Property (gw) and perturbations

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Abstract

The property (gw) is a variant of generalized Weyls theorem, for a bounded operator T acting on a Banach space. In this note we consider the preservation of property (gw) under a finite rank perturbation commuting with T, whenever T is isoloid, polaroid, or T has analytical core $K(\lambda_0 I - T) = \{0\}$ for some $\lambda_0 \in \mathbb{C}$. The preservation of property (gw) is also studied under commuting nilpotent or under algebraic perturbations. The theory is exemplified in the case of some special classes of operators.

1 Introduction

Throughout this paper let $\mathbf{B}(\mathcal{X})$, denote, the algebra of bounded linear operators acting on an infinite dimensional Banach space $\mathcal{X}$. If $T \in \mathbf{B}(\mathcal{X})$ we shall write $\ker(T)$ and $\mathcal{R}(T)$ (or $ran(T)$) for the null space and range of T, respectively. Also, let $\alpha(T) := \dim \ker(T)$, $\beta(T) := \dim \mathcal{R}(T)$, and let $\sigma(T), \sigma_a(T), \sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in \mathbf{B}(\mathcal{X})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The *index* of a Fredholm operator is given by

$$ind(T) := \alpha(T) - \beta(T).$$

An operator T is called a *Weyl* if it is a Fredholm of index 0, and *Browder* if it is Fredholm "of finite ascent and descent"; equivalently, [33, Theorem 7.9.3] if T is Fredholm and $T - \lambda I$ (Abbreviate $T - \lambda$) is invertible for sufficiently small $\lambda \neq 0$

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in $\mathbb{C}$.

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$, and if such integer does not exist we put $d(T) = \infty$. The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$$

$$\sigma_W(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T) \subseteq \sigma_W(T) \subseteq \sigma_b(T) \subseteq \sigma_F(T) \cup acc\sigma(T)$$

where we write $accK$ for the accumulation points of $K \subseteq \mathbb{C}$.

For a bounded operator T and nonnegative integer n, define $T_{[n]}$ to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_{[0]} = T$). If for some n the range $\mathcal{R}(T^n)$ is closed and $T_{[n]}$ is an upper (resp. a lower) semi-Fredholm operator, then T is called an upper (resp. a lower) *semi-B-Fredholm* operator. In this case the index of T is defined as the index of the semi-Fredholm operator $T_{[n]}$, see [18, 19]. Moreover, if $T_{[n]}$ is a Fredholm operator, then T is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-Fredholm operator. An operator $T \in \mathbf{B}(\mathcal{X})$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. the semi-B-Fredholm spectrum $\sigma_{SBF}(T)$ and the B-Weyl spectrum σ_{BW} of T are defined by

$$\sigma_{SBF}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a semi-}B\text{-Fredholm operator}\},$$

$$\sigma_{BW} := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a } B\text{-Weyl operator}\}.$$

If we write $isoK = K \setminus accK$, then we let

$$E_0(T) := \{\lambda \in iso\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$$

and

$$\pi_0(T) := \sigma(T) \setminus \sigma_b(T).$$

Given $T \in \mathbf{B}(\mathcal{X})$, we say that Weyl's theorem holds for T (or that T satisfies Weyl's theorem, in symbol, $T \in \mathcal{W}$), see [26] if

$$\sigma(T) \setminus \sigma_W(T) = E_0(T),$$

and that Browder's theorem holds for T (in symbol, $T \in \mathcal{B}$) if

$$\sigma(T) \setminus \sigma_W(T) = \pi_0(T).$$

Recall that an operator $T \in \mathbf{B}(\mathcal{X})$ is a *Drazin invertible* if and only if it has a finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is nilpotent operator and T_1 is invertible operator, see [36, Proposition A]. The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$, where $\pi(T)$ is the set of all poles. Define

$$E(T) := \{\boldsymbol{\lambda} \in iso\sigma(T) : 0 < \alpha(T - \boldsymbol{\lambda})\},$$

we also say that the *generalized Weyl's theorem* holds for T (in symbol, $T \in g\mathcal{W}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

and that the *generalized Browder's theorem* holds for T (in symbol, $T \in g\mathcal{B}$) if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T).$$

It is Known [21, 22, 23] that

$$g\mathcal{W} \subseteq g\mathcal{B} \cup \mathcal{W} \qquad \text{and that} \qquad g\mathcal{B} \cup \mathcal{W} \subseteq \mathcal{B}.$$

Moreover, given $T \in g\mathcal{B}$, then it is clear $T \in g\mathcal{W}$ if and only if $E(T) = \pi(T)$, see [21, 23].

Let $SF_+(\mathcal{X})$ be the class of all *upper semi-Fredholm* operators, $SF_+^-(\mathcal{X})$ be the class of all $T \in SF_+(\mathcal{X})$ with $ind(T) \leq 0$, and for any $T \in \mathbf{B}(\mathcal{X})$ let

$$\sigma_{SF_+^-}(T) := \left\{\lambda \in \mathbb{C} : T - \lambda I \notin SF_+^-(\mathcal{X})\right\}.$$

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma_a(T)$. According to [42], we say that T satisfies *a-Weyl's theorem*(and we write $T \in a\mathcal{W}$) if

$$\sigma_{SF_+^-}(T) = \sigma_a(T) \setminus E_0^a(T),$$

and that a-Browder's theorem holds for T (in symbol, $T \in a\mathcal{B}$) if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_0^a(T),$$

where $\pi_0^a(T)$ is the set of all left poles of finite rank.

