On the rotation index of bar billiards and Poncelet's porism

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Abstract

We present some new results on the relations between the rotation index of bar billiards of two nested circles C_R and C_r, of radii R and r and with distance d between their centers, satisfying Poncelet's porism property. The rational indices correspond to closed Poncelet transverses, without or with self-intersections. We derive an interesting series arising from the theory of special functions. This relates the rotation number $\frac{1}{3}$, of a triangle of Poncelet transverses, to a double series involving R, r, and d. We also provide a Steiner-type formula which gives a necessary condition for a bar billiard to be a pentagon with self-intersections and rotation index $\frac{2}{5}$. Finally we show that, close to a pair of circles having Poncelet's porism property for index $\frac{1}{3}$, there exist always circle pairs having indices $\frac{1}{4}$ they and $\frac{1}{6}$; in the case $\frac{1}{4}$ they are even unique.

1 Introduction

A famous and much studied problem from the 19th century, Poncelet's closure theorem, has many facets and applications, see the excellent survey [3]. We want to present some new results on circular versions of it. So let us consider two circles C_R and C_r in the plane, and let the circle C_r of radius r lie inside the second circle C_R of radius $R > r$ in such a way that the distance between their centers is equal to d. From any point on C_R, draw a tangent to C_r and extend it to

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C_R in the opposite direction. From this new intersection point we draw another tangent to C_r, etc. For all tangents, the resulting *Poncelet transverse* will be called a *bar billiard*, since it models the well known traditional game (see, e.g., [20]). Usually, players of this game score points by knocking balls into the holes while avoiding toppling a skittle in the middle of the table. Here the role of the skittle is played by the circle C_r, and it must be "toppled" by the tangent line. The general behavior of such billiards is very interesting. In this paper we will concentrate on the so-called Poncelet porism for this specific setting. Namely, we know that such a bar billiard has the *Poncelet porism property*; that is, if there is a starting point for which a Poncelet transverse is closed, then the transverse will also close for any other starting point on the circle. One can find an extensive bibliography on the Poncelet porism property in [3], and we point out also interesting papers by B. Mirman (cf. [12] and [13]). For triangular Poncelet transverses, the *closure theorem of Poncelet* has a long history also in Elementary Geometry; see [11, § 2.4], [1, ch. 16], and [9]; further basic references are [2], [9], [14], [19], and [8]. The interested reader should also consult the recent paper by Cima, Gasull and Manosa [5].

Note that in our setting we allow self-intersections of the Poncelet transverses, like also other authors do (see, e.g., [10] and [7]). It seems that the problem of finding relations between the radii R and r and the distance d, similar to the investigations in [6], was only considered by Radić in [16] and [17]. But he did not give any explicit formula, even in simplest cases, and it seems to be hard to do this by using his method. Here we will formulate the problem and apply the method used in [4] to give an explicit formula for a pentagon with self-intersections. Our considerations give hope to derive some interesting and new series expressing fractions of the form $\frac{n\pi}{m}$, where $\frac{n}{m} \in]0, \frac{1}{2}[\cup]\frac{1}{2}, 1]$.

It is well-known that such a bar billiard is a homeomorphism of a circle, and everything is known about the dynamics of this homeomorphism (see, e.g., [18]). In particular, we know that such a bar billiard is conjugated to either a rational rotation or an irrational rotation. The rational cases correspond to Poncelet's porism, and any rational number, except for $\frac{1}{2}$, is related to a closed polygon, either without or with self-intersections. In the present paper we produce a series relating the rotation number $\frac{1}{3}$ given by a triangle with a double series involving R, r, and d. Similar formulas can be produced for all rational numbers from the interval $]0, 1[$, except for $\frac{1}{2}$, and seem to be new and interesting. The fractions $\frac{1}{n}$ and $\frac{n-1}{n}$ correspond to the polygons considered by Poncelet and his followers, the other fractions $\frac{p}{q}$, where $p \neq 1, n-1$ and the numbers p and q are coprime, come from suitable polygons with self-intersections. As an example, we find a relation between R, r, d to characterize a pentagon with self-intersections having corresponding rational rotation index equal to $\frac{2}{5}$. By using this formula, one could produce a suitable series absolutely converging to $\frac{2}{5}$ and being similar to the one given for $\frac{1}{3}$ (this is not presented in our paper). It seems to be possible to give a relationship between R, r, and d for obtaining an irrational rotation index and, then, to produce a similar series converging to this number.

