A Generalization of Rickart Modules

Burcu Ungor Sait Halıcıoglu Abdullah Harmanci

Abstract

Let R be an arbitrary ring with identity and M a right R-module with $S = \mathrm{End}_R(M)$. In this paper we introduce π-Rickart modules as a generalization of Rickart modules. π-Rickart modules are also a dual notion of dual π-Rickart modules and extends that of generalized right principally projective rings to the module theoretic setting. The module M is called *π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $r_M(f^n) = \mathrm{Ker} f^n = eM$. We obtain several results about generalized right principally projective rings by using π-Rickart modules. Moreover, we investigate relations between a π-Rickart module and its endomorphism ring.

1 Introduction

Throughout this paper R denotes an associative ring with identity and modules are unitary right R-modules. For a module M, $S = \mathrm{End}_R(M)$ is the ring of all right R-module endomorphisms of M. In this work, for the (S, R)-bimodule M, $l_S(.)$ and $r_M(.)$ are the left annihilator of a subset of M in S and the right annihilator of a subset of S in M, respectively. A ring is called *reduced* if it has no nonzero nilpotent elements. By considering the right R-module M as an (S, R)-bimodule the reduced ring concept was considered for modules in [1]. The module M is called *reduced* if for any $f \in S$ and $m \in M$, $fm = 0$ implies $fM \cap Sm = 0$. In [10] *Baer rings* are introduced as rings in which the right (equivalently, left) annihilator of every nonempty subset is generated by an idempotent.

Received by the editors in April 2012 - In revised form in September 2013.
Communicated by S. Caenepeel.
2010 *Mathematics Subject Classification* : 13C99, 16D40, 16D80.
Key words and phrases : Rickart module, π-Rickart module, Fitting module, generalized right principally projective ring.

Motivated by Kaplansky's work [10] on Baer rings, principally projective rings were introduced by Hattori [5] to study the torsion theory, that is, a ring is called *left (right) principally projective* if every principal left (right) ideal is projective. This is equivalent to the left (right) annihilator of any element of the ring is generated by an idempotent as a left (right) ideal, i.e., the ring is *left (right) Rickart*. Clearly, every Baer ring is left and right Rickart. The concept of left (right) Rickart rings has been comprehensively studied in the literature. The concept of Baer rings was extended by Rizvi and Roman [17] to the general module theoretic setting, that is, an R-module M is called *Baer* if for any R-submodule N of M, $l_S(N) = Se$ with $e^2 = e \in S$. Also, the notion of Rickart modules initially appeared in Rizvi and Roman [18] and was further studied in [1] and [13]. A module M is said to be *Rickart* if for any $f \in S$, $r_M(f) = \mathrm{Ker}f = eM$ for some $e^2 = e \in S$. Clearly, every Baer module is Rickart. In [14] Lee, Rizvi and Roman introduced a dual notion of Rickart property for modules. A module M is called *dual Rickart* if for any $f \in S$, $\mathrm{Im}f = eM$ for some $e^2 = e \in S$. It is obvious that a ring R is Rickart (dual Rickart) as an R-module if and only if it is a right Rickart (von Neumann regular) ring.

Regarding a generalization of Baer rings as well as principally projective rings, Hirano introduced the notion of generalized left (right) principally projective rings in [6]. A ring R is called *generalized left (right) principally projective* if for each $x \in R$, there exists a positive integer n such that Rx^n (x^nR) is projective, equivalently, for any $x \in R$, the left (right) annihilator of x^n is generated by an idempotent for some positive integer n. There are some sources for additional information on these rings such as [6], [9] and [16]. This class of rings generalizes also π-regular rings. Recently, the present authors studied the notion of dual π-Rickart modules in [20] as a generalization of the notion of dual Rickart modules. On the other hand, the concept of dual π-Rickart modules extended that of π-regular rings to the general module theoretic setting. A module M is called *dual π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $\mathrm{Im}f^n = eM$. In [20], a relation between the concepts of dual π-Rickart property for modules and generalized left principally projectivity for rings was obtained such as if M is a dual π-Rickart module, then S is a generalized left principally projective ring. Motivated by these works, in this paper, we define the dual notion of dual π-Rickart property for modules, namely π-Rickart modules. Additionally, the notion of a π-Rickart module coincides with that of a generalized right principally projective ring. In [9, Proposition 9], it is proved that if R is a generalized right principally projective ring, then so is eRe for any $e = e^2 \in R$. It is a natural question to come to mind that if R is a generalized right principally projective ring and $e = e^2 \in R$, what kind of generalized principally projectivity does the R-module eR have? One of the motivations to study the concept of π-Rickart modules is this question. In Corollary 2.13, we showed that eR is π-Rickart where R is a generalized right principally projective ring and $e = e^2 \in R$ as an answer of the question. If the endomorphism ring S of a module M is π-regular, then it is generalized right principally projective. According to Proposition 3.1 and Proposition 3.9, the π-regularity of S implies that M is a π-Rickart module, and if M is a π-Rickart module, then S is a generalized right principally projective ring. Inserting the π-Rickart property for a module between two properties of its endomorphism ring makes this concept more interesting. In other respects, some of the results in this paper can be applied to the rings of matrices. For instance, in Theorem 3.17, we characterize the matrix ring $M_n(R)$ being π-regular for every positive integer n where R is a right self-injective ring by using π-Rickart property of finitely generated projective modules. Similarly, Proposition 3.1 and Proposition 3.4 can be applied to a ring of matrices which satisfy the generalized right principally projective property. These make the concept of π-Rickart modules more attractive. In addition, this paper helps improve knowledge about generalized right principally projective rings by some results such as Corollary 2.9, Corollary 2.14, Corollary 2.31, Corollary 3.3, Corollary 3.16 and Corollary 3.24.

In what follows, we denote by $\mathbb{Z}$ and $\mathbb{Z}_n$ integers and the ring of integers modulo n, respectively. $J(R)$ and $M_n(R)$ denote the Jacobson radical of a ring R and $n \times n$ matrix ring over R, respectively.

