edge of a cycle of $\mathcal{H}'$. Therefore each of the n right cosets of $\{1, \lambda\}$ is an edge of I and hence, by the *pigeon hole principle*, $I \setminus [\infty, \overline{\infty}]$ necessarily coincides with the set of right cosets of $\{1, \lambda\}$ in G. We also deduce that λ is the only involution of G otherwise, with the same reasoning, we would have other edges not covered by the cycles of $\mathcal{H}'$. Thus G is binary and $\{1, \lambda\}$ is normal in G.

Now take any cycle C of $\mathcal{H}'$, let $\mathrm{Stab}(C)$ be its $\widehat{G}$-stabilizer, and let $\mathrm{Orb}(C)$ be its $\widehat{G}$-orbit. Of course $\mathrm{Orb}(C)$ is entirely contained in $\mathcal{H}'$ and hence its length is at most equal to $|\mathcal{H}'|$, which a trivial counting argument shows to be equal to n. It follows that $\mathrm{Stab}(C)$ is not trivial. A non-identity element τ_g of $\mathrm{Stab}(C)$ fixes C and also fixes the two vertices ∞ and $\overline{\infty}$ so that $[\infty, \overline{\infty}]$ is necessarily a diameter of C and τ_g acts on C as a reflection in this diameter. We deduce, in particular, that g is an involution and hence, recalling that G is binary, we have $\mathrm{Stab}(C) = \{1, \tau_\lambda\}$. Thus $\mathrm{Orb}(C)$ has length $\frac{|G|}{2} = n$, that is the size of $\mathcal{H}'$. We conclude that $\mathcal{H}'$ coincides with $\mathrm{Orb}(C)$.

From the above paragraph, the neighbors x and y of $\overline{\infty}$ in C are switched by τ_λ, i.e., we have $y = \lambda x$. It follows that the pairs of neighbors of $\overline{\infty}$ in the cycles of $\mathcal{H}'$ are the cosets of $\{1, \lambda\}$ in G, i.e., the edges of $I \setminus [\infty, \overline{\infty}]$. We conclude that removing $\overline{\infty}$ from each cycle of $\mathcal{H}'$ and joining its neighbors we get a set $\mathcal{H}$ of $(2n+1)$-cycles which is an HCS$(2n+1)$ with vertex-set $G \cup \{\infty\}$. It follows that $\mathcal{H}$ is 1-rotational and that $\mathcal{H}' = \mathcal{H}_+$. ∎

If $\mathcal{H}$ is a 1-rotational HCS$(2n+1)$ and C is its starter cycle, then C_+ will be naturally called the starter cycle of $\mathcal{H}_+$.

Remembering how $\mathcal{H}_+$ is obtained from $\mathcal{H}$, it is evident from the above proposition that $[\infty, \overline{\infty}]$ is a diameter of every cycle of a 2-pyramidal HCS.

In view of the general result concerning 1-rotational HCSs mentioned above, we can state the following result.

Corollary 2.2. *There exists a* 2*-pyramidal HCS*$(2n+2)$ *under a group* G *of order* $2n$ *if and only if* G *is symmetrically sequenceable.*

3 The full automorphism group of a 2-pyramidal HCS

It is easy to see that, up to isomorphism, there is exactly one HCS(4) and exactly one HCS(6), both pictured in Figure 2. They are 2-pyramidal under $\mathbb{Z}_2$ and $\mathbb{Z}_4$, respectively and their full automorphism groups are both isomorphic to the *dihedral group* of order 8. This is clear if we consider that the HCS(4) is just a 4-cycle, whereas the full automorphism group of the HCS(6) is generated by the translation τ_1 and by the reflection β in the *axis a* of the diameter $[\infty, \overline{\infty}]$.

For $n > 2$ we prove that the full automorphism group of a 2-pyramidal HCS$(2n+2)$ under G is just G.

In this case, differently from the HCS(4) and the HCS(6), the reflection β in the axis of the diameter $[\infty, \overline{\infty}]$ is not an automorphism of a 2-pyramidal HCS$(2n+2)$ with $n > 2$.