2 The homeomorphism of a circle

Let C_R and C_r be two nested circles with centers $(0,0)$ and $(d,0)$ as well as radii R and r, respectively, such that $d > 0$ and $d + r < R$. For the circle C_R we consider a positive natural parametrization given by $z(t) = (R\cos t, R\sin t)$, where $t \in \mathbb{R}$. From any point $(R\cos t, R\sin t) = Re^{it}$ on C_R, draw the first tangent to C_r with respect to the parameter t and extend it to C_R in the opposite direction to obtain the point $(R\cos\varphi(t), R\sin\varphi(t)) = Re^{i\varphi(t)}$, where

$$Re^{i\varphi(t)} = h(t)e^{it} + g(t)ie^{it} \tag{2.1}$$

with

$$\begin{aligned}
h(t) \;&=\; -\frac{R}{\left(d^2+R^2-2dR\cos t\right)^2}\Big(R^2\left(3d^2-2r^2+R^2\right)+\\
&+\; d^2\left(d^2-2r^2+3R^2\right)\cos 2t - 2dR\left(d^2-2r^2+2R^2+d^2\cos 2t\right)\cos t+\\
&+\; 4dr(R-d\cos t)\sqrt{d^2-r^2+R^2-2dR\cos t}\,\sin t\Big)
\end{aligned}$$

and

$$\begin{aligned}
g(t) \;&=\; \frac{2R}{\left(d^2+R^2-2dR\cos t\right)^2}\Big(rR^2\sqrt{d^2-r^2+R^2-2dR\cos t}+\\
&+\; d\Big(dr\sqrt{d^2-r^2+R^2-2dR\cos t}\cos 2t-\\
&-\; R\left(2d^2-2r^2+R^2+d^2\cos 2t\right)\sin t+\\
&+\; \left(-2rR\sqrt{d^2-r^2+R^2-2dR\cos t}+d\left(d^2-2r^2+3R^2\right)\sin t\right)\cos t\Big)\Big).
\end{aligned}$$

Now we take $Re^{i\varphi(t)}$ as starting point of the next tangent, which ends at $Re^{i\varphi(\varphi(t))}$, and so on, as it is illustrated in Fig. 1. The above construction gives a homeo-

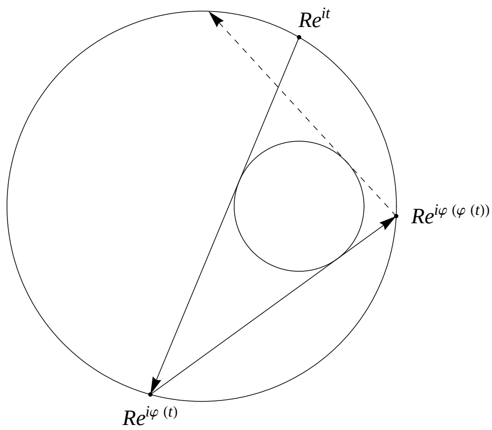

Figure 1: Construction of a bar billiard

morphism Φ of the circle C_R which has a natural lift $\varphi\colon \mathbb{R} \to \mathbb{R}$. The case of the second possible tangent to the circle C_r will be discussed at the end of this section.

Having differentiated the equation (2.1), we get

$$\varphi'(t) = \frac{(R^2 - d^2)\sqrt{d^2 - r^2 + R^2 - 2dR\cos t} + 2dr\sin t}{(d^2 + R^2 - 2dR\cos t)\sqrt{d^2 - r^2 + R^2 - 2dR\cos t}}, \tag{2.2}$$

with

$$\cos\varphi(0) = \frac{2r^2 + 2dR - R^2 - d^2}{(d-R)^2}, \tag{2.3}$$

$$\sin\varphi(0) = \frac{2r\sqrt{-r^2 + d^2 - 2dR + R^2}}{(d-R)^2} \tag{2.4}$$

for $0 \le \varphi(0) < \pi$, since $\varphi(0)$ cannot exceed π by geometric reasons. Hence

$$\begin{aligned} \varphi(t) &= \int\limits_0^t \frac{(R^2 - d^2)\sqrt{d^2 - r^2 + R^2 - 2dR\cos\tau} + 2dr\sin\tau}{(d^2 + R^2 - 2dR\cos\tau)\sqrt{d^2 - r^2 + R^2 - 2dR\cos\tau}}\, d\tau \\ &+ \arccos\left(\frac{2r^2}{(d-R)^2} - 1\right). \end{aligned} \tag{2.5}$$