2 π-Rickart Modules

In this section, we introduce the concept of π-Rickart modules and supply an example to show that all π-Rickart modules need not be Rickart. Although every direct summand of a π-Rickart module is π-Rickart, we give an example to show that a direct sum of π-Rickart modules need not be π-Rickart. It is shown that the class of some abelian π-Rickart modules is closed under direct sums. Now we begin with our main definition.

Definition 2.1. Let M be an R-module with $S = \mathrm{End}_R(M)$. The module M is called *π-Rickart* if for any $f \in S$, there exist $e^2 = e \in S$ and a positive integer n such that $r_M(f^n) = eM$.

For the sake of brevity, in the sequel, S will stand for the endomorphism ring of the module M considered.

Remark 2.2. *R is a π-Rickart R-module if and only if it is a generalized right principally projective ring.*

Every module of finite length, every nonsingular injective (or extending) and every Rickart module is a π-Rickart module. Also every quasi-projective strongly co-Hopfian module, every quasi-injective strongly Hopfian module is π-Rickart. Every finitely generated module over a right Artinian ring is π-Rickart (see Theorem 2.30), every free module which its endomorphism ring is generalized right principally projective is π-Rickart (see Corollary 3.5), every finitely generated projective regular module is π-Rickart (see Corollary 3.7) and every finitely generated projective module over a commutative π-regular ring is π-Rickart (see Proposition 3.11).

One may suspect that every π-Rickart module is Rickart. But the following example illustrates that this is not the case.

Example 2.3. Let M denote $\mathbb{Z} \oplus \mathbb{Z}_2$ as a $\mathbb{Z}$-module. It can be easily determined that $S = \mathrm{End}_{\mathbb{Z}}(M)$ is $\begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$. For any $f = \begin{bmatrix} a & 0 \\ \bar{b} & \bar{c} \end{bmatrix} \in S$, consider the following cases.

`Case 1`. Assume that $a = 0$, $\overline{b} = \overline{0}$, $\overline{c} = \overline{1}$ or $a = 0$, $\overline{b} = \overline{c} = \overline{1}$. In both cases f is an idempotent, and so $r_M(f) = (1 - f)M$.

`Case 2`. If $a \neq 0$, $\overline{b} = \overline{0}$, $\overline{c} = \overline{1}$ or $a \neq 0$, $\overline{b} = \overline{c} = \overline{1}$, then $r_M(f) = 0$.

`Case 3`. If $a \neq 0$, $\overline{b} = \overline{c} = \overline{0}$ or $a \neq 0$, $\overline{b} = \overline{1}$, $\overline{c} = \overline{0}$, then $r_M(f) = 0 \oplus \mathbb{Z}_2$.

`Case 4`. If $a = 0$, $\overline{b} = \overline{1}$, $\overline{c} = \overline{0}$, then $f^2 = 0$. Hence $r_M(f^2) = M$.

Therefore M is π-Rickart, but it is not Rickart by [13, Example 2.5].

Our next endeavor is to find conditions under which a π-Rickart module is Rickart. We show that reduced rings play an important role in this direction.

Proposition 2.4. *If M is a Rickart module, then it is π-Rickart. The converse holds if S is a reduced ring.*

Proof. The first assertion is clear. For the second, let $f \in S$. Since M is π-Rickart, $r_M(f^n) = eM$ for some positive integer n and $e^2 = e \in S$. If $n = 1$, then there is nothing to do. Assume that $n > 1$. Since S is a reduced ring, e is central and so $(fe)^n = 0$. It follows that $fe = 0$. Hence $eM \leq r_M(f)$. On the other hand, always $r_M(f) \leq r_M(f^n) = eM$. Therefore M is Rickart. ∎

Reduced modules are studied in [1] and it is shown that if M is a reduced module, then S is a reduced ring. Hence we have the following.

Corollary 2.5. *If M is a reduced module, then it is Rickart if and only if it is π-Rickart.*

We obtain the following well known result (see [9, Lemma 1] and [16, Proposition 1]) as a consequence of Proposition 2.4.

Corollary 2.6. *Let R be a reduced ring. Then R is right Rickart if and only if it is generalized right principally projective.*

Lemma 2.7. *If M is a π-Rickart module, then every non-nil left annihilator in S contains a nonzero idempotent.*

Proof. Let $I = l_S(N)$ be a non-nil left annihilator where $\emptyset \neq N \subseteq M$ and choose $f \in I$ be a non-nilpotent element. Since M is π-Rickart, $r_M(f^n) = eM$ for some idempotent $e \in S$ and a positive integer n. In addition $e \neq 1$. Due to $r_M(I) \subseteq r_M(f^n)$, we have $(1 - e)r_M(I) = 0$. It follows that $1 - e \in l_S(r_M(I)) = l_S(r_M(l_S(N))) = l_S(N) = I$. This completes the proof. ∎

We now give a relation among π-Rickart modules, Rickart modules and Baer modules by using Lemma 2.7.

Theorem 2.8. *Let M be a module. If S has no infinite set of nonzero orthogonal idempotents and $J(S) = 0$ (in particular, if S is semisimple), then the following are equivalent.*

(1) *M is a π-Rickart module.*

(2) *M is a Rickart module.*

(3) *M is a Baer module.*

Proof. It is enough to show that (1) implies (3). Consider any left annihilator $I = l_S(N)$ where $\emptyset \neq N \subseteq M$. If I is nil, then $I \subseteq J(S)$, and so $I = 0$. Thus we may assume that I is not nil. By [11, Proposition 6.59], S satisfies DCC on left direct summands, and so among all nonzero idempotents in I, choose $e \in I$ such that $S(1-e) = l_S(eM)$ is minimal. We claim that $I \cap l_S(eM) = 0$. Note that $I \cap l_S(eM) = l_S(N \cup eM)$. If $I \cap l_S(eM)$ is nil, then there is nothing to do. Now we assume that $I \cap l_S(eM)$ is not nil. If $I \cap l_S(eM) \neq 0$, then there exists $0 \neq f = f^2 \in I \cap l_S(eM)$ by Lemma 2.7. Since $fe = 0$, $e + (1-e)f \in I$ is an idempotent, say $g = e + (1-e)f$. Then $ge = e$, and so $g \neq 0$. Also $fg = f$. This implies that $l_S(gM) \subsetneq l_S(eM)$. This contradicts to the choice of e. Hence $I \cap l_S(eM) = 0$. Due to $\varphi(1-e) \in I \cap l_S(eM)$ for any $\varphi \in I$, we have $\varphi = \varphi e$. Thus $I \subseteq Se$, and clearly $Se = I = l_S(N)$. Therefore M is Baer. ■