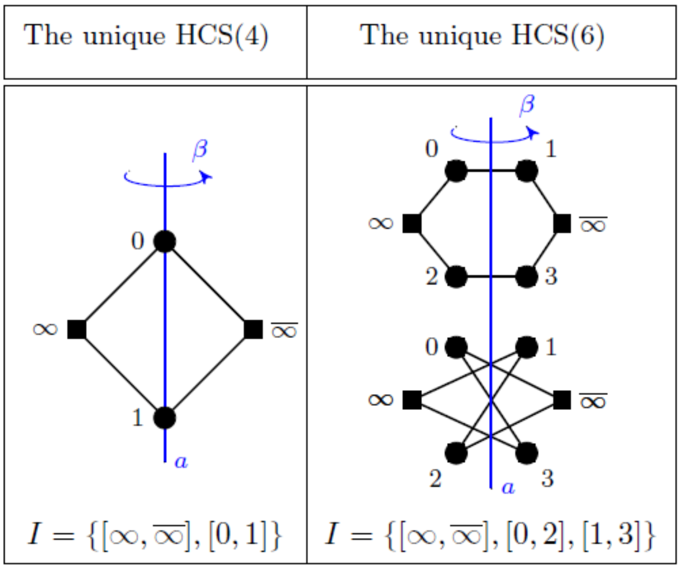

Figure 2:

Theorem 3.1. *If $n > 2$, the full automorphism group of a 2-pyramidal HCS$(2n+2)$ under G is isomorphic to G itself.*

Proof. Let $\mathcal{H}$ be a 2-pyramidal HCS$(2n+2)$ under G and let C be its starter cycle. Denote by A the full automorphism group of $\mathcal{H}$.

We know that $\widehat{G} := \{\tau_g \mid g \in G\}$ is a subgroup of A isomorphic to G and that $\widehat{G}$ is transitive on $\mathcal{H}$ so that A is transitive on $\mathcal{H}$ as well. Thus, if $\widehat{G}_0$ and A_0 are the stabilizers of C under $\widehat{G}$ and A respectively, we have

$$|\mathcal{H}| = |\widehat{G} : \widehat{G}_0| = |A : A_0| \tag{3.1}$$

by the orbit-stabilizer theorem.

Since $n > 2$, the only edge of the removed 1-factor that is a diameter of every cycle is $[\infty, \overline{\infty}]$, hence A fixes $\{\infty, \overline{\infty}\}$. There are exactly four symmetries of C which preserve $\{\infty, \overline{\infty}\}$; they are the identity, the reflection τ_λ in the diameter $[\infty, \overline{\infty}]$, the reflection α in the axis of $[\infty, \overline{\infty}]$, and the rotation $\tau_\lambda\alpha$ through 180 degrees. Therefore either $A_0 = \widehat{G}_0 = \{\tau_1, \tau_\lambda\}$ or $A_0 = \{\tau_1, \tau_\lambda, \alpha, \tau_\lambda\alpha\}$. By (3.1), we have $A = \widehat{G}$ in the former case and $|A : \widehat{G}| = 2$ in the latter.

Suppose that $A_0 \neq \widehat{G}_0$ so that $\alpha \in A_0$. Note that α swaps the infinities and that α has order 2, so that $\alpha^2(g) = g$ for every $g \in G$. For $|A : \widehat{G}| = 2$, we have that $\widehat{G}$ is normal in A. Thus, for every $g \in G$, there exists a suitable $\phi(g) \in G$ such that $\alpha\tau_g\alpha^{-1} = \tau_{\phi(g)}$. Given g_1, g_2 in G, we can write:

$$\tau_{\phi(g_1g_2)} = \alpha\tau_{g_1g_2}\alpha^{-1} = \alpha\tau_{g_1}\alpha^{-1}\alpha\tau_{g_2}\alpha^{-1} = \tau_{\phi(g_1)}\tau_{\phi(g_2)} = \tau_{\phi(g_1)\phi(g_2)}.$$