Thus, after some calculations we get

$$\begin{aligned} \varphi(t) &= \arccos\left(-1 + \frac{2r^2}{(d-R)^2}\right) - 2\arctan\left(\frac{\sqrt{d^2 - r^2 - 2dR + R^2}}{r}\right) + \\ &+ 2\arctan\left(\frac{\sqrt{d^2 - r^2 + R^2 - 2dR\cos t}}{r}\right) + 2\arctan\left(\frac{(d-R)\cot\frac{t}{2}}{d+R}\right) + \\ &+ 2\pi\left\lfloor\frac{t}{2\pi}\right\rfloor + \pi\operatorname{sgn} t \end{aligned} \tag{2.6}$$

with

$$\varphi(0) = \arccos\left(\frac{2r^2}{(d-R)^2} - 1\right) \tag{2.7}$$

and $\varphi(t + 2\pi) = \varphi(t) + 2\pi$.

At this moment we can comment the existence of an invariant measure on C_R for the homeomorphism φ. Let us denote the intervals AB and CD as in Fig. 2. Note that $|AB| = \sqrt{d^2 - r^2 + R^2 - 2dR\cos t}$ and produce a 1-form $\omega(t) = f(t)dt$ on C_R, where

$$f(t) = \frac{1}{|AB|} = \frac{1}{\sqrt{d^2 - r^2 + R^2 - 2dR\cos t}}. \tag{2.8}$$

Then, simply, we have reproved a theorem given in [10] and [7].

Corollary 2.1. *The measure ω for the bar billiard considered above is invariant in both cases, namely with and without self-intersections.*

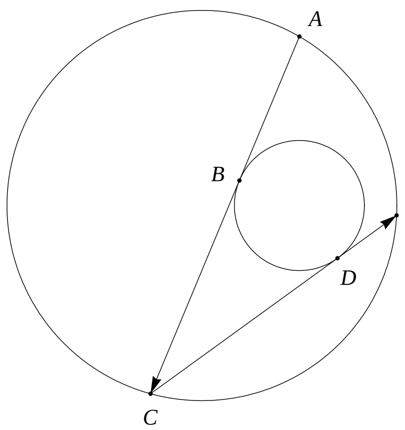

Figure 2: Construction of an invariant measure

Proof. We have to show that $\varphi^*\omega = \omega$, or that $f(\varphi(t))\varphi'(t) = f(t)$ for $t \in \mathbb{R}$. We have that $f(\varphi(t)) = \frac{1}{|CD|}$, and thus we have to show the equality

$$\varphi'(t) = \frac{|CD|}{|AB|}. \tag{2.9}$$

From the expression of $|AB|$ and formula (2.1) we get

$$|CD| = \sqrt{d^2 - r^2 + R^2 - 2d(A(t)\cos t - B(t)\sin t)}.$$

Hence, using the formula (2.2) together with the above expressions for $|AB|$ and $|CD|$, we get after some calculations that the claimed formula follows. ∎

Recall that the rotation index (see, for instance, [18]) of a homeomorphism φ of the circle is given by a number

$$\varrho_0(\varphi, t) = \frac{1}{2\pi} \lim_{n\to\infty} \frac{\varphi^n(t) - t}{n}. \tag{2.10}$$

It is well-known that the above limit exists and is not depending on t. Therefore this limit is usually denoted by $\varrho_0(\varphi)$. Since the index does not depend on t, we have that if $\varphi^n(t_0) = t_0 + 2\pi$ for some particular t_0, then $\varphi^n(t) = t + 2\pi$ for all $t \in \mathbb{R}$, and so we have the *Poncelet porism* in this setting. If $\varphi^n(t) = t + 2\pi$, then we have an n-gon inscribed to C_R, circumscribed about C_r and having no self-intersections with

