Existence of periodic solutions for a nonautonomous differential equation

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Abstract

We consider the nonautonomous differential equation of second order $x'' + a(t)x - b(t)x^l + c(t)x^{2k+1} = 0$, where $a(t), b(t), c(t)$ are T-periodic functions and $2 \leq l < 2k+1$. This is a generalization of a biomathematical model of an aneurysm in the circle of Willis. We prove the existence of a T-periodic solution for this equation, using a saddle-point theorem.

1 Introduction

We consider the nonautonomous differential equation

$$x'' + a(t)x - b(t)x^l + c(t)x^{2k+1} = 0, \tag{1}$$

where $2 \leq l < 2k+1$ and $a(t), b(t), c(t)$ are T-periodic functions, subject to the constraints $0 < a \leq a(t) \leq A$, $0 < c \leq c(t) \leq C$ and $|b(t)| \leq B$, for $a, A, b, B, C > 0$. Equation (1) comes from biomathematics, see for example [1].

The existence of periodic solutions to (1) for $(k, l) = (3, 2)$ was previously considered in [2], [3] and [4], using different methods. Here we aim in the general case adopting methods similar to these in [3], basically a saddle point theorem due to Silva ([5]). The key point in the generalization of the results in [3] for arbitrary l and k is the proof of the existence of real positive zeroes for a $2k$ degree polynomial.

In this paper we prove the existence of a nontrivial T-periodic solution for (1) using a saddle point theorem, under some specific assumptions on a, A, c, C, B to be given latter. For simplicity, we fix $T = 2\pi$.

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2 Description of the main result

The starting point is to associate weak solutions of (1) with critical points of a functional. If $F(\alpha, \beta, \gamma)$ is a function of class C^2, $I : C^1[a, b] \to \mathbb{R}$ is defined as

$$I(x) = \int_a^b F(x(t), x'(t), t)\, dt,$$

and I has an extremum at $x_0 \in S$, then x_0 satisfies the Euler-Lagrange equation

$$\frac{\partial F}{\partial \alpha}(x_0(t), x_0'(t), t) - \frac{d}{dt}\frac{\partial F}{\partial \beta}(x_0(t), x_0'(t), t) = 0,$$

for $t \in [a, b]$.

Now we recall the Palais-Smale (PS) condition. Let H be a Hilbert space. A differentiable functional $J \in C^1(H, \mathbb{R})$ is said to satisfies the (PS) condition if every sequence $(x_n)_n$ in H such that $(J(x_n))_n$ is bounded and $J'(x_n) \to 0$ in H has a convergent subsequence in H.

The following saddle-point theorem is due to Silva ([5]).

Theorem 1. *Let $E = X_1 \oplus X_2$ a real Banach space, with X_1 finite dimensional. Suppose $f \in C^1(E, \mathbb{R})$ with*
(i) $f(x) \leq 0$, for $x \in X_1$,
(ii) $f(x) \geq 0$ for $x \in X_2$ with $||x|| = \rho$, for some $\rho > 0$,
(iii) $f(x) \leq \beta$ for $x = x_1 + \lambda\xi \in X_1 \oplus \langle\xi\rangle$, for some unitary $\xi \in X_2$ and $\beta > 0$, with $\lambda > 0$.

If f satisfies (PS) condition, then f has a nonzero critical point in E.

Now we present a criteria on the existence of positive zeroes for a polynomial. For a proof, see [6].

Theorem 2. *Let $p(t) = a_0 + a_1 t + \ldots + a_m x^m$ be a polynomial. Consider the sequence $\overline{p} = (a_0, a_1, \ldots, a_m)$. Then the number of positive roots of p does not exceed the number of the changes of sign in the sequence $\overline{p}$.*

Our main theorem is

Theorem 3. *Consider $l, k \in \mathbb{Z}$ with $2 \leq l < 2k + 1$ and let $a(t), b(t), c(t)$ be mensurable 2π-periodic functions with $|b(t)| \leq B$, $0 < c \leq c(t) \leq C$, $m^2 < a \leq a(t) \leq A < (m+1)^2$, $\dfrac{c(l+1)}{B(2k+2)} > 1$ and $\dfrac{(a - m^2)(l+1)}{2B} > 1$ for some integer $m \geq 0$. Then the equation*

$$x'' + a(t)x - b(t)x^l + c(t)x^{2k+1} = 0$$

has a nontrivial 2π periodic solution.

Example 4. *Consider equation (1) with $a(t) = 6 + \sin(t)$, $b(t) = \cos(t)$, $c(t) = 2k + 3 + \sin(t)$ and $m = 2$. Define $a = 5$, $A = 7$, $B = 1$, $c = 2k + 2$ and $C = 2k + 4$. As $l \geq 2$, Theorem 3 guarantees that the equation*

$$x'' + (6 + \sin(t))x - \cos(t)x^l + (2k + 3 + \sin(t))x^{2k+1} = 0$$

has a nontrivial 2π-periodic solution, for every $2 \leq l < 2k + 1$.