Corollary 2.9. *Let R be a ring. If R has no infinite set of nonzero orthogonal idempotents and $J(R) = 0$, then the following are equivalent.*

(1) *R is a generalized right principally projective ring.*

(2) *R is a right Rickart ring.*

(3) *R is a Baer ring.*

Corollary 2.10. *If M is a Noetherian (Artinian) module and $J(S) = 0$, then the following are equivalent.*

(1) *M is a π-Rickart module.*

(2) *M is a Rickart module.*

(3) *M is a Baer module.*

Proof. S has no infinite set of nonzero orthogonal idempotents in case M is either Noetherian or Artinian. The rest is clear from Theorem 2.8. ■

Modules which contain π-Rickart modules need not be π-Rickart, as the following example shows.

Example 2.11. Let R denote the ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ and M the right R-module $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. Let $f \in S$ be defined by $f \begin{bmatrix} x & y \\ r & s \end{bmatrix} = \begin{bmatrix} 2x+3r & 2y+3s \\ 0 & 0 \end{bmatrix}$, where $\begin{bmatrix} x & y \\ r & s \end{bmatrix} \in M$. Then $r_M(f) = \left\{ \begin{bmatrix} 3k & 3z \\ -2k & -2z \end{bmatrix} : k, z \in \mathbb{Z} \right\}$. Since $r_M(f)$ is not a direct summand of M and $r_M(f) = r_M(f^n)$ for any integer $n \geq 2$, M is not a π-Rickart module. On the other hand, consider the submodule $N = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ of M. Then $\text{End}_R(N) = \begin{bmatrix} \mathbb{Z} & 0 \\ 0 & 0 \end{bmatrix}$. It is easy to show that N is a Rickart module and so it is π-Rickart.

In [13, Theorem 2.7], it is shown that every direct summand of a Rickart module is Rickart. We now prove that every direct summand of a π-Rickart module inherits this property.

Theorem 2.12. *Every direct summand of a π-Rickart module is also π-Rickart.*

Proof. Let $M = N \oplus P$ be a module with $S = \mathrm{End}_R(M)$ and $S_N = \mathrm{End}_R(N)$. For any $f \in S_N$, define $g = f \oplus 0_{|_P}$ and so $g \in S$. By hypothesis, there exist a positive integer n and $e^2 = e \in S$ such that $r_M(g^n) = eM$ and $g^n = f^n \oplus 0_{|_P}$. Let $M = eM \oplus Q$. Since $P \subseteq eM$, there exists $L \leq eM$ such that $eM = P \oplus L$. So we have $M = eM \oplus Q = P \oplus L \oplus Q$. Let $\pi_N : M \to N$ be the natural projection of M onto N. Then $\pi_{N|_{Q\oplus L}} : Q \oplus L \to N$ is an isomorphism. Hence $N = \pi_N(Q) \oplus \pi_N(L)$. We claim that $r_N(f^n) = \pi_N(L)$. We get $g^n(L) = 0$ since $g^n(P \oplus L) = 0$. But for all $l \in L$, $l = \pi_N(l) + \pi_P(l)$. Since $g^n(l) = g^n\pi_N(l) + g^n\pi_P(l)$ and $g^n(l) = 0$ and $g^n\pi_P(l) = 0$ and $g^n\pi_N(l) = f^n\pi_N(l)$, we have $f^n\pi_N(L) = 0$ so $\pi_N(L) \subseteq r_N(f^n)$. For the reverse inclusion, let $n \in r_N(f^n)$. Assume that $n \notin \pi_N(L)$ and we reach a contradiction. Then $n = n_1 + n_2$ for some $n_1 \in \pi_N(L)$ and some $0 \neq n_2 \in \pi_N(Q)$ and so there exists $q \in Q$ such that $\pi_N(q) = n_2$. Since $Q \cap r_M(g^n) = 0$, we have $g^n(q) = (f^n \oplus 0_{|_P})(q) \neq 0$. Due to $q = \pi_N(q) + \pi_P(q)$ and $g^n\pi_P(q) = (f^n \oplus 0_{|_P})\pi_P(q) = 0$, we get $f^n(q) = g^n(q) = f^n\pi_N(q) \neq 0$. This implies $n \notin r_N(f^n)$ which is the required contradiction. Hence $r_N(f^n) \leq \pi_N(L)$. Therefore $r_N(f^n) = \pi_N(L)$. ∎

Corollary 2.13. *Let R be a generalized right principally projective ring with any idempotent e of R. Then eR is a π-Rickart module.*

Corollary 2.14. *Let $R_1 \oplus R_2$ be a generalized right principally projective ring with direct sum of the rings R_1 and R_2. Then the rings R_1 and R_2 are also generalized right principally projective.*

We now characterize generalized right principally projective rings in terms of π-Rickart modules.