Let $SBF_+(\mathcal{X})$ be the class of all *upper semi-B-Fredholm* operators, and $SBF_+^-(\mathcal{X})$ the class of all $T \in SBF_+(\mathcal{X})$ such that $ind(T) \leq 0$, and

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{X})\}.$$

Recall that an operator $T \in \mathbf{B}(\mathcal{X})$ satisfies the *generalized a-Weyl's theorem* (in symbol, $T \in ga\mathcal{W}$) if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T),$$

where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$.

Define a set $LD(\mathcal{X})$ by

$$LD(\mathcal{X}) := \left\{T \in \mathbf{B}(\mathcal{X}) : a(T) < \infty \quad \text{and} \quad \mathcal{R}(T^{a(T)+1}) \quad \text{is closed}\right\}.$$

An operator $T \in \mathbf{B}(\mathcal{H})$ is called *left Drazin invertible* if $a(T) < \infty$ and $\mathcal{R}(T^{a(T)+1})$ is closed (see [23, Definition 2.4]). The left Drazin spectrum is given by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not left Drazin invertible}\}.$$

Recall [23, Definition 2.5] that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is left Drazin invertible operator and $\lambda \in \sigma_a(T)$ is a left pole of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. We will denote $\pi^a(T)$ the set of all left pole of T. We have $\sigma_{LD}(T) = \sigma_a(T) \setminus \pi^a(T)$. Note that if $\lambda \in \pi^a(T)$, then it is easily seen that $T - \lambda$ is an operator of topological uniform descent. Therefore, it follows from ([21, Theorem 2.5]) that λ is isolated in $\sigma_a(T)$. Following [23] if $T \in \mathbf{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ is an isolated in $\sigma_a(T)$, then $\lambda \in \pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF_+^-}(T)$ and $\lambda \in \pi_0^a(T)$ if and only if $\lambda \notin \sigma_{SF_+^-}(T)$.

We will say that *generalized a-Browder's theorem* holds for T (in symbol $T \in ga\mathcal{B}$) if

$$\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \pi^a(T).$$

It is Known [23, 21, 42]that

$$g\mathcal{W} \cup g\mathcal{B} \cup a\mathcal{W} \cup ga\mathcal{B} \subseteq ga\mathcal{W} \quad \text{and that} \quad a\mathcal{B} \cup \mathcal{W} \subseteq a\mathcal{W} \quad \text{and that } \mathcal{B} \subseteq a\mathcal{B}.$$

This article also deals with the single valued extension property. This property has a basic role in the local spectral theory, see the recent monograph of Laursen and Neumann [39] or Aiena [3]. In this article consider a localized version of this property, recently studied by several authors [1, 4, 11, **?**], and previously by Finch [31].

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [31] we say that $T \in \mathbf{B}(\mathcal{X})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathcal{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.

An operator $T \in \mathbf{B}(\mathcal{X})$ has the SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. The identity theorem for analytic functions ensures that for every $T \in \mathbf{B}(\mathcal{X})$, both T and T^* have the SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, that both T and T^* have the SVEP at every isolated point of $\sigma(T) = \sigma(T^*)$. The SVEP is inherited by the restrictions to closed invariant subspaces, i.e., if $T \in \mathbf{B}(\mathcal{X})$ has the SVEP at λ_0 and M is closed T-invariant subspace then $T|_M$ has SVEP at λ_0.

The *quasinilpotent part* $H_0(T - \lambda I)$ and the *analytic core* $K(T - \lambda I)$ of $T - \lambda I$ are defined by

$$H_0(T - \lambda I) := \{x \in \mathcal{X} : \lim_{n \longrightarrow \infty} \|(T - \lambda I)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$\begin{aligned} K(T - \lambda I) = \{x \in \mathcal{X} &: \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0 \text{ for which} \\ &x = x_0, (T - \lambda I)x_{n+1} = x_n \text{ and } \quad \|x_n\| \leq \delta^n \|x\| \text{ for all } \quad n = 1, 2, \cdots\}. \end{aligned}$$

We note that $H_0(T-\lambda I)$ and $K(T-\lambda I)$ are generally non-closed hyper-invariant subspaces of $T-\lambda I$ such that $(T-\lambda I)^{-p}(0) \subseteq H_0(T-\lambda I)$ for all $p = 0, 1, \cdots$ and $(T-\lambda I)K(T-\lambda I) = K(T-\lambda I)$. Recall that if $\lambda \in iso(\sigma(T))$, then $H_0(T-\lambda I) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow \mathcal{X}$ that satisfies $(T-\mu I)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$(see [29]). From [2], the following implication holds for every $T \in \mathbf{B}(\mathcal{X})$,

$$H_0(T-\boldsymbol{\lambda} I) \text{ is closed} \Longrightarrow T \text{ has SVEP at } \boldsymbol{\lambda}.$$

Definition 1.1. ([42]) An operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property (w) if

$$\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_0(T).$$

In [6], it is shown that the property (w) implies Weyls theorem. For $T \in \mathbf{B}(\mathcal{H})$, let $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T)$ and $\Delta_a^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T)$. If T^* has the SVEP, then it is known from [39] that $\sigma(T) = \sigma_a(T)$ and from [12] we have $\sigma_{BW}(T) = \sigma_{SBF_+^-}(T)$. Thus $E(T) = E^a(T)$ and $\Delta^g(T) = \Delta_a^g(T)$.