$$\varrho_0(\varphi) = \frac{1}{n}. \tag{2.11}$$

Note that if $\varphi^n(t) = t + 2\pi$, then the inverse homeomorphism $\varphi^{-1}(t)$ has the index $\frac{n-1}{n}$ since it corresponds to $\varphi^{n-1}(t)$ and to the choice of the second tangent to the smaller circle at the beginning of our construction. Moreover, it has the same graph, but with opposite orientation with respect to the starting homeomorphism $\varphi(t)$. In general, $\varrho_0(\varphi^k) = k\varrho_0(\varphi) = \frac{k}{n}$ for $k = 1, 2, \ldots, n-1$; then for the homeomorphism φ, with $\varphi^n(t) = t + 2\pi$, we have associated closed polygons corresponding to the consecutive iterations $\varphi^2(t), \varphi^3(t), \ldots, \varphi^{n-1}(t)$. If the reduced form of the index $\frac{k}{n}$ is $\frac{k_0}{n_0}$, then $\varphi^k(t)$ determines a closed polygon without self-intersections if the numerator k_0 is either 1 or $n_0 - 1$, and the remaining cases give the n_0-gons with self-intersections. If the index $\frac{a}{b}$ is in reduced form and $a < \frac{b}{2}$, then $\frac{b-a}{b}$ is given by the second tangent to the circle C_r, and both resulting polygons coincide but have opposite orientations. In general, if $\varphi^n(t) = t + 2k\pi$ and the natural numbers n and k are relatively prime, then $\varrho_0(\varphi) = \frac{k}{n}$.

In Section 4 we find the relation between r, R, and d in order to have a pentagon with self-intersections satisfying the Poncelet porism property and having the rotation index $\frac{2}{5}$. It seems that up to now nobody investigated such polygons, and also the analysis of relations similar to those given in [6], [3], and [4] would be interesting.

3 A series related to a homeomorphism with rotation index $\frac{1}{3}$

As it was proved in [10], the 1-form ω is a measure on C_R for which a measure of the circular arc from $(R, 0)$ to Re^{it} is given by the formula

$$\gamma = \frac{2\pi \int\limits_0^t \omega(t)}{\int\limits_0^{2\pi} \omega(t)}. \tag{3.1}$$

Thus, the measure of the whole circle C_R is equal to 2π, and the rotation index of the homeomorphism $\varphi(t)$ described in Section 2 is given by

$$\varrho_0(\varphi) = \frac{\int\limits_0^{\varphi(0)} \omega(t)}{\int\limits_0^{2\pi} \omega(t)}. \tag{3.2}$$

In this section we consider the case $\varrho_0(\varphi) = \frac{1}{3}$ as a typical one and model the situation. An analogous study can be carried out for any other fraction different to $\frac{1}{2}$, under the condition that one knows the relation between r, R, and d which gives the suitable polygon. It is well-known (see, for example, [6] and [21]) that, in order to have a closed cycle which is a triangle and which gives a Poncelet porism for C_R and C_r, we should choose r, R, and d suitably to satisfy the following relation. It is called *Steiner relation* for a bicentric triangle:

$$R^2 - 2Rr - d^2 = 0. \tag{3.3}$$

However, we begin with some general manipulations of the formula (3.2).

Starting with its denominator, and after a few substitutions, we get

$$\int_0^{2\pi} \omega(t)dt = \int_0^{2\pi} \frac{dt}{\sqrt{d^2 - r^2 + R^2 - 2dR\cos t}} =$$

$$\frac{4}{\sqrt{(d+R-r)(d+R+r)}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}},$$

where

$$k^2 = \frac{4dR}{(d+R-r)(d+R+r)}. \tag{3.4}$$

Note that under our geometric assumptions $d > 0$, $d + r < R$, and $0 < r < R$ we have that $0 < k^2 < 1$, and so the last formula is the complete elliptic integral of first type.

Doing similarly for the numerator, we obtain

$$\int_0^{\varphi(0)} \omega(t)dt = \int_0^{\varphi(0)} \frac{dt}{\sqrt{d^2 - r^2 + R^2 - 2dR\cos t}} = \frac{2}{\sqrt{(d+R-r)(d+R+r)}}\cdot$$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}} - \frac{2}{\sqrt{(d+R-r)(d+R+r)}} \int_0^{\frac{\pi}{2}-\frac{\varphi(0)}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}}.$$

Finally, we get

$$\varrho_0(\varphi) = \frac{1}{2} - \frac{1}{2}\frac{\int_0^{\frac{\pi}{2}-\frac{\varphi(0)}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}}}{\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}}}, \tag{3.5}$$

where $\varphi(0) = \arccos\left(\frac{2r^2}{(d-R)^2} - 1\right)$.