3 Proof of Theorem 3

Let us denote

$$E = H^1(0,2\pi) = \left\{ \sum_{k\in\mathbb{Z}} c_k e^{ikt};\ c_{-k} = \overline{c_k},\ \sum_{k\in\mathbb{Z}} (1+k^2)c_k^2 < \infty \right\}$$

normed with

$$||x||^2 = \int_0^{2\pi} (x'^2 + x^2)\, dx.$$

Note that for $x = \sum_{k\in\mathbb{Z}} c_k e^{ikt} \in E$ we have that

$$||x||^2 = 2\pi \sum_{k\in\mathbb{Z}} (1+k^2)c_k^2.$$

Consider

$$F(x,x',t) = \frac{1}{2}x'^2 - \frac{1}{2}a(t)x^2 + \frac{1}{l+1}b(t)x^{l+1} - \frac{1}{2k+2}c(t)x^{2k+2}.$$

Recall that critical points of

$$f(x) = \int_0^{2\pi} F(x,x',t)\, dt$$

are weak solutions of

$$x'' + a(t)x - b(t)x^l + c(t)x^{2k+1} = 0.$$

Consider

$$X_1 = \{x \in E;\ x = \sum_{k\leq m^2} c_k e^{ikt}\}$$

and

$$X_2 = \{x \in E;\ x = \sum_{k\geq (m+1)^2} c_k e^{ikt}\}.$$

As we want to use Theorem 1, we shall estimate $f(x)$ for $x \in X_1$ and for $x \in X_2$.

Let $x \in X_1$.

$$\begin{aligned} f(x) &\leq \pi \sum_{k^2\leq m^2} c_k^2(k^2 - a) + \int_0^{2\pi} \left(\frac{1}{l+1}b(t)x^{l+1} - \frac{1}{2k+2}c(t)x^{2k+2} \right) dx \\ &\leq \pi(m^2 - a) \sum_{k^2\leq m^2} c_k^2 + \int_0^{2\pi} \left(\frac{1}{l+1}B|x|^{l+1} - \frac{1}{2k+2}cx^{2k+2} \right) dt \qquad (2) \\ &\leq \int_0^{2\pi} x^2 \left(\frac{m^2-a}{2} + \frac{1}{l+1}B|x|^{l-1} - \frac{1}{2k+2}c|x|^{2k} \right) \end{aligned}$$

Let $q(x) = \frac{m^2-a}{2} + \frac{1}{l+1}B|x|^{l-1} - \frac{1}{2k+2}c|x|^{2k}$. The hypotheses of Theorem 3 guarantees that $q(x) \leq 0$ for all $x \in X_1$, so the last integral in (2) is negative and the first item of Theorem 1 is true.

Now let $x \in X_2$.

$$\begin{aligned} f(x) &\geq \pi \sum_{k^2\geq(m+1)^2} c_k^2(k^2 - A) + \int_0^{2\pi} \left(\frac{1}{l+1} b(t)x^{l+1} - \frac{1}{2k+2} c(t)x^{2k+2} \right) dt \\ &\geq \int_0^{2\pi} x^2 \left(\frac{(m+1)^2 - A}{2} - \frac{1}{l+1} B|x|^{l-1} - \frac{1}{2k+2} C|x|^{2k} \right) dt \end{aligned}$$

To verify (ii) in Theorem 1, we have to find $\rho > 0$ such that $f(x) \geq 0$ for $x \in X_2$ and $||x|| \leq \rho$. This is equivalent to find a positive solution of $\Delta(x) = 0$, where

$$\Delta(x) = \frac{(m+1)^2 - A}{2} - \frac{1}{l+1} Bx^{l-1} - \frac{1}{2k+2} Cx^{2k}.$$

The next lemma applied to the polynomial $\Delta(x)$ guarantee that it has a positive root.

Lemma 5. *A polynomial $p(x) = p_0 - p_1 x^{l-1} - p_2 x^{2k}$, with $p_0, p_1, p_2 > 0$ and $l < 2k+1$, always has a unique positive root.*

Proof. As we have just one change of sign in the sequence of the coefficients of p, follows from Theorem 2 that p has at most one positive real root. As $p(0) > 0$ and $\lim_{y\to+\infty} p(y) = -\infty$, p has exactly one real root. ∎