Definition 1.2. ([16]) An operator $T \in \mathbf{B}(\mathcal{X})$ is said to satisfy property (gw) if

$$\Delta_a^g(T) = E(T).$$

The following diagram resume the relationships between generalized a-Weyls theorem, generalized Weyl's theorem, a-Weyls theorem, generalized a-Browders theorem, a-Browders theorem, property (gw) and property (w), see [5, 7, 8, 10, 16, 28].

$$\begin{array}{ccccccc}
\text{Property (gw)} & \Longrightarrow & g\mathcal{W} & \Longleftarrow & ga\mathcal{W} & \Longrightarrow & ga\mathcal{B} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Updownarrow \\
\text{Property (w)} & \Longrightarrow & \mathcal{W} & \Longleftarrow & a\mathcal{W} & \Longrightarrow & a\mathcal{B}
\end{array}$$

2 Results

We begin this section by some results about the structural of $ga\mathcal{B}$ and $ga\mathcal{W}$.

Theorem 2.1. *Let $T \in \mathbf{B}(\mathcal{X})$. Then the following statements are equivalent:*
(i) $T \in ga\mathcal{B}$;
(ii) $\sigma_{SBF_+^-}(T) = \sigma_{lD}(T)$;
(iii) $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E^a(T)$;
(iv) $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$;
(v) $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$.

Proof. (i)$\Rightarrow$ (ii). Suppose that $T \in ga\mathcal{B}$. Then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $\lambda \in \pi^a(T)$, and so $T-\lambda I$ is left Drazin invertible. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{lD}(T)$, and hence $\sigma_{lD}(T) \subseteq \sigma_{SBF_+^-}(T)$. On the other hand, since $\sigma_{SBF_+^-}(T) \subseteq \sigma_{lD}(T)$ is always verified for any operator T [21, Lemma 2.12.].

(ii)⇒ (i). We assume that $\sigma_{SBF_+^-}(T) = \sigma_{lD}(T)$ and we will establish that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$. Suppose first that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{lD}(T)$, and so $T - \lambda I$ is left Drazin invertible. Therefore, $d = a(T) < \infty$ and $ran(T^{d+1})$ is closed. Since $\lambda \in \sigma_a(T)$, we have $\lambda \in \pi^a(T)$. Thus $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi^a(T)$.
Conversely, suppose that $\lambda \in \pi^a(T)$. Then $T - \lambda I$ is left Drazin invertible but not bounded below. Since λ is an isolated point of $\sigma_a(T)$, then $T - \lambda \in SBF_+^-(\mathcal{X})$. Therefore, $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Thus $\pi^a(T) \supseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.
(ii)⇒ (iii). Let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $\lambda \in \sigma_a(T) \setminus \sigma_{lD}(T)$, and so $T - \lambda I$ is left Drazin invertible but not bounded below. Therefore, $\lambda \in E^a(T)$. Thus $\sigma_a(T) \subseteq \sigma_{SBF_+^-}(T) \cup E^a(T)$. Since the other inclusion is always true, we must have $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E^a(T)$.
(iii)⇒ (ii). Suppose $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E^a(T)$. To show that $\sigma_{SBF_+^-}(T) = \sigma_{lD}(T)$. it suffices to show that $\sigma_{SBF_+^-}(T) \subseteq \sigma_{lD}(T)$. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T - \lambda I \in SBF_+^-(\mathcal{X})$ but not invertible. Since $\sigma_a(T) = \sigma_{SBF_+^-}(T) \cup E^a(T)$, we see that $\lambda \in E^a(T)$. In particular, λ is an isolated point of $\sigma_a(T)$. Hence $T - \lambda I$ is left Drazin invertible, and so $\sigma_{SBF_+^-}(T) = \sigma_{lD}(T)$.
(i)⇔ (iv). Suppose $T \in ga\mathcal{B}$. Then $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \pi^a(T)$. Let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $\lambda \in \pi^a(T)$, and so λ is an isolated point of $\sigma_a(T)$. Therefore, $\lambda \in \sigma_a(T) \setminus acc(\sigma_a(T))$, and hence $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$.
Conversely, let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Since $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$, it follows that $\lambda \in iso(\sigma_a(T))$ and $T - \lambda I \in SBF_+^-(\mathcal{X})$. It follows from [21, Theorem 2.8.] that $\lambda \in \pi^a(T)$. Therefore, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi^a(T)$. For the converse, suppose $\lambda \in \pi^a(T)$. Then λ is a left pole of the resolvent of T, and so λ is an isolated point of $\sigma_a(T)$. Therefore, $\lambda \in \sigma_a(T) \setminus acc(\sigma_a(T))$. It follows from [21, Theorem 2.11.] that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Thus $\pi^a(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, and so $T \in ga\mathcal{B}$.
(iv)⇔ (v). Suppose that $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$, and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Then $T - \lambda \in SBF_+^-(\mathcal{X})$ but not bounded below. Since $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$, λ is an isolated point of $\sigma_a(T)$. It follows from [21, Theorem 2.8.] that λ is a left pole of of the resolvent of T. Therefore, $\lambda \in \pi^a(T)$, and hence $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$.
Conversely, suppose that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$ and let $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subseteq E^a(T)$. Then $\lambda \in E^a(T)$, and so λ is an isolated point of $\sigma_a(T)$. Therefore, $\lambda \in \sigma_a(T) \setminus acc(\sigma_a(T))$, which implies that $acc(\sigma_a(T)) \subseteq \sigma_{SBF_+^-}(T)$. ∎

The next result gives simple necessary and sufficient conditions for an operator $T \in ga\mathcal{B}$ to belong to the smaller class $ga\mathcal{W}$.