Now we are going to use some special functions and related series, to obtain finally our formula. In this context we have

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, k^2\right) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n}. \tag{3.6}$$

In order to consider the second integral, we write $\varphi_0 = \frac{\pi}{2} - \frac{\varphi(0)}{2}$ and perform a few natural substitutions, and so we get

$$\int_0^{\varphi_0} \frac{dx}{\sqrt{1-k^2\sin^2 x}} = \frac{\sin\varphi_0}{2}\int_0^1 \frac{1}{\sqrt{1-k^2\sin^2\varphi_0 t}} \cdot \frac{1}{\sqrt{1-\sin^2\varphi_0 t}} \cdot \frac{1}{\sqrt{t}}dt\,, \tag{3.7}$$

where one of the integrands can be written by means of series as follows:

$$\frac{1}{\sqrt{1-\sin^2\varphi_0 t}} = \sum_{k=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} t^k \sin^{2k}\varphi_0. \tag{3.8}$$

Hence

$$\int_0^{\varphi_0} \frac{dx}{\sqrt{1-k^2\sin^2 x}} =$$

$$= \frac{\sin\varphi_0}{2} \sum_{m=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \sin^{2m}\varphi_0 \int_0^1 (1-k^2\sin^2\varphi_0 t)^{-\frac{1}{2}} \cdot t^{-\frac{1}{2}+m} \cdot (1-t)^0 dt =$$

$$= \frac{\sin\varphi_0}{2} \sum_{m=0}^{\infty} \frac{(2m-1)!!}{(2m)!!} \sin^{2m}\varphi_0 B\left(\frac{1}{2}+m, 1\right) F\left(\frac{1}{2}, \frac{1}{2}+m, \frac{3}{2}+m, k^2\sin^2\varphi_0\right) =$$

$$= \sin\varphi_0 \sum_{m=0}^{\infty} \left(\frac{(2m-1)!!}{(2m)!!} \sin^{2m}\varphi_0 \cdot \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!(2m+2n+1)} k^{2n} \sin^{2n}\varphi_0 \right). \tag{3.9}$$

Thus, we have proved

Theorem 3.1. *For any bar billiard φ we have the following formula for the rotation index:*

$$\varrho_0(\varphi) = \frac{1}{2} - \frac{\sin\varphi_0 \sum\limits_{m=0}^{\infty} \left(\frac{(2m-1)!!}{(2m)!!} \sin^{2m}\varphi_0 \cdot \sum\limits_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!(2m+2n+1)} k^{2n}\sin^{2n}\varphi_0 \right)}{\pi \sum\limits_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n}}, \tag{3.10}$$

where

$$\varphi_0 = \frac{\pi}{2} - \frac{\arccos\left(\frac{2r^2}{(d-R)^2} - 1\right)}{2}. \tag{3.11}$$

Let us go back to the special case of the bicentric triangle, for which we have the relation

$$r = \frac{R^2 - d^2}{2R}. \tag{3.12}$$

This relation we substitute into formula (3.10), in order to obtain some interesting formula for $\frac{\pi}{3}$.

First we have

$$\varphi_0 = \frac{\pi}{2} - \frac{\arccos\left(\frac{-1+2t+t^2}{2}\right)}{2}. \tag{3.13}$$

We substitute also $t = \frac{d}{R}$ and get immediately

$$\sin\varphi_0 = \frac{1}{2}(1+t). \tag{3.14}$$

Similarly substituting formula (3.12) into (3.4), we get

$$k^2 = \frac{-16t}{(-3+t)(1+t)^3}. \tag{3.15}$$

Finally, using the above form in formula (3.10), we come to the result

$$\frac{1}{3} = \frac{1}{2} - \frac{\frac{1}{2}(1+t)\sum\limits_{m=0}^{\infty}\left(\frac{(2m-1)!!}{(2m)!!}\cdot\left(\frac{1+t}{2}\right)^{2m}\cdot\sum\limits_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!(2m+2n+1)}\left(\frac{4t}{(3-t)(1+t)}\right)^{n}\right)}{\pi\sum\limits_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^{2}\cdot\left(\frac{16t}{(3-t)(1+t)^{3}}\right)^{n}}. \tag{3.16}$$

Thus, we are able to formulate one of the two main results of our paper.

Corollary 3.1. *For any $t \in (0,1)$ we have*

$$\frac{\pi}{3} = \frac{(1+t)\sum\limits_{m=0}^{\infty}\frac{(2m-1)!!}{(2m)!!}\cdot\left(\frac{1+t}{2}\right)^{2m}\cdot\sum\limits_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!(2m+2n+1)}\left(\frac{4t}{(3-t)(1+t)}\right)^{n}}{\sum\limits_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^{2}\cdot\left(\frac{16t}{(3-t)(1+t)^{3}}\right)^{n}}. \tag{3.17}$$

Note that, similarly, we could obtain adequate formulae for any other closed bicentric polygon. It would be interesting to give similar formulae for non-closed Poncelet transverses.