It follows that ϕ is a permutation on G such that $\phi(g_1g_2) = \phi(g_1)\phi(g_2)$ for every pair of elements $g_1, g_2 \in G$, i.e., ϕ is an automorphism of G. Set $\alpha(1) = h$ so that we have $\alpha\tau_g\alpha^{-1}(h) = \alpha(\tau_g(1)) = \alpha(g)$. By definition of $\phi(g)$, we also have $\alpha\tau_g\alpha^{-1}(h) = \tau_{\phi(g)}(h) = \phi(g)h$. Thus we have:

$$\alpha(g) = \phi(g)h \quad \forall\, g \in G. \tag{3.2}$$

In particular, we have
$$1 = \alpha^2(1) = \alpha(h) = \phi(h)h. \tag{3.3}$$
Applying (3.2) twice and taking into account that ϕ is an automorphism of G, we have
$$g = \alpha^2(g) = \alpha(\phi(g)h) = \phi(\phi(g)h)h = \phi^2(g)\phi(h)h \quad \forall\, g \in G$$
and then, by (3.3), we have
$$g = \phi^2(g) \quad \forall\, g \in G. \tag{3.4}$$
Now let ψ be the automorphism of $\mathcal{H}$ defined by $\psi = \tau_{h^{-1}}\alpha$. In view of (3.2) we have $\psi(g) = \tau_{h^{-1}}\alpha(g) = h^{-1}\phi(g)h$, for all $g \in G$. Thus we can write
$$\psi^2(g) = \psi(h^{-1}\phi(g)h) = h^{-1}\phi(h^{-1}\phi(g)h)h$$
and then, recalling again that ϕ is an automorphism of G,
$$\psi^2(g) = h^{-1}\phi(h^{-1})\phi^2(g)\phi(h)h.$$
On the other hand we have $\phi(h) = h^{-1}$ by (3.3) and $\phi^2(g) = g$ by (3.4) so that we have
$$\psi^2(g) = g \quad \forall g \in G. \tag{3.5}$$
Let $\mathrm{Fix}(\psi)$ and $\mathrm{Fix}(\tau_\lambda\psi)$ be the sets of vertices which are fixed by ψ and $\tau_\lambda\psi$, respectively. If $|\mathrm{Fix}(\psi)| \geq 3$, then there is an edge $[x, y]$ with endpoints in $\mathrm{Fix}(\psi)$ not belonging to the removed 1-factor I and hence there is a cycle $C(x, y)$ of $\mathcal{H}$ containing $[x, y]$. For $\psi(x) = x$ and $\psi(y) = y$, we have that ψ fixes $C(x, y)$, i.e., ψ is a symmetry of $C(x, y)$. It follows that ψ is the identity since there is no non-trivial symmetry of a cycle having three fixed vertices. On the other hand we see that ψ swaps the infinities so that we have a contradiction. We conclude that $\mathrm{Fix}(\psi)$ has size at most two. Similarly, one can prove that $\mathrm{Fix}(\tau_\lambda\psi)$ has size at most two.

Thus the set $\mathrm{Fix}(\psi) \cup \mathrm{Fix}(\tau_\lambda\psi)$ has size at most four and then, having $|G| \geq 6$, there is some $g \in G$ such that $\psi(g) \neq g$ and $\psi(g) \neq \lambda g$. This means that $e := [g, \psi(g)]$ is an edge not belonging to I. Now note that e is fixed by ψ in view of (3.5) and hence ψ also fixes the cycle of $\mathcal{H}$ containing e. On the other hand, using (3.2), we see that $\psi(1) = 1$ contradicting the fact that a non-trivial symmetry of a cycle of even length which fixes an edge has no fixed vertex.

The conclusion is that we have $A_0 = \widehat{G}_0$ and hence $A = \widehat{G}$ which is the assertion. ∎

It is interesting to establish which groups, up to isomorphism, are the full automorphism group of a combinatorial design of a given type. Although this problem is often hard to solve in general, it has been settled when the design is, for example, one of the following: a Steiner triple or quadruple system [19]; a non-Hamiltonian 2-factorization of the complete graph [5]; an even cycle system [14]; an odd cycle system [17]. For HCSs of even order we have the following partial answer.