Theorem 2.15. *Let R be a ring. Then R is generalized right principally projective if and only if every cyclic projective R-module is π-Rickart.*

Proof. The sufficiency is clear. For the necessity, let M be a cyclic projective R-module. Then $M \cong I$ for some direct summand right ideal I of R. By Remark 2.2, R is π-Rickart as an R-module. Also by Theorem 2.12, I is π-Rickart, and so is M. ∎

Proposition 2.16. *Let R be a ring and consider the following conditions.*

(1) *Every free R-module is π-Rickart.*

(2) *Every projective R-module is π-Rickart.*

(3) *Every flat R-module is π-Rickart.*

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). *Also* (2) $\Rightarrow$ (3) *holds for finitely presented modules.*

Proof. (3) ⇒ (2) ⇒ (1) Clear. (1) ⇒ (2) Let M be a projective R-module. Then M is a direct summand of a free R-module F. By (1), F is π-Rickart, and so is M due to Theorem 2.12.

(2) ⇒ (3) is clear from the fact that finitely presented flat modules are projective. ∎

Lemma 2.17. *Let M be a module and $f \in S$. If $r_M(f^n) = eM$ for some central idempotent $e \in S$ and a positive integer n, then $r_M(f^{n+1}) = eM$.*

Proof. It is clear that $r_M(f^n) \leq r_M(f^{n+1})$. For the reverse inclusion, let $m \in r_M(f^{n+1})$. Then $fm \in r_M(f^n) = eM$, and so $fm = efm$. Since e is central, $f^n m = f^{n-1}fm = f^{n-1}efm = f^{n-1}fem = f^n em = 0$. Hence $m \in r_M(f^n)$ and so $r_M(f^{n+1}) \leq r_M(f^n)$. ∎

The next example reveals that a direct sum of π-Rickart modules need not be π-Rickart.

Example 2.18. Let R denote the ring $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ and M the right R-module $\begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. Consider the submodules $N = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ of M. It is easy to check that every nonzero endomorphism of N and K is a monomorphism. Therefore N and K are π-Rickart modules but, as was claimed in Example 2.11, $M = N \oplus K$ is not π-Rickart.

A ring R is called *abelian* if every idempotent is central, that is, $ae = ea$ for any $a, e^2 = e \in R$. A module M is called *abelian* [19] if $fem = efm$ for any $f \in S$, $e^2 = e \in S$, $m \in M$. Note that M is an abelian module if and only if S is an abelian ring. In [9, Proposition 7], it is shown that the class of abelian generalized right principally projective rings is closed under direct sums. We extend this result as follows.

Proposition 2.19. *Let M_1 and M_2 be π-Rickart R-modules. If M_1 and M_2 are abelian and $Hom_R(M_i, M_j) = 0$ for $i \neq j$, then $M_1 \oplus M_2$ is a π-Rickart module.*

Proof. Let $M = M_1 \oplus M_2$, $S_i = \mathrm{End}_R(M_i)$ for $i = 1, 2$ and $S = \mathrm{End}_R(M)$. We may write S as the direct sum $S = S_1 \oplus S_2$ of the rings S_1 and S_2. By this notation S acts on M as $(f_1 \oplus f_2)(m_1 + m_2) = f_1(m_1) + f_2(m_2)$, where $f_1 \oplus f_2 \in S$, $m_1 + m_2 \in M$. Let $f = f_1 \oplus f_2 \in S$. By hypothesis, there exist positive integers n, m and $e_1^2 = e_1 \in S_1$ with $r_{M_1}(f_1^n) = e_1 M_1$ and $e_2^2 = e_2 \in S_2$ with $r_{M_2}(f_2^m) = e_2 M_2$. Without loss of generality, we may assume that $n \leq m$. By Lemma 2.17, we have $r_{M_1}(f_1^n) = r_{M_1}(f_1^m) = e_1 M_1$. Let $e = e_1 \oplus e_2$. Then e is an idempotent in S and $r_M(f^m) = eM$. ∎

Recall that a module M is called *duo* if every submodule of M is fully invariant, i.e., for a submodule N of M, $f(N) \leq N$ for each $f \in S$. Our next aim is to find some conditions under which a fully invariant submodule of a π-Rickart module is also π-Rickart.

Lemma 2.20. *Let M be a module and N a fully invariant submodule of M. If M is π-Rickart and every endomorphism of N can be extended to an endomorphism of M, then N is π-Rickart.*

Proof. Let $S = \text{End}_R(M)$ and $f \in \text{End}_R(N)$. By hypothesis, there exists $g \in S$ such that $g|_N = f$ and M being π-Rickart implies that there exist a positive integer n and an idempotent e of S such that $r_M(g^n) = eM$. Then $r_N(f^n) = N \cap r_M(g^n)$. Since N is fully invariant, we have $r_N(f^n) = eN$, and so $r_N(f^n)$ is a direct summand of N. Therefore N is π-Rickart. ∎

Corollary 2.21. *If R is a generalized right principally projective ring and I is an ideal of R which every endomorphism of I is extended to an endomorphism of R, then I is also generalized right principally projective.*

The following result is an immediate consequence of Lemma 2.20.

Proposition 2.22. *Let M be a quasi-injective module and $E(M)$ denote the injective hull of M. If $E(M)$ is π-Rickart, then so is M.*

Proposition 2.23. *Let M be a quasi-injective duo module. If M is π-Rickart, then every submodule of M is also π-Rickart.*

Proof. Let M be a π-Rickart module and N a submodule of M and $f \in \text{End}_R(N)$. By quasi-injectivity of M, f extends to an endomorphism g of M. Then $r_M(g^n) = eM$ for some positive integer n and $e^2 = e \in S$. Since N is fully invariant under g, the proof follows from Lemma 2.20. ∎

Rizvi and Roman [17] introduced that a module M is *$\mathcal{K}$-nonsingular* if for any $f \in S$, $r_M(f)$ is essential in M implies $f = 0$. They proved that every Rickart module is $\mathcal{K}$-nonsingular. For π-Rickart modules, we now give a generalization of the notion of $\mathcal{K}$-nonsingularity. The module M is called *generalized $\mathcal{K}$-nonsingular*, if $r_M(f)$ is essential in M for any $f \in S$, then f is nilpotent. It is clear that every $\mathcal{K}$-nonsingular module is generalized $\mathcal{K}$-nonsingular. The converse holds if the module is rigid. A ring R is called *π-regular* if for each $a \in R$ there exist a positive integer n and an element x in R such that $a^n = a^n x a^n$.