4 Relations for Poncelet's porism with self-intersections in a special case

In this section we are going to show how to construct a pentagon with self-intersections, yielding the rotation index equal to $\frac{2}{5}$ and satisfying the Poncelet porism property. A method and the formulae below are taken from [4] and can be extended to any desired bicentric polygon with self-intersections.

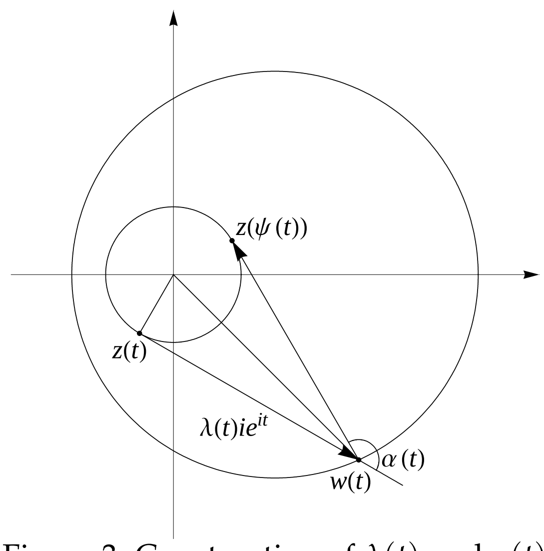

Figure 3: Construction of $\lambda(t)$ and $\alpha(t)$.

We begin with the parametrization of both the circles C_R and C_r. Now the center of the smaller circle is at $(0,0)$, and the larger one has its center at $(d,0)$, where again $d > 0$ and $d + r < R$. Let the circle C_r be parametrized by the equation $z(t) = re^{it}$, while the larger circle is parametrized by $w(t) = re^{it} + \lambda(t)ie^{it}$, where $t \in \mathbb{R}$. Thus

$$\lambda(t) = \sqrt{R^2 - (d\cos t - r)^2} - d\sin t. \tag{4.1}$$

Then, using the notation from Figure 3, we get

$$\tan\frac{\alpha(t)}{2} = \frac{\lambda(t)}{r}. \tag{4.2}$$

Then the counterpart of the homeomorphism $\varphi(t)$ in this framework is determined by

$$\psi(t) = t + \alpha(t). \tag{4.3}$$

Let $\psi^0(t) = t$, $\psi^n(t) = \psi(\psi^{n-1}(t))$ for $n \in \mathbb{N}$. From formula (4.2) we get

$$e^{i\alpha(t)} = \frac{r^2 - \lambda^2(t)}{r^2 + \lambda^2(t)} + i\frac{2\lambda(t)r}{r^2 + \lambda^2(t)}, \tag{4.4}$$

and hence

$$e^{i\psi(t)} = \left(\frac{r + i\lambda(t)}{|r + i\lambda(t)|}\right)^2 e^{it}. \tag{4.5}$$

If for any $n \in \mathbb{N} \cup \{0\}$ we write

$$\mu_n(t) = \frac{r + i\lambda(\psi^n(t))}{|r + i\lambda(\psi^n(t))|}, \tag{4.6}$$

then we get

$$e^{i\psi^n(t)} = (\mu_0(t)\mu_1(t)\ldots\mu_{n-1}(t))^2 e^{it}. \tag{4.7}$$

Of course, the polygon corresponding to $\psi(t)$ satisfies the Poncelet porism property iff for some fraction in reduced form $\frac{k}{n}$, where $k, n \in \mathbb{N}$, $n \geq 3$, $k < n$, we have

$$\psi^n(t) = t + 2k\pi.$$

It follows (cf. Theorem 2.1 in [4]) that if the resulting polygon either without or with self-intersection is closed, then

$$(\mu_0(t)\mu_1(t)\ldots\mu_{n-1}(t))^2 = 1, \tag{4.8}$$

and hence we have

$$\mathrm{Im}(\mu_0(t)\mu_1(t)\ldots\mu_{n-1}(t)) = 0. \tag{4.9}$$

In what follows, we derive a formula relating the radii R and r and the distance d in the considered pair of nested circles, with a pentagonal bar billiard having self-intersections for which $\varrho_0(\psi) = \frac{2}{5}$.