Theorem 2.2. *Let $T \in ga\mathcal{B}$. The following statements are equivalent:*
(i) $T \in ga\mathcal{W}$.
(ii) $\sigma_{SBF_+^-}(T) \cap E^a(T) = \emptyset$.
(iii) $\pi^a(T) = E^a(T)$.

Proof. (i)⇒(ii). Assume $T \in ga\mathcal{W}$, that is, $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$. It then easily that $\sigma_{SBF_+^-}(T) \cap E^a(T) = \emptyset$, as required for (ii).
(ii)⇒(iii). Let $\lambda \in E^a(T)$. The condition in (ii) implies that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$, and since $T \in ga\mathcal{B}$, we must have $\lambda \in \pi^a(T)$. It follows that $E^a(T) \subseteq \pi^a(T)$, and since the reverse inclusion always holds, we obtain (iii).
(iii)⇒(i). Since $T \in ga\mathcal{B}$, we know that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$, and since we are assuming $E^a(T) = \pi^a(T)$, it follows that $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$, that is, $T \in ga\mathcal{W}$. ■

Theorem 2.3. *([16]) Let $T \in \mathbf{B}(\mathcal{X})$. The following statements are equivalent:*
i) *T satisfies property* (gw)*;*
ii) *generalized a-Browders theorem holds for T and $\pi^a(T) = E(T)$.*

The following example show that property (gw) is not intermediate between generalized Weyl's theorem and generalized *a*-Weyl's theorem.

Example 2.4. Let T be the hyponormal operator given by the direct sum of the 1-dimensional zero operator and the unilateral right shift R on $\ell^2(\mathbb{N})$. Then $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ the closed unit disc in $\mathbb{C}$. Moreover, 0 is an isolated point of $\sigma_a(T) = C(0,1) \cup \{0\}$, $C(0,1)$ the unit circle of $\mathbb{C}$, $0 \in E^a(T)$ and $\sigma_{SBF_+^-}(T) = C(0,1)$ while $0 \notin \pi^a(T) = \emptyset$ since $a(T) = a(R) = \infty$. Hence, T does not satisfy generalized a-Weyls theorem. On the other hand $E(T) = \emptyset$, since $\sigma(T)$ has no isolated points, so $\pi^a(T) = E(T)$. Since every hyponormal operator has SVEP we also know that generalized a-Browders theorem holds for T , so from Theorem 2.3 we see that property (gw) holds for T.

The following example shows that generalized a-Weyls theorem and generalized Weyls theorem does not imply property (gw).

Example 2.5. Let $R \in \ell^2(\mathbb{N})$ be the unilateral right shift and let U defined by $U(x_1, x_2, \cdots) = (0, x_2, x_3, ...)$, $(x_n) \in \ell^2(\mathbb{N})$. If $T = R \oplus U$, then $\sigma(T) = D(0,1)$ the closed unit disc in $\mathbb{C}$, $iso\sigma(T) = \emptyset$ and $\sigma_a(T) = C(0,1) \cup \{0\}$, where $C(0,1)$ is unit circle of $\mathbb{C}$. It follows from [6, Example 2.14] that $\sigma_{SF_+^-}(T) = C(0,1)$. This implies that

$$\sigma_{SBF_+^-}(T) = C(0,1) \quad \text{and} \quad \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{0\}$$

Moreover, we have $E(T) = \emptyset$ and $E^a(T) = \{0\}$. Hence T satisfies generalized *a*- Weyls theorem and so T satisfies generalized Weyls theorem. But T does not satisfy property (gw).

The class of operators $T \in \mathbf{B}(\mathcal{X})$ for which $K(T) = \{0\}$ was introduced and studied by M. Mbekhta in [40]. It was shown that for such operators, the spectrum is connected and the SVEP holds.

Theorem 2.6. *Let $T \in \mathbf{B}(\mathcal{X})$. If there exists λ such that $K(T - \lambda) = \{0\}$, then $f(T) \in ga\mathcal{B}$, for every $f \in Hol(\sigma(T))$. Moreover, if in addition* $\ker(T - \lambda) = 0$, *then property* (gw) *holds for $f(T)$*

Proof. Since T has the SVEP, then by Theorem 3.2 of [14] , generalized a- Browder's theorem holds for $f(T)$. Let $\gamma \in \sigma(f(T))$, then

$$f(z) - \gamma I = P(z)g(z),$$

where g is complex-valued analytic function on a neighborhood of $\sigma(T)$ without any zeros in $\sigma(T)$ while P is a complex polynomial of the form $P(z) = \prod_{j=1}^{n}(z - \lambda_j I)^{k_j}$ with distinct roots $\lambda_1, \cdots, \lambda_n \in \sigma(T)$. Since $g(T)$ is invertible, then we deduce that

$$\ker(f(T) - \gamma I) = \ker(P(T)) = \bigoplus_{j=1}^{n} \ker(T - \lambda_j I)^{k_j}.$$