Corollary 3.2. *If G is a symmetrically sequenceable group, then there exists an HCS of even order whose full automorphism group is isomorphic to G.*

Proof. By Corollary 2.2 and Theorem 3.1, it is enough to prove the assertion for groups G of order not greater than 4, namely only for $G = \mathbb{Z}_2$ and $G = \mathbb{Z}_4$ considering that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not symmetrically sequenceable. Consider the following two HCS(10) with vertex-set $\mathbb{Z}_{10}$ and removed 1-factor $I = \{[i, i+5] \mid 0 \leq i \leq 4\}$:

$$\begin{aligned}
\mathcal{H} = \{&(0,1,3,9,7,5,2,4,8,6),\ (0,2,3,6,4,5,9,1,8,7),\\
&(0,3,4,7,6,5,1,2,9,8),\ (0,4,1,7,3,5,8,2,6,9)\};\\
\mathcal{H}' = \{&(0,1,3,7,9,5,4,2,8,6),\ (0,2,9,1,8,5,3,6,4,7),\\
&(0,3,4,1,2,5,7,6,9,8),\ (0,4,8,7,1,5,6,2,3,9)\}.
\end{aligned}$$

One can check that the only non-trivial automorphism of $\mathcal{H}$ is the permutation $(16)(27)(38)(49)$ so that $Aut(\mathcal{H})$ is isomorphic to $\mathbb{Z}_2$. Also, $Aut(\mathcal{H}')$ is generated by the permutation $(1267)(3984)$ and hence it is isomorphic to $\mathbb{Z}_4$. The assertion follows. ∎

In particular, by the mentioned result by Anderson and Ihrig we can claim that for any binary solvable group except Q_8 there exists an HCS of even order whose full automorphism group is G.

4 Enumeration of 2-pyramidal HCSs

Let $\mathcal{H} = \{C_1, \ldots, C_n\}$ be a 1-rotational HCS$(2n+1)$ under G and consider the set of Hamiltonian cycles $\mathcal{H}^* = \{C_1^*, \ldots, C_n^*\}$ where, for $1 \leq i \leq n$, the cycle C_i^* is obtained from C_i by simply moving ∞ between the two vertices at distance n from it. We note that $\mathcal{H}^*$ is again a 1-rotational HCS$(2n+1)$ under G; we call it the *twin* of $\mathcal{H}$. As an example, Figure 3 shows $W(7)$, that is the Walecki 1-rotational HCS(7), and its twin $W(7)^*$.

Using the above terminology, a result on 3-*perfect* HCS$(2n+1)$ recently, and independently, obtained in [13] and [16] can be stated as follows.

Theorem 4.1. *For $n \geq 3$, the twin of the Waleki HCS$(2n+1)$ is 3-perfect.*

For convenience of the reader, we recall that an HCS$(2n+1)$ is said to be i-perfect with $1 \leq i \leq n$ if for every pair of vertices x and y there is exactly one cycle of the HCS in which x and y are at distance i.

The Walecki HCS$(2n+1)$ is not 3-perfect and hence, by Theorem 4.1, it is not isomorphic to its twin. We are going to show that this result holds in general.

Lemma 4.2. *If $\mathcal{H}$ is a 1-rotational HCS$(2n+1)$ with $n \geq 3$, then $\mathcal{H}$ and its twin $\mathcal{H}^*$ are not isomorphic.*

Proof. Assume that $\mathcal{H}$ is isomorphic to its twin $\mathcal{H}^*$. In this case, reasoning as in [13, Theorem 4.3], there is an isomorphism α between $\mathcal{H}$ and $\mathcal{H}^*$ sending ∞ into ∞. Consider the permutation β on $G \cup \{\infty, \overline{\infty}\}$ switching the two infinities and acting as α on G. Given $C = (\infty, g_1, \ldots, g_{2n}) \in \mathcal{H}$, we have $C_+^* = (\overline{\infty}, g_1, \ldots, g_n, \infty, g_{n+1}, \ldots, g_{2n}) \in \mathcal{H}_+^*$ and thus we see that $\beta(C_+^*) = \alpha(C)_+$.