Lemma 2.24. *Let M be a module. If S is a π-regular ring, then M is generalized $\mathcal{K}$-nonsingular.*

Proof. Let $f \in S$ with $r_M(f)$ essential in M. By hypothesis, there exist a positive integer n and $g \in S$ such that $f^n = f^n g f^n$. Then $g f^n$ is an idempotent of S and so $r_M(f^n)$ is a direct summand of M. Since $r_M(f)$ is essential in M, $r_M(f^n)$ is also essential in M. Hence $r_M(g f^n) = M$ and so $g f^n = 0$. Therefore $f^n g f^n = f^n = 0$. ∎

Proposition 2.25. *Every π-Rickart module is generalized $\mathcal{K}$-nonsingular.*

Proof. Let M be a π-Rickart module and $f \in S$ with $r_M(f)$ essential in M. Then $r_M(f^n) = eM$ for some $e^2 = e \in S$ and a positive integer n. Hence $r_M(f^n)$ is essential in M. Thus $r_M(f^n) = M$ and so $f^n = 0$. ∎

Corollary 2.26. *If R is a generalized right principally projective ring, then it is generalized $\mathcal{K}$-nonsingular as an R-module.*

Our next purpose is to find out the conditions when a π-Rickart module M is torsion-free as an S-module. So we consider the set $T({}_SM) = \{m \in M \mid fm = 0$ for some nonzero $f \in S\}$ of all torsion elements of a module M with respect to S. The subset $T({}_SM)$ of M need not be a submodule of the modules ${}_SM$ and M_R in general. If S is a commutative domain, then $T({}_SM)$ is an (S,R)-submodule of M.

Proposition 2.27. *Let M be a module with a commutative domain S. If M is π-Rickart, then $T({}_SM) = 0$ and every nonzero element of S is a monomorphism.*

Proof. Let $0 \neq f \in S$. Then there exist a positive integer n and $e^2 = e \in S$ such that $r_M(f^n) = eM$. Hence $f^n e = 0$. Since S is a domain, we have $e = 0$ and so $r_M(f^n) = 0$. This implies that $\mathrm{Ker} f = 0$. Thus f is a monomorphism. On the other hand, if $m \in T({}_SM)$ there exists $0 \neq f \in S$ such that $fm = 0$. f being a monomorphism implies $m = 0$, and so $T({}_SM) = 0$. ■

We close this section with the relations among strongly Hopfian modules, Fitting modules and π-Rickart modules. Recall that a module M is called *Hopfian* if every surjective endomorphism of M is an automorphism, while M is called *strongly Hopfian* [7] if for any endomorphism f of M, the ascending chain $\mathrm{Ker} f \subseteq \mathrm{Ker} f^2 \subseteq \cdots \subseteq \mathrm{Ker} f^n \subseteq \cdots$ stabilizes. We now give a relation between abelian and strongly Hopfian modules by using π-Rickart modules as follows. It is obtained from Lemma 2.17 and [7, Proposition 2.5].

Proposition 2.28. *Every abelian π-Rickart module is strongly Hopfian.*

A module M is said to be *a Fitting module* [7] if for any $f \in S$, there exists an integer $n \geq 1$ such that $M = \mathrm{Ker} f^n \oplus \mathrm{Im} f^n$. In this direction we have the following result.

Remark 2.29. *Every Fitting module is π-Rickart.*

The following provides another source of examples for π-Rickart modules.

Theorem 2.30. *Every finitely generated module over a right Artinian ring is π-Rickart.*

Proof. Let R be a right Artinian ring and M a finitely generated R-module. By [2, Proposition 10.18, Proposition 10.19 and Theorem 15.20], M is a Fitting module. Thus Remark 2.29 completes the proof. ■

By applying Theorem 2.30 to the ring itself we obtain the next result.

Corollary 2.31. *Every right Artinian ring is generalized right principally projective.*

3 The Endomorphism Ring of a π-Rickart Module

In this section we study some relations between π-Rickart modules and their endomorphism rings. We prove that endomorphism ring of a π-Rickart module is always generalized right principally projective, the converse holds either the module is flat over its endomorphism ring or it is 1-epiretractable. Also modules whose endomorphism rings are π-regular are characterized.

Proposition 3.1. *If M is a π-Rickart module, then S is a generalized right principally projective ring.*

Proof. If $f \in S$, then $r_M(f^n) = eM$ for some $e^2 = e \in S$ and positive integer n. If $g \in r_S(f^n)$, then $gM \leq r_M(f^n) = eM$. This implies that $g = eg \in eS$, and so $r_S(f^n) \leq eS$. Let $h \in S$. Due to $f^n ehM \leq f^n eM = 0$, we have $f^n eh = 0$. Hence $eS \leq r_S(f^n)$. Therefore $r_S(f^n) = eS$. ■

The next known result (see [9, Proposition 9]) is a consequence of Theorem 2.12 and Proposition 3.1.

Corollary 3.2. *If R is a generalized right principally projective ring, then so is eRe for any $e^2 = e \in R$.*

Corollary 3.3 is an application of Theorem 2.30 and Proposition 3.1 for the matrix rings over Artinian rings.

Corollary 3.3. *Let R be a right Artinian ring. Then $M_n(R)$ is generalized right principally projective for every positive integer n.*

Proof. Clear by Theorem 2.30 since $M_n(R)$ is the endomorphism ring of the R-module R^n for any positive integer n. ■

A module M is called *n-epiretractable* [4] if every n-generated submodule of M is a homomorphic image of M. We now show that 1-epiretractable modules allow us to get the converse of Proposition 3.1.