On the other hand, it follows from [40, Proposition 2.1] that $\sigma_p(T) \subseteq \{\lambda\}$. If we assume that $\ker(T - \lambda I) = 0$, then $T - \lambda I$ is an injective and consequently $\sigma_p(T) = \emptyset$. Hence $\ker(f(T) - \lambda I) = 0$. Therefore, $\sigma_p(f(T)) = \emptyset$. To prove that property (gw) holds for $f(T)$, by Theorem 2.3 it then suffices to prove that

$$\pi^a(f(T)) = E(f(T)).$$

Obviously, the condition $\sigma_p(f(T)) = \emptyset$ entails that

$$E(f(T)) = E^a(f(T)) = \emptyset.$$

On the other hand, the inclusion $\pi^a(f(T)) \subseteq E^a(f(T))$ holds for every operator $T \in \mathbf{B}(\mathcal{X})$. So also $\pi^a(f(T)) = \emptyset$. By Theorem 2.6 of [16] it then follows that property (gw) holds for $f(T)$. ■

Theorem 2.7. *Let T be a bounded linear operator on $\mathcal{X}$ satisfying the SVEP. If $T - \lambda I$ has finite descent at every $\lambda \in E^a(T)$, then property* (gw) *holds for $f(T^*)$, for every $f \in Hol(\sigma(T))$.*

Proof. Let $\lambda \in E^a(T)$, then $p = d(T - \lambda I) < \infty$ and since T has the SVEP it follows that $a(T - \lambda I) = d(T - \lambda I) = p$ and hence λ is a pole of the resolvent of T of order p, consequently λ is an isolated point in $\sigma_a(T)$. Then $\mathcal{X} = K(T - \lambda I) \oplus H_0(T - \lambda I)$, with $K(T - \lambda I) = \mathcal{R}(T - \lambda I)^p$ is closed, Therefore, $\lambda \in \pi^a(T)$. Hence, T is a-polaroid. Now the result follows now from Theorem 2.11 of [16]. ■

A bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to be *polaroid* (respectively, *a-polaroid*) if $iso\sigma(T) = \emptyset$ or every isolated point of $\sigma(T)$ is a pole of the resolvent of T (respectively, if $iso\sigma_a(T) = \emptyset$ or every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T).
In [41] Oudghiri introduced the class $H(p)$ of operators on Banach spaces for which there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$H_0(\lambda I - T) = \ker(T - \lambda I)^p \qquad \text{for all } \lambda \in \mathbb{C}.$$

Let $P(\mathcal{X})$ be the class of all operators $T \in \mathbf{B}(\mathcal{X})$ having the property $H(p)$. The class $P(\mathcal{X})$ contains the classes of subscalar, algebraically totally paranormal and transaloid operators on a Banach space, $*$-totally paranormal, M-hyponormal,

p-hyponormal $(0 < p < 1)$ and log-hyponormal operators on a Hilbert space (see [25, 26, 27, 32, 35]).

It is known that if $H_0(T - \lambda I)$ is closed for every complex number λ, then T has the SVEP (see [3, 38]). So that, the SVEP is shared by all the operators of $P(\mathcal{X})$. Moreover, T is polaroid, see [5, Lemma 3.3].

Theorem 2.8. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ is generalized scalar. Then T satisfies property* (gw) *if and only if T satisfies generalized Weyl's theorem*

Proof. If T is generalized scalar then both T and T^* has SVEP. Moreover, T is polaroid since every generalized scalar has the property $H(p)$. Then T satisfies property (gw) by Theorem 2.10 of [16]. The equivalence then follows from [16, Theorem 2.7]. ■

Theorem 2.9. *Let $T \in P(\mathcal{X})$ be such that $\sigma(T) = \sigma_a(T)$ then property* (gw) *holds for $f(T)$, for every $f \in Hol(\sigma(T))$.*

Proof. Since $\sigma(T) = \sigma_a(T)$, it follows that

$$E^a(T) = \sigma_p(T) \cap iso(\sigma_a(T)) = \sigma_p(T) \cap iso(\sigma(T)) = E(T).$$

Let $\lambda \in E^a(T) = E(T)$, Since $T \in \mathcal{P}(\mathcal{X})$, then there exists $d_\lambda \in \mathbb{N}$ such that $H_0(T - \lambda I) = \ker(T - \lambda I)^{d_\lambda}$. Since λ is isolated in $\sigma(T)$ then, by [3, Theorem 3.74],

$$\mathcal{X} = H_0(T - \lambda I) \oplus K(T - \lambda I) = \ker(T - \lambda I)^{d_\lambda} \oplus K(T - \lambda I),$$

from which we obtain

$$\mathcal{R}((T - \lambda I)^{d_\lambda}) = (T - \lambda I)^{d_\lambda}(K(T - \lambda I)) = K(T - \lambda I),$$

so

$$\mathcal{X} = \ker(T - \lambda I)^{d_\lambda} \oplus \mathcal{R}((T - \lambda I)^{d_\lambda}),$$

which implies, by [3, Theorem 3.6], that $a(T - \lambda I) = d(T - \lambda I) \leq d_\lambda$, hence λ is a pole of the resolvent, so that T is polaroid. As T^* has the SVEP and T is polaroid, then $f(T)$ satisfies property (gw) for every $f \in Hol(\sigma(T))$ by Theorem 2.11 of [16]. ■