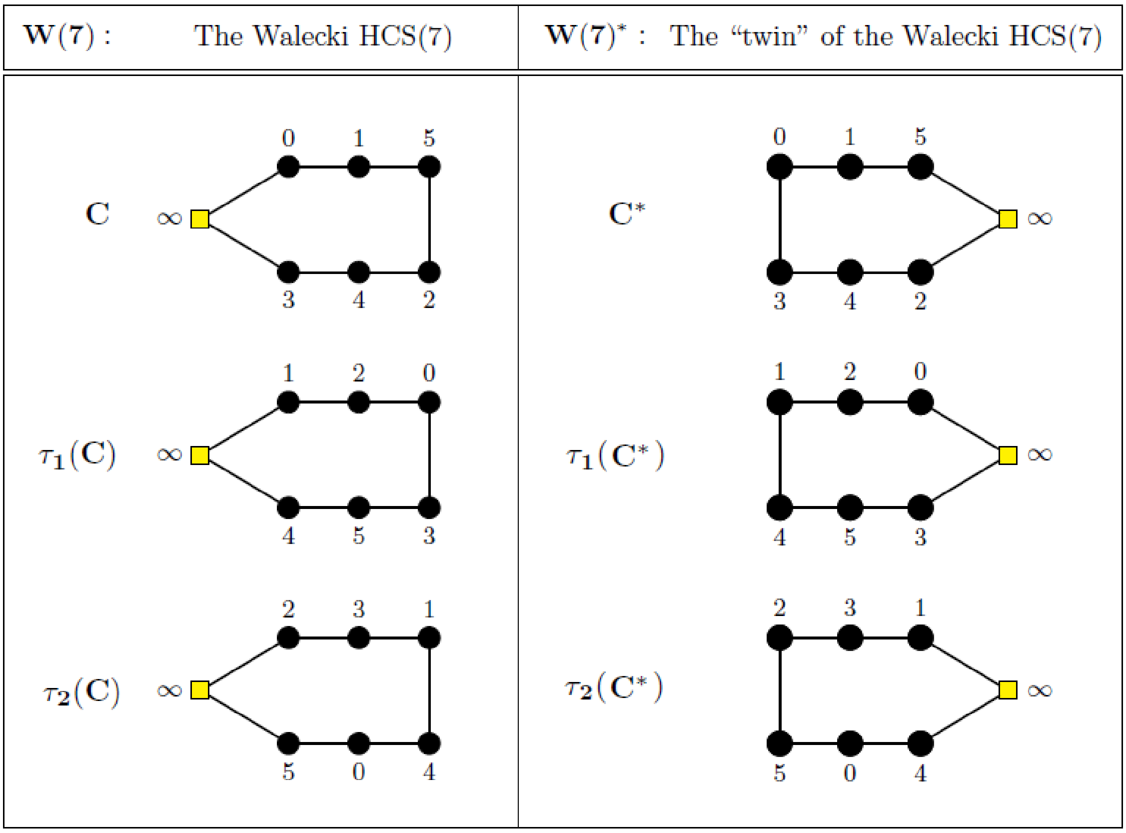

Figure 3:

Thus, considering that $\alpha(C) \in \mathcal{H}^*$, every cycle $C^*_+ \in \mathcal{H}^*_+$ is turned by β into a cycle still belonging to $\mathcal{H}^*_+$, i.e., β is an automorphism of $\mathcal{H}^*_+$. This is a contradiction since, by Theorem 3.1, any automorphism of a 2-pyramidal HCS$(2n+2)$ with $n \geq 3$ fixes both the infinities. ■

Theorem 4.3. *For $n \geq 3$, the number of non-isomorphic 1-rotational HCS$(2n+1)$ is twice the number of non-isomorphic 2-pyramidal HCS$(2n+2)$.*

Proof. For a given Hamiltonian cycle system $\mathcal{H}$, let us denote by $[\mathcal{H}]$ its isomorphism class. Then denote by $\mathbb{H}_{1rot}(2n+1)$ the set of all isomorphism classes of 1-rotational HCS$(2n+1)$ under any group and by $\mathbb{H}_{2pyr}(2n+2)$ the set of all isomorphism classes of 2-pyramidal HCS$(2n+2)$ under any group. The map

$$f : [\mathcal{H}] \in \mathbb{H}_{1rot}(2n+1) \mapsto [\mathcal{H}_+] \in \mathbb{H}_{2pyr}(2n+2)$$

is clearly well defined and it is surjective by Proposition 2.1.