Let $\lambda_n = \lambda(\psi^n(0))$. Then, using the notations from Figure 4, we get

$$|BC| = \lambda_0, |DE| = \lambda_1, |FG| = \lambda_2, |HI| = \lambda_3, |JA| = \lambda_4.$$

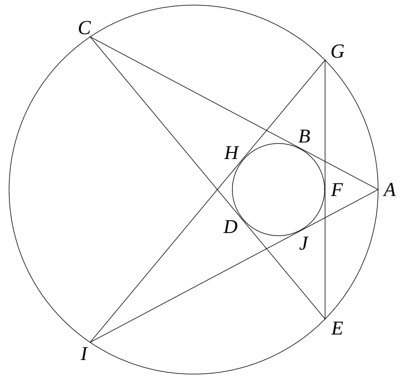

Figure 4: Notations to be used in the proof of the formula relating R, r, and d

Again from this figure and simple geometric observations we obtain that $\lambda_0 = \lambda_3$ and $\lambda_1 = \lambda_2$. Moreover, we get

$$\lambda_1 = \lambda_2 = \sqrt{R^2 - (d+r)^2}, \tag{4.10}$$

$$\lambda_4 = \sqrt{(R-d)^2 - r^2}, \tag{4.11}$$

$$\lambda_0 = \lambda_3 = \frac{(d+R)\sqrt{(R-d)^2 - r^2}}{R-d}. \tag{4.12}$$

From formula (4.9) we have

$$0 = \operatorname{Im}(\mu_0(0)\mu_1(0)\mu_2(0)\mu_3(0)\mu_4(0)), \tag{4.13}$$

and hence the imaginary part of the expression

$$\left(r + i\sqrt{R^2 - (d+r)^2}\right)^2 \left(r + i\sqrt{(R-d)^2 - r^2}\right) \left(r + i\frac{(d+R)\sqrt{(R-d)^2 - r^2}}{R-d}\right)^2 \tag{4.14}$$

is equal to 0, giving the required formula

$$\begin{aligned}
&2d^4r^2\sqrt{R^2-(d+r)^2} - 8d^3r^2R\sqrt{R^2-(d+r)^2} + 4d^2r^2R^2\sqrt{R^2-(d+r)^2} \\
&+ 8r^4R^2\sqrt{R^2-(d+r)^2} + 8dr^2R^3\sqrt{R^2-(d+r)^2} - 6r^2R^4\sqrt{R^2-(d+r)^2} \\
&- d^6\sqrt{(R-d)^2-r^2} - 2d^5r\sqrt{(R-d)^2-r^2} - 2d^4r^2\sqrt{(R-d)^2-r^2} \\
&+ 3d^4R^2\sqrt{(R-d)^2-r^2} + 4d^3rR^2\sqrt{(R-d)^2-r^2} + 8d^2r^2R^2\sqrt{(R-d)^2-r^2} \\
&+ 8dr^3R^2\sqrt{(R-d)^2-r^2} + 8r^4R^2\sqrt{(R-d)^2-r^2} - 3d^2R^4\sqrt{(R-d)^2-r^2} \\
&- 2drR^4\sqrt{(R-d)^2-r^2} - 6r^2R^4\sqrt{(R-d)^2-r^2} + R^6\sqrt{(R-d)^2-r^2} = 0.
\end{aligned} \tag{4.15}$$

Thus, this is a kind of Steiner formula giving the relation between the radii R and r and the offset d, which is a necessary condition for a pentagonal bar billiard with self-intersections for which $\varrho_0(\psi) = \frac{2}{5}$.

5 From the index $\frac{1}{3}$ to the indices $\frac{1}{4}$ and $\frac{1}{6}$

In this final section we are going to show that, “close” to a pair of circles having Poncelet’s porism property for index $\frac{1}{3}$, there exist pairs of circles having indices $\frac{1}{4}$ and $\frac{1}{6}$, in the first case even unique ones. This is conjectured to be true for arbitrary indices; see the final conjecture.