Theorem 2.10. *Let T a bounded operator on $\mathcal{X}$. If there exists a function $g \in Hol(\sigma(T))$ non constant in any connected component of its domain, and such that $g(T^*) \in P(\mathcal{X}^*)$, then property* (gw) *holds for $f(T)$, for every $f \in Hol(\sigma(T))$.*

Proof. Suppose that $g(T^*) \in P(\mathcal{X}^*)$, then by [41, Theorem 3.4], we have $T^* \in P(\mathcal{X}^*)$. Since T^* has the SVEP, then as it had been already mentioned, we have

$$\sigma_a(T) = \sigma(T), \quad \sigma_{SBF_+^-}(T) = \sigma_{BW}(T), \quad E^a(T) = E(T) \quad \text{and} \quad \Delta_a^g(T) = \Delta_a(T),$$

it suffices to show that $\pi^a(T) = E^a(T)$. For this let $\lambda \in E^a(T)$, then λ is isolated eigenvalue of $\sigma_a(T)$. So $\mathcal{X}^* = H_0(T^* - \bar{\lambda}) \oplus K(T^* - \bar{\lambda})$, where the direct sum is topological. Since $T^* \in P(\mathcal{X}^*)$, then there exists $d_\lambda \in \mathbb{N}$ such that $H_0(T^* - \bar{\lambda} I) = \ker(T^* - \bar{\lambda} I)^{d_\lambda}$, and hence $\mathcal{X}^* = \ker(T^* - \bar{\lambda})^{d_\lambda} \oplus K(T^* - \bar{\lambda})$. Since

$$\mathcal{R}((T - \bar{\lambda} I)^{d_\lambda}) = (T - \bar{\lambda})^{d_\lambda}(K(T - \bar{\lambda} I)) = K(T - \bar{\lambda} I),$$

so
$$\mathcal{X} = \ker(T - \bar{\lambda}I)^{d_\lambda} \oplus \mathcal{R}((T - \bar{\lambda}I)^{d_\lambda}),$$
which implies, by [3, Theorem 3.6], that $a(T^* - \bar{\lambda}I) = d(T - \bar{\lambda}I) \leq d_\lambda$, hence $\bar{\lambda}$ is a pole of the resolvent of T^*, so that T^* is polaroid. Hence we have $\mathcal{X}^* = \ker((T^* - \bar{\lambda}I)^{d_\lambda} \oplus \mathcal{R}(T^* - \bar{\lambda}I)^{d_\lambda})$ and $\mathcal{R}(T^* - \bar{\lambda}I)^{d_\lambda})$ is closed. Therefore, $\mathcal{R}(T - \lambda I)^{n_0})$ is closed and $\mathcal{X} = \ker((T^* - \bar{\lambda}I)^{d_\lambda})^\perp \oplus \mathcal{R}(T^* - \bar{\lambda}I)^{d_\lambda})^\perp = \ker((T - \lambda I)^{d_\lambda}) \oplus \mathcal{R}(T - \lambda I)^{d_\lambda})$. So $\lambda \in \pi^a(T)$. As T^* has the SVEP and T is polaroid, then $f(T)$ satisfies property (gw) for every $f \in Hol(\sigma(T))$ by Theorem 2.11 of [16]. ■

As an easy consequence of the previous theorem, we have the following corollary

Corollary 2.11. *If $T^* \in P(\mathcal{X}^*)$, then property* (gw) *holds for for $f(T)$, for every $f \in Hol(\sigma(T))$.*

Example 2.12. Property (gw), as well as generalized Weyl's theorem, is not transmitted from T to its dual T^*. To see this, consider the weighted right shift $T \in \underline{(}\ell^2(\mathbb{N}))$, defined by
$$T(x_1, x_2, \cdots) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$
Then
$$T^*(x_1, x_2, \cdots) := (\frac{x_2}{2}, \frac{x_3}{3}, \cdots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}).$$
Both T and T^* are quasi-nilpotent, and hence are decomposable, T satisfies generalized Weyls theorem since $\sigma(T) = \sigma_{BW}(T) = \{0\}$ and $E(T) = \pi(T) = \emptyset$ and hence T has property (gw). On the other hand, we have $\sigma(T^*) = \sigma_a(T^*) = \sigma_{SBF_+^-}(T^*) = E^a(T^*) = \sigma_{BW}(T^*) = E(T^*) = \{0\}$ and $\pi^a(T^*) = \emptyset$, so T^* does not satisfy generalized Weyl's theorem (and nor generalized a-Weyl's theorem). Since T^* has SVEP, then T^* does not satisfy property (gw).

Lemma 2.13. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ satisfying property* (gw) *and F is a finite operator commuting with T such that $\sigma_a(T + F) = \sigma_a(T)$. Then $\pi^a(T + F) \subseteq E(T + F)$.*

Proof. Let $\lambda \in \pi^a(T + F)$ be arbitrary given. Then $\lambda \in iso\sigma_a(T + F)$ and $\lambda \notin \sigma_{LD}(T + F)$ and so $T + F - \lambda I$ is left Drazin invertible. Hence $m = a(T + F - \lambda I) < \infty$ and $\mathcal{R}((T + F - \lambda)^{m+1})$ is closed. Since $(T + F - \lambda)^{m+1}$ has closed range, the condition $\lambda \in \sigma_a(T + F)$ entails that $\alpha((T + F - \lambda)^{m+1}) > 0$. Therefore,, in order to show that $\lambda \in E(T + F)$, we need only to prove that λ is an isolated of $\sigma(T + F)$.