Now we check item (iii) of Theorem 1. Consider

$$\xi(x) = \frac{1}{\sqrt{\pi(1+(m+2)^2)}} \cos(m+2)x.$$

Note that $\xi(x) \in X_2$. Also observe that $||\xi|| = 1$. If $\overline{x} = \sum_{k^2\leq m^2} c_k e^{ikt} + \lambda\xi$, with $\lambda \geq 0$, then, by the equivalence of the p-norms in the finite dimensional space $X_1 \oplus \langle\xi\rangle$, we have that

$$\begin{aligned} f(\overline{x}) &\leq \pi t^2(m+1)^2 + \frac{B}{l+1} \int_0^{2\pi} |\overline{x}|^{l+1}\, dt - \frac{c}{2k+2} \int_0^{2\pi} \overline{x}^{2k+2}\, dt \\ &\leq \frac{(m+1)^2}{2} ||\overline{x}||_2^2 + \frac{B}{l+1} ||\overline{x}||_{l+1}^{l+1} - \frac{c}{2k+2} ||u||_{2k+2}^{2k+2} \\ &\leq \frac{(m+1)^2}{2} ||\overline{x}||_2^2 + \delta_1 \frac{B}{l+1} ||\overline{x}||_2^{l+1} - \delta_2 \frac{c}{2k+2} ||\overline{x}||_2^{2k+2}, \end{aligned}$$

with $\delta_1, \delta_2 > 0$. Put $\varphi(t) = \dfrac{(m+1)^2}{2} t^2 + \delta_1 \dfrac{B}{l+1} t^{l+1} - \delta_2 \dfrac{c}{2k+2} t^{2k+2}$. As $\lim_{t\to\pm\infty} \varphi(t) = -\infty$, φ is bounded from above by a constant $\beta > 0$, so $f(\overline{x}) \leq \beta$ and item (iii) of Theorem 1 is verified.

Now it rests to check (PS) condition. Denote $g(t,x) = c(t)x^{2k+1} - b(t)x^l$. Consider $G(t,x) = \int_0^x g(t,y)\, dy$. Take $\theta = l+1$. Then

$$\begin{aligned} \theta G(t,x) &= \frac{l+1}{2k+2} c(t)x^{2k+2} - b(t)x^{l+1} \\ &\leq c(t)x^{2k+2} - b(t)x^{l+1} \\ &= xg(t,x). \end{aligned}$$

Note also that $G(t,x) > 0$ for larger values of x. This shows that g satisfies the Rabinowitz's condition.

Let (u_n) be a (PS) sequence for f in E. There exists a constant c_0 such that for $\nu \in (2,\theta)$

$$
\begin{aligned}
c_0 + \frac{1}{\nu}\|u_n\| &\geq f(u_n) - \frac{1}{\nu}(f'(u_n), u_n) \\
&\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\nu}\right)(m^2+1)\|u_n\|_2^2 \\
&\quad + \frac{1}{\nu}\int_0^{2\pi} u_n g(x,u_n)dx - \int_0^{2\pi} G(x,u_n)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\nu}\right)(m^2+1)\|u_n\|_2^2 \\
&\quad + \left(\frac{\theta}{\nu} - 1\right)\int_0^{2\pi}\left(\frac{1}{2k+2}c(t)x^{2k+2} - \frac{1}{l+1}b(t)x^{l+1}\right)dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\nu}\right)(m^2+1)\|u_n\|_2^2 \\
&\quad + \frac{c}{2k+2}\left(\frac{\theta}{\nu} - 1\right)\|u_n\|_{2k+2}^{2k+2} - \frac{B}{l+1}\left(\frac{\theta}{\nu} - 1\right)\|u_n\|_{l+1}^{l+1} \\
&\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \left(\frac{1}{2} - \frac{1}{\nu}\right)(m^2+1)\beta_1\|u_n\|_{2k+2}^2 \\
&\quad + \frac{c}{2k+2}\left(\frac{\theta}{\nu} - 1\right)\|u_n\|_{2k+2}^{2k+2} - \frac{B}{l+1}\left(\frac{\theta}{\nu} - 1\right)\beta_2\|u_n\|_{2k+2}^{l+1},
\end{aligned}
$$

with $\beta_1, \beta_2 > 0$. So the sequence (x_n) is bounded in E because $2k+2 > l+1 > 2$. Assume the weak convergence $x_n \rightharpoonup u$ in E, the strong convergence $x_n \to u$ in $L^2(0,2\pi)$ and the uniform convergence $x_n \to u$ in $C[0,2\pi]$.

Note that x is a critical point of f, as

$$
\begin{aligned}
0 &= \lim_{n\to\infty}(f'(x_n), y) \\
&= \lim_{n\to\infty}\int_0^{2\pi}\left(x_n'y' - a(t)x_n y + b(t)x_n^l y - c(x)x_n^{2k+1}y\right)dt \\
&= \int_0^{2\pi}\left(x'y' - a(t)xy + b(t)x^l y - c(t)x^{2k+1}y\right)dt,
\end{aligned}
$$

so

$$
\begin{aligned}
||x_n||^2 &= (f'(x_n), x_n) + \int_0^{2\pi}\left((a(t)+1)x_n^2 + x_n g(t,x_n)\right)dt \\
&\to \int_0^{2\pi}\left((a(t)+1)x^2 + xg(t,x)\right)dt \\
&= ||x||.
\end{aligned}
$$

As weak convergence and convergence of the square of the norm implies convergence, the proof is complete.

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