Proposition 3.4. *Let M be a 1-epiretractable module. Then M is π-Rickart if and only if S is a generalized right principally projective ring.*

Proof. The necessity holds from Proposition 3.1. For the sufficiency, let $f \in S$. Since S is generalized right principally projective, there exist a positive integer n and $e^2 = e \in S$ such that $r_S(f^n) = eS$. Then $f^n e = 0$, and so $eM \leq r_M(f^n)$. In order to show the reverse inclusion, let $0 \neq m \in r_M(f^n)$. The module M being 1-epiretractable implies that there exists $0 \neq g \in S$ with $gM = mR$, and so $m = gm_1$ for some $m_1 \in M$. On the other hand, $f^n gM = f^n mR = 0$, and so $f^n g = 0$. Thus $g \in r_S(f^n) = eS$. It follows $g = eg$. Hence we have $m = gm_1 = egm_1 = em \in eM$. Therefore $r_M(f^n) = eM$. ■

Corollary 3.5. *A free module is π-Rickart if and only if its endomorphism ring is generalized right principally projective.*

A module M is called *regular* (in the sense of Zelmanowitz [24]) if for any $m \in M$ there exists a right R-homomorphism $M \stackrel{\phi}{\to} R$ such that $m = m\phi(m)$. Every cyclic submodule of a regular module is a direct summand, and so it is 1-epiretractable. Then we have the following result.

Corollary 3.6. *Let M be a regular module. Then S is generalized right principally projective if and only if M is π-Rickart.*

Corollary 3.7. *Every finitely generated projective regular module is π-Rickart.*

Proof. Let M be a finitely generated projective regular module. By [22, Theorem 3.6], the endomorphism ring of M is generalized right principally projective. Hence by Corollary 3.6, M is π-Rickart. ■

Let $\mathcal{U}$ be a nonempty set of R-modules. Recall that for a module L, the submodule $\mathrm{Tr}(\mathcal{U}, L) = \sum\{Imh | h \in Hom(U, L), U \in \mathcal{U}\}$ is called the *trace of* $\mathcal{U}$ *in* L. If $\mathcal{U}$ consists of a single module U we simply write $\mathrm{Tr}(U, L)$. The following result shows that the converse of Proposition 3.1 is also true for flat modules over their endomorphism rings. On the other hand, Theorem 3.8 generalizes the result [23, 39.10].

Theorem 3.8. *Let M be a module and $f \in S$. Then we have the following.*

(1) If f^nS is a projective right S-module for some positive integer n, then $Tr(M, r_M(f^n))$ is a direct summand of M.

(2) If M is a flat left S-module and S is a generalized right principally projective ring, then M is π-Rickart as an R-module.

Proof. (1) Assume that f^nS is a projective right S-module for some positive integer n. Then there exists $e^2 = e \in S$ with $r_S(f^n) = eS$. We show $\mathrm{Tr}(M, r_M(f^n)) = eM$. Since $f^neM = 0$, $eM \leq \mathrm{Tr}(M, r_M(f^n))$. Let $g \in \mathrm{Hom}(M, r_M(f^n))$. Hence $gM \leq r_M(f^n)$ or $f^ngM = 0$ or $f^ng = 0$. Thus $g \in r_S(f^n) = eS$ and so $eg = g$. It follows that $gM \leq egM \leq eM$ or $\mathrm{Hom}(M, r_M(f^n))M \leq eM$.

(2) Assume that M is a flat left S-module and S is a generalized right principally projective ring. If $f \in S$, then f^nS is a projective right S-module since $r_S(f^n) = eS$ for some positive integer n and $e^2 = e \in S$. As in the proof of (1), we have $\mathrm{Tr}(M, r_M(f^n)) = eM$. Since M is a flat left S-module and $f^n \in S$, $r_M(f^n)$ is M-generated by [23, 15.9]. Again by [23, 13.5(2)], $\mathrm{Tr}(M, r_M(f^n)) = r_M(f^n)$. Thus $r_M(f^n) = eM$. ■

Recall that a ring R is said to be *von Neumann regular* if for any $a \in R$, there exists $b \in R$ with $a = aba$. For a module M, it is shown that if S is a von Neumann regular ring, then M is a Rickart module (see [13, Theorem 3.17]). We obtain a similar result for π-Rickart modules.

Proposition 3.9. *Let M be a module. If S is a π-regular ring, then M is a π-Rickart module.*

Proof. Let $f \in S$. Since S is π-regular, there exist a positive integer n and an element g in S such that $f^n = f^ngf^n$. Then gf^n is an idempotent of S. Now we show that $r_M(f^n) = (1 - gf^n)M$. For $m \in M$, we have $f^n(1 - gf^n)m = (f^n - f^ngf^n)m = (f^n - f^n)m = 0$. Hence $(1 - gf^n)M \leq r_M(f^n)$. For the other side, if $m \in r_M(f^n)$, then $gf^nm = 0$. This implies that $m = (1 - gf^n)m \in (1 - gf^n)M$. Therefore $r_M(f^n) = (1 - gf^n)M$. ■

Now we recall some known facts that will be needed about π-regular rings.

Lemma 3.10. *Let R be a ring. Then*

(1) *If R is π-regular, then eRe is also π-regular for any $e^2 = e \in R$.*

(2) *If $M_n(R)$ is π-regular for any positive integer n, then so is R.*

(3) *If R is a commutative ring, then R is π-regular if and only if $M_n(R)$ is π-regular for any positive integer n.*

Proof. (1) Let R be a π-regular ring, $e^2 = e \in R$ and $a \in eRe$. Then $a^n = a^n r a^n$ for some positive integer n and $r \in R$. Since $a^n = a^n e = e a^n$, we have $a^n = a^n(ere)a^n$. Therefore eRe is π-regular.

(2) is clear from (1).

(3) Let R be a commutative π-regular ring. By [12, Ex.4.15], every prime ideal of R is maximal, and so every finitely generated R-module is co-Hopfian from [21]. Then for any positive integer n, $M_n(R)$ is π-regular by [3, Theorem 1.1]. The rest is known from (2). ■

Proposition 3.11. *Let R be a commutative π-regular ring. Then every finitely generated projective R-module is π-Rickart.*

Proof. Let M be a finitely generated projective R-module. So the endomorphism ring of M is $eM_n(R)e$ for some positive integer n and an idempotent e in $M_n(R)$. Since R is commutative π-regular, $M_n(R)$ is π-regular, and so is $eM_n(R)e$ by Lemma 3.10. Hence M is π-Rickart by Proposition 3.9. ■

The converse of Proposition 3.9 may not be true in general, as the following example shows.