We know that $\lambda \in iso\sigma_a(T)$. We also have $\lambda I - (T + F) - F = \lambda I - T \in ga\mathcal{B}$ so that $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$.

Now, by assumption T satisfies property (gw), so, by Theorem 2.3, $\pi^a(T) = E(T)$. Moreover, T satisfies generalized Weyl's theorem and hence, by [20, Corollary 2.6],
$$E(T) = \pi(T) = \sigma(T) \setminus \sigma_{BW}(T).$$

Therefore, $T - \lambda I \in g\mathcal{B}$ and hence also $T + F - \lambda I \in g\mathcal{B}$, so

$$0 < a(T + F - \lambda I) = d(T + F - \lambda I) < \infty$$

and hence λ is a pole of the resolvent of $T + F$. Consequently, λ an isolated point of $\sigma(T + F)$, as desired. ∎

Recall that a bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to be *isoloid* (respectively, *a-isoloid*) if every isolated point of $\sigma(T)$ (respectively, every isolated point of $\sigma_a(T)$) is an eigenvalue of T. Every a-isoloid operator is isoloid. This is easily seen: if T is a-isoloid and $\lambda \in iso\sigma(T)$ then $\lambda \in \sigma_a(T)$ or $\lambda \notin \sigma_a(T)$. In the first case $T - \lambda I$ is bounded below, in particular upper semi-Fredholm. The SVEP of both T and T^* at λ then implies that $a(T - \lambda I) = d(T - \lambda I) < \infty$, so λ is a pole. Obviously, also in the second case λ is a pole, since by assumption T is a-isoloid. However, the converse is not true . Consider the following example: Let $U \oplus Q$, where U is the unilateral forward shift on ℓ^2 and Q is an injective quasinilpotent on ℓ^2, respectively. Then $\sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and $\sigma_a(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\}$. Therefore, T is isoloid but not a-isoloid.

Theorem 2.14. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ is a-isoloid and F is a finite rank operator commuting with T such that $\sigma_a(T + F) = \sigma_a(T)$. If T satisfies property* (gw), *then $T + F$ satisfies property* (gw).

Proof. Suppose that T satisfies property (gw). Then, by Theorem 2.3, $T \in ga\mathcal{B}$, and hence also $T + K \in ga\mathcal{B}$.

By Theorem 2.3, in order to show that $T + K$ satisfies property (gw) it suffices only to prove the equality $\pi^a(T + F) = E(T + F)$. The inclusion $\pi^a(T + F) \subseteq E(T + F)$ follows from Lemma 2.13, so we need only to show the opposite inclusion $\pi^a(T + F) \supseteq E(T + F)$.

We first show the inclusion

$$E(T + F) \subseteq \pi(T). \tag{2.1}$$

Let $\lambda \in E(T + F)$. By assumption $\lambda \in iso\sigma(T + F)$ and $\alpha(T + F - \lambda I) > 0$ so $\lambda \in iso\sigma_a(T + F)$, and hence $\lambda \in iso\sigma_a(T)$. Since T satisfies property (gw) we then conclude that λ is an isolated point of $\sigma(T)$. Furthermore, Since T is a-isoloid, we have also $0 < \alpha(T - \lambda I)$. Therefore, the inclusion $E(T + F) \subseteq \pi(T)$ is proved. Now, since property (gw) entails that T satisfies generalized Weyl's theorem, by [20, Corollary 2.6], we then have $E(T + F) \subseteq \pi(T + F) = \pi(T)$ and hence the inclusion 2.1 is established. Consequently, if $\lambda \in E(T + F)$, then $T - \lambda I \in g\mathcal{B}$. By Theorem 2.1 of [37] it then follows that $T + F - \lambda I \in g\mathcal{B}$, hence

$$\lambda \in \sigma(T + F) \setminus \sigma_{BW}(T + F) = \pi(T + F) \subseteq \pi^a(T + F),$$

so the proof is achieved. ∎

In the sequel we shall consider nilpotent perturbations of operators satisfying property (gw). It easy to check that if N is a nilpotent operator commuting with T , then $\sigma(T) = \sigma(T + N)$ and $\sigma_a(T) = \sigma_a(T + N)$.

Lemma 2.15. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ satisfying property* (gw) *and N is a nilpotent operator commuting with T. Then $\pi^a(T+N) \subseteq E(T+N)$.*

Proof. Suppose that $\lambda \in \pi^a(T+N)$. Then

$$\lambda \in \sigma_a(T+N) \setminus \sigma_{SBF_+^-}(T+N) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T).$$

Since T satisfies property (gw) we then have, by Theorem 2.3, $\pi^a(T) = E(T)$. Hence λ is an isolated point of $\sigma(T) = \sigma(T^*)$ and Therefore, both T and T^* have SVEP at λ. Since $T - \lambda I \in ga\mathcal{B}$ it then follows that $0 < m = a(T-\lambda I) = d(T-\lambda I) < \infty$. Furthermore, since $\lambda \in E(T)$ we also have $\alpha(T-\lambda I) > 0$, thus $T - \lambda I \in ga\mathcal{B}$ and hence also $T + N - \lambda I \in ga\mathcal{B}$, by Theorem 2.1 of [37]. Hence λ is an isolated point of $\sigma(T+N)$ and $\alpha(T+N-\lambda I) > 0$.