We consider a circular annulus C_rC_R formed by two circles C_r and C_R. The circles C_r, C_R are given by the equations

$$x^2 + y^2 = r^2, \qquad (x-d)^2 + y^2 = R^2, \tag{5.1}$$

respectively, where

$$0 < d < R - r. \tag{5.2}$$

Recall a suitable form of Poncelet’s closure theorem (see [3]) used in the forthcoming considerations:

If there exists one circuminscribed (i.e., simultaneously inscribed in the outer circle and circumscribed about the inner circle) n-gon in a circular annulus, then any point of the outer circle is the vertex of some circuminscribed n-gon.

If Poncelet’s closure property holds for $n = 3$ (that is, if the rotation index of the associated homeomorphism is equal to $\frac{1}{3}$ in the annulus C_rC_R), then condition (3.3) is satisfied. For our purpose it is convenient to denote this condition by $\mathrm{Pct}(C_rC_R, 3)$. Similarly, the conditions $\mathrm{Pct}(C_rC_R, 4)$ and $\mathrm{Pct}(C_rC_R, 6)$ have the forms

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2), \tag{5.3}$$

and

$$3(R^2 - d^2)^4 = 4r^2(R^2 + d^2)(R^2 - d^2)^2 + 16r^4d^2R^2, \tag{5.4}$$

respectively (cf. [3]).

It is easy to see that for each fixed $\lambda \in [0, 1]$ the circle $\mathcal{C}_\lambda$ given by the equation

$$(x - \lambda d)^2 + y^2 = [(1-\lambda)r + \lambda R]^2 \tag{5.5}$$

lies in the annulus C_rC_R, and that $\mathcal{C}_0 = C_r$, $\mathcal{C}_1 = C_R$.

In this framework we prove two theorems which show how to pass from the rotation index $\frac{1}{3}$ to $\frac{1}{4}$ and from $\frac{1}{3}$ to $\frac{1}{6}$ in a natural way.

Theorem 5.1. *If the condition* $\mathrm{Pct}(C_rC_R, 3)$ *is satisfied, then there exists a unique* $\lambda \in]0, 1[$ *such that the condition* $\mathrm{Pct}(C_r\mathcal{C}_\lambda, 4)$ *is satisfied.*

Proof. Taking the pattern from formula (5.5), we substitute d and R in (5.3) by the offset λd and the radius $(1-\lambda)r + \lambda R$, respectively. We get then

$$\left[\left((1-\lambda)r + \lambda R\right)^2 - \lambda^2 d^2\right]^2 = 2r^2\left[\left((1-\lambda)r + \lambda R\right)^2 + \lambda^2 d^2\right].$$

Using (3.3), we obtain

$$r\lambda^4 + 4(R-r)\lambda^3 + 4r\lambda^2 - r = 0. \tag{5.6}$$

The left-hand side of (5.6) is a polynomial that we denote by $w(\lambda)$. Thus we have that $w(0) = -r$, $w(1) = 4R$, and $w'(\lambda) > 0$ for $\lambda > 0$. Hence there exists exactly one $\lambda \in]0, 1[$ such that $w(\lambda) = 0$. ∎

Theorem 5.2. *If the condition* $\mathrm{Pct}(C_r C_R, 3)$ *is satisfied, then there exists a value* $\lambda \in]0,1[$ *such that the condition* $\mathrm{Pct}(C_r C_\lambda, 6)$ *is satisfied.*

Proof. As before, we substitute d and R in (5.4) by the offset λd and the radius $(1-\lambda)r + \lambda R$, respectively, and next we use (3.3). Simple calculations lead us to the following equation:

$$3\left[(1-\lambda)^2 r + 2\lambda R\right]^4 - 4\left[(1-\lambda)^2 r^2 + 2\lambda r R - 4\lambda^r R + 2\lambda^2 R^2\right]\left[(1-\lambda)^2 r + 2\lambda R\right]^2$$
$$-16\lambda^2\left(R^2 - 2rR\right)\left[(1-\lambda)r + \lambda R\right]^2 = 0.$$

If we denote by $f(\lambda)$ the polynomial on the left-hand side of the above equation, then we get $f(0) = -r^4$ and $f(1) = 64rR^3$. Thus, Darboux's theorem (in this framework also studied in [14]) implies the existence of a $\lambda \in]0,1[$ such that $f(\lambda) = 0$. ■

Theorems 5.1 and 5.2 suggest the following

Conjecture. *If the condition* $\mathrm{Pct}(C_r C_R, k)$ *is satisfied, then there exists a value* $\lambda \in]0,1[$ *such that the condition* $\mathrm{Pct}(C_r C_\lambda, n)$ *is satisfied for any* $n > k$.

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