Also note that we have $f([\mathcal{H}]) = f([\mathcal{H}^*])$ for every $[\mathcal{H}] \in \mathbb{H}_{1rot}(2n+1)$. In fact, if $\mathcal{H}$ is a 1-rotational HCS$(2n+1)$ under G, then the transposition $(\infty\ \overline{\infty})$ is an isomorphism between $\mathcal{H}_+$ and $\mathcal{H}^*_+$.

Assume that $\mathcal{H}$ and $\mathcal{H}'$ are 1-rotational HCS$(2n+1)$ (under G and G', respectively) such that $[\mathcal{H}_+] = [\mathcal{H}'_+]$ so that there exists an isomorphism $\alpha : G \cup \{\infty, \overline{\infty}\} \to G' \cup \{\infty, \overline{\infty}\}$ between $\mathcal{H}_+$ and $\mathcal{H}'_+$. In both $\mathcal{H}_+$ and $\mathcal{H}'_+$, $[\infty, \overline{\infty}]$ is the only removed edge which is a diameter of every cycle, so α must fix the 2-set $\{\infty, \overline{\infty}\}$ and hence we can define the map $\beta : G \cup \{\infty\} \to G' \cup \{\infty\}$ by setting $\beta(\infty) = \infty$ and $\beta(g) = \alpha(g)$ for every $g \in G$.

Take a cycle $C = (\infty, g_1, \ldots, g_n, g_{n+1}, \ldots, g_{2n}) \in \mathcal{H}$ and distinguish two cases according to whether α fixes $\{\infty, \overline{\infty}\}$ pointwise or not.

1st case: α fixes $\{\infty, \overline{\infty}\}$ pointwise. We have:

$$\alpha(C_+) = (\infty, \alpha(g_1), \ldots, \alpha(g_n), \overline{\infty}, \alpha(g_{n+1}), \ldots, \alpha(g_{2n})) = \beta(C)_+$$

and hence, considering that $\alpha(C_+) \in \mathcal{H}'_+$, we have $\beta(C) \in \mathcal{H}'$. Thus β turns every cycle C of $\mathcal{H}$ into a cycle of $\mathcal{H}'$, i.e., β is an isomorphism between $\mathcal{H}$ and $\mathcal{H}'$.

2nd case: α swaps ∞ and $\overline{\infty}$. Here we have:

$$\alpha(C_+) = (\overline{\infty}, \alpha(g_1), \ldots, \alpha(g_n), \infty, \alpha(g_{n+1}), \ldots, \alpha(g_{2n})) = \beta(C^*)_+$$

and hence, considering that $\alpha(C_+) \in \mathcal{H}'_+$, we have $\beta(C^*) \in \mathcal{H}'$. Thus β turns every cycle C^* of $\mathcal{H}^*$ into a cycle of $\mathcal{H}'$, i.e., β is an isomorphism between $\mathcal{H}^*$ and $\mathcal{H}'$.

Thus the equality $[\mathcal{H}_+] = [\mathcal{H}'_+]$ implies that $[\mathcal{H}']$ is either $[\mathcal{H}]$ or $[\mathcal{H}^*]$ which are distinct isomorphism classes by Lemma 4.2. We conclude that the pre-image under f of any isomorphism class a of 2-pyramidal HCS$(2n+2)$ always has size two and hence the size of $\mathbb{H}_{1rot}(2n+1)$ is twice the size of $\mathbb{H}_{2pyr}(2n+1)$, that is the assertion. ∎

In [13] the last three authors determined a formula enumerating all 1-rotational HCS$(2n+1)$ up to isomorphism. Even though our formula heavily depends on some hardly computable parameters, it allowed us to claim that for any $n \geq 6$ there are at least $2^{\lceil 3n/4 \rceil}$ non-isomorphic 1-rotational HCS$(2n+1)$. Hence, by Theorem 4.3, we can state the following result.

Theorem 4.4. *If $n \geq 6$, then there exists at least $2^{\lceil 3n/4 \rceil - 1}$ non-isomorphic 2-pyramidal HCS$(2n+2)$.*

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