Example 3.12. Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module. Then it can be easily shown that $\mathbb{Z}$ is a π-Rickart module, but its endomorphism ring is not π-regular.

A ring R is called *strongly π-regular* if for every element a of R there exist a positive integer n (depending on a) and an element x of R such that $a^n = a^{n+1}x$, equivalently, an element y of R such that $a^n = ya^{n+1}$. Due to Armendariz, Fisher and Snider [3], the module M is a Fitting module if and only if S is a strongly π-regular ring. Theorem 3.13 fully characterizes the endomorphism ring of a module with C_2 condition to be π-regular. A module M has *C_2 condition* if any submodule N of M which is isomorphic to a direct summand of M is a direct summand. A ring R is called *right C_2* if the right R-module R has C_2 condition. In [13, Theorem 3.17], it is proven that the module M is Rickart with C_2 condition if and only if S is von Neumann regular. The C_2 condition allows us to show the converse of Proposition 3.9.

Theorem 3.13. *Let M be a module with C_2 condition. Then M is π-Rickart if and only if S is a π-regular ring.*

Proof. The sufficiency holds from Proposition 3.9. For the necessity, let $0 \neq f \in S$. Since M is π-Rickart, $\mathrm{Ker} f^n$ is a direct summand of M for some positive integer n. Let $M = \mathrm{Ker} f^n \oplus N$ for some $N \leq M$. It is clear that $f^n|_N$ is a monomorphism. By the C_2 condition, $f^n N$ is a direct summand of M. On the other hand, there exists $0 \neq g \in S$ such that $gf^n|_N = 1_N$. Hence $(f^n - f^n g f^n)M = (f^n - f^n g f^n)$ $(\mathrm{Ker} f^n \oplus N) = (f^n - f^n g f^n)N = 0$. Thus $f^n = f^n g f^n$, and so S is a π-regular ring. ∎

The following is a consequence of Proposition 3.11 and Theorem 3.13.

Corollary 3.14. *Let R be a commutative right C_2 ring. Then the following are equivalent.*

(1) *R is a π-regular ring.*

(2) *Every finitely generated projective R-module is π-Rickart.*

Since every quasi-injective module has C_2 condition, the next result is obtained immediately.

Corollary 3.15. *Let M be a quasi-injective module. Then M is π-Rickart if and only if S is a π-regular ring.*

It is known that every π-regular ring is generalized right principally projective. The next result shows when the converse of this statement is true.

Corollary 3.16. *Let R be a right C_2 ring. Then R is generalized right principally projective if and only if it is π-regular.*

The following is an application to the ring of matrices which satisfy π-regularity in view of Proposition 3.9 and Theorem 3.13. In Theorem 2.15, we determine the necessary and sufficient condition for every cyclic projective module being π-Rickart. Now we deal with not only cyclic but also finitely generated projective modules.

Theorem 3.17. *Let R be a right self-injective ring. Then the following are equivalent.*

(1) *$M_n(R)$ is π-regular for every positive integer n.*

(2) *Every finitely generated projective R-module is π-Rickart.*

Proof. (1) $\Rightarrow$ (2) Let M be a finitely generated projective R-module. Then $M \cong eR^n$ for some positive integer n and $e^2 = e \in M_n(R)$. Hence S is isomorphic to $eM_n(R)e$. By (1), S is π-regular. Thus M is π-Rickart due to Proposition 3.9.

(2) $\Rightarrow$ (1) $M_n(R)$ can be viewed as the endomorphism ring of a projective module R^n for a positive integer n. By (2), R^n is π-Rickart, and by hypothesis, it is quasi-injective. Then $M_n(R)$ is π-regular by Corollary 3.15. ∎

The proof of Lemma 3.18 may be in the context. We include it as an easy reference.

Lemma 3.18. *Let M be a module. Then S is a π-regular ring if and only if for each $f \in S$, there exists a positive integer n such that $\mathrm{Ker} f^n$ and $\mathrm{Im} f^n$ are direct summands of M.*

Theorem 3.19. *Let M be a π-Rickart module. Then the right singular ideal $Z_r(S)$ of S is nil, moreover, $Z_r(S) \subseteq J(S)$.*

Proof. Let $f \in Z_r(S)$. Since M is π-Rickart, $r_M(f^n) = eM$ for some positive integer n and $e = e^2 \in S$. By Proposition 3.1, $r_S(f^n) = eS$. Since $r_S(f^n)$ is essential in S as a right ideal, $r_S(f^n) = S$. This implies that $f^n = 0$, and so $Z_r(S)$ is nil. On the other hand, for any $g \in S$ and $f \in Z_r(S)$, according to previous discussion, $(fg)^n = 0$ for some positive integer n. Hence $1 - fg$ is invertible. Thus $f \in J(S)$. Therefore $Z_r(S) \subseteq J(S)$. ■

The next known result (see [16, Proposition 2]) can be obtained as a consequence of Theorem 3.19.

Corollary 3.20. *The right singular ideal $Z_r(R_R)$ of a generalized right principally projective ring R is a nil ideal.*

Proposition 3.21. *The following are equivalent for a module M.*

(1) *Each element of S is either a monomorphism or nilpotent.*

(2) *M is an indecomposable π-Rickart module.*

Proof. (1) ⇒ (2) Let $e = e^2 \in S$. If e is nilpotent, then $e = 0$. If e is a monomorphism, then $e(m - em) = 0$ implies $em = m$ for any $m \in M$. Hence $e = 1$, and so M is indecomposable. Also for any $f \in S$, $r_M(f) = 0$ or $r_M(f^n) = M$ for some positive integer n. Therefore M is π-Rickart.

(2) ⇒ (1) Let $f \in S$. Then $r_M(f^n)$ is a direct summand of M for some positive integer n. Since M is indecomposable, we see that $r_M(f^n) = 0$ or $r_M(f^n) = M$. This implies that f is a monomorphism or nilpotent. ■

In [15], a module M is called *morphic* if for every $f \in S$, $M/fM \cong \mathrm{Ker}f$, and a ring R is *right morphic* if every $a \in R$ satisfies $R/aR \cong r_R(a)$. Also in [8], a ring R is said to be *left π-morphic* if for every $a \in R$, there exists a positive integer n such that $R/Ra^n \cong l_R(Ra^n)$. A right π-morphic ring is defined similarly. We end this paper by observing some results about the notion of morphic modules.

Theorem 3.22. *Consider the following conditions for a module M.*

(1) *S is a local ring with nil Jacobson radical.*

(2) *M is an indecomposable π-Rickart module.*

Then (1) ⇒ (2). *If M is a morphic module, then* (2) ⇒ (1).

Proof. (1) ⇒ (2) Clearly, each element of S is either a monomorphism or nilpotent. Then M is indecomposable π-Rickart due to Proposition 3.21.

(2) ⇒ (1) Let $f \in S$. Then $r_M(f^n) = eM$ for some positive integer n and an idempotent e in S. If $e = 0$, then f is a monomorphism. Since M is morphic, f is invertible by [15, Corollary 2]. If $e = 1$, then $f^n = 0$. Hence $1 - f$ is invertible. This implies that S is a local ring. Now let $0 \neq f \in J(S)$. Since f is not invertible, there exists a positive integer n such that $r_M(f^n) = M$. Therefore $J(S)$ is nil. ■

The next result can be obtained from Theorem 3.22 and [8, Lemma 2.11].

Corollary 3.23. *Let M be an indecomposable π-Rickart module. If M is morphic, then S is a left and right π-morphic ring.*

Corollary 3.24. *Consider the following conditions for a ring R.*

(1) *Each element of R is either regular or nilpotent.*

(2) *R is indecomposable generalized right principally projective.*

(3) *R is local with $J(R)$ nil.*

Then (3) $\Rightarrow$ (2) $\Leftrightarrow$ (1). *If R is a morphic ring, then* (2) $\Rightarrow$ (3).

Acknowledgements. The authors would like to thank the referee for several helpful suggestions which contributed to improve the presentation of this paper. The first author thanks the Scientific and Technological Research Council of Turkey (TUBITAK) for the financial support.

References

[1] N. Agayev, S. Halicioglu and A. Harmanci, *On Rickart modules*, Bull. Iran. Math. Soc. 38(2)(2012), 433-445.

[2] F. W. Anderson and K. R. Fuller, Rings and categories of modules, Springer-Verlag, New York, 1992.

[3] E. P. Armendariz, J. W. Fisher and R. L. Snider, *On injective and surjective endomorphisms of finitely generated modules*, Comm. Algebra 6(7)(1978), 659-672.

[4] A. Ghorbani and M. R. Vedadi, *Epi-Retractable modules and some applications*, Bull. Iran. Math. Soc. 35(1)(2009), 155-166.

[5] A. Hattori, *A foundation of the torsion theory over general rings*, Nagoya Math. J. 17(1960), 147-158.

[6] Y. Hirano, *On generalized p.p.-rings*, Math. J. Okayama Univ. 25(1)(1983), 7-11.

[7] A. Hmaimou, A. Kaidi and E. Sanchez Campos, *Generalized fitting modules and rings*, J. Algebra 308(1)(2007), 199-214.

[8] Q. Huang and J. Chen, *π-Morphic rings*, Kyungpook Math. J. 47(2007), 363-372.

[9] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra 167(1)(2002), 37-52.

[10] I. Kaplansky, Rings of operators, W. A. Benjamin Inc. New York-Amsterdam, 1968.

[11] T. Y. Lam, Lectures on modules and rings, Springer-Verlag, New York, 1999.

[12] T. Y. Lam, A first course in noncommutative rings, Springer-Verlag, New York, 2001.

[13] G. Lee, S. T. Rizvi and C. S. Roman, *Rickart modules*, Comm. Algebra 38(11)(2010), 4005-4027.

[14] G. Lee, S. T. Rizvi and C. S. Roman, *Dual Rickart modules*, Comm. Algebra 39(2011), 4036-4058.

[15] W. K. Nicholson and E. Sanchez Campos, *Morphic modules*, Comm. Algebra 33(2005), 2629-2647.

[16] M. Ohori, *On non-commutative generalized p.p. rings*, Math. J. Okayama Univ. 26(1984), 157-167.

[17] S. T. Rizvi and C. S. Roman, *Baer and quasi-Baer modules*, Comm. Algebra 32(2004), 103-123.

[18] S. T. Rizvi and C. S. Roman, *Baer property of modules and applications*, Advances in Ring Theory (2005), 225-241.

[19] J. E. Roos, *Sur les catégories spectrales localement distributives*, C. R. Acad. Sci. Paris Ser. A-B 265(1967), 14-17.

[20] B. Ungor, Y. Kurtulmaz, S. Halicioglu and A. Harmanci, *Dual π-Rickart modules*, Rev. Colombiana Mat. (46)(2)(2012), 167-183.

[21] W. V. Vasconcelos, *On finitely generated flat modules*, Trans. Amer. Math. Soc. 138(1969), 505-512.

[22] R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. 155, 1(1971), 233-256.

[23] R. Wisbauer, Foundations of module and ring theory, Gordon and Breach, Reading, 1991.

[24] J. M. Zelmanowitz, *Regular modules*, Trans. Amer. Math. Soc. 163(1972), 341-355.

Department of Mathematics, Ankara University, Turkey
email:bungor@science.ankara.edu.tr

Department of Mathematics, Ankara University, Turkey
email:halici@ankara.edu.tr

Department of Mathematics, Hacettepe University, Turkey
email:harmanci@hacettepe.edu.tr