On the other hand, $(T+N-\lambda I)^{m+1}$ has closed range and since $\lambda \in \sigma_a(T+N)$ it then follows that $\alpha(T+N-\lambda I) > 0$. Thus $\lambda \in E(T+N)$. ■

Theorem 2.16. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ is a-isoloid and N is a nilpotent operator that commutes with T. If T satisfies property* (gw)*, then $T + N$ satisfies property* (gw)*.*

Proof. Observe first that $a\mathcal{B} \Leftrightarrow ga\mathcal{B}$ by Theorem 2.2 of [15], $\mathcal{B} \Leftrightarrow g\mathcal{B}$ by Theorem 2.1 of [15]. Then it follows from Theorem 1.2 of [7] that $\sigma_{LD}(T+N) = \sigma_{LD}(T)$ and $\sigma_{SBF_+^-}(T+N) = \sigma_{SBF_+^-}(T)$. Since $T \in ga\mathcal{B}$, by Theorem 1.3 of [24], it then follows that $\sigma_{LD}(T+N) = \sigma_{SBF_+^-}(T+N)$, i.e. $T+N \in ga\mathcal{B}$. By Theorem 2.6 of [16] and Lemma 2.15 we have only prove the inclusion

$$E(T+N) \subseteq \pi^a(T+N). \tag{2.2}$$

Let $\lambda \in E(T+N)$ be arbitrary given. There is no harm if we assume $\lambda = 0$. Clearly, $0 \in iso\sigma(T+N) = iso\sigma(T)$. Let $s \in \mathbb{N}$ be such that $N^s = 0$. If $x \in \ker(T+N)$, then

$$T^s x = (-1)^s T^s x = 0,$$

then $\ker(T+N) \subseteq \ker(T^s)$. Since by assumption $\alpha(T+N) > 0$ it then follows that $\alpha(T^s) > 0$ and this is obviously implies that $\alpha(T) > 0$. Therefore, $0 \in E(T)$ and consequently $E(T+N) \subseteq E(T)$. Now, since $T \in g\mathcal{W}$ we have

$$E(T) = \pi(T) \subseteq \pi^a(T).$$

The inclusion 2.2 will be then proved if we show that $\pi^a(T+N) = \pi^a(T)$. But this is immediate, since $\sigma_a(T+N) = \sigma_a(T)$ and $\sigma_{LD}(T+N) = \sigma_{SBF_+^-}(T+N)$, so the proof is achieved. ■

Recall that $T \in \mathbf{B}(\mathcal{H})$ is said to be a *Riesz operator* if $T - \lambda I$ is a Fredholm operator for all $\lambda \neq 0$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. A bounded operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *finite-isoloid* if every isolated spectral point is an eigenvalue having finite multiplicity.

Theorem 2.17. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ and Q is a quasi-nilpotent operator that commutes with T. Then*

$$\sigma_{SBF_+^-}(T+Q) = \sigma_{SBF_+^-}(T).$$

Proof. It is well known that if $T \in SBF_+(\mathcal{X})$ and K is a Riesz operator commuting with T, then $T + \lambda K \in SBF_+(\mathcal{X})$ for all $\lambda \in \mathbb{C}$. Suppose that $\lambda \notin \sigma_{SBF_+^-}(T)$. There is no harm if we suppose that $\lambda = 0$. Then $T \in SBF_+^-(\mathcal{X})$ and hence $T + \mu Q \in SBF_+(\mathcal{X})$ for all $\mu \in \mathbb{C}$. Clearly, T and $T + Q$ belong to the same component of the open set $SBF_+(\mathcal{X})$, so $ind(T) = ind(T + Q) \leq 0$, and hence $0 \notin \sigma_{SBF_+^-}(T + Q)$. This shows $\sigma_{SBF_+^-}(T + Q) \subseteq \sigma_{SBF_+^-}(T)$. By symmetry then

$$\sigma_{SBF_+^-}(T) = \sigma_{SBF_+^-}(T + Q - Q) \subseteq \sigma_{SBF_+^-}(T + Q),$$

so the equality $\sigma_{SBF_+^-}(T + Q) = \sigma_{SBF_+^-}(T)$ is proved. ∎

Theorem 2.18. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ and Q an injective quasi-nilpotent operator that commutes with T. If T satisfies property* (gw)*, then $T + Q$ satisfies property* (gw).

Proof. Since T satisfies property (gw) from Theorem 2.17 we have

$$\sigma_a(T + Q) \setminus \sigma_{SBF_+^-}(T + Q) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E(T). \tag{2.3}$$

To show property (gw) for $T + Q$ it suffices to prove that

$$E(T) = E(T + Q) = \emptyset.$$

Suppose that $E(T) \neq \emptyset$ and let $\lambda \in E(T)$. From Equation 2.3 we know that $T - \lambda I \in SBF_+^-(\mathcal{X})$, and hence by Lemma 2.11 of [7] it then follows that $\alpha(T - \lambda I) = 0$, a contradiction.

To show that $E(T + Q) = \emptyset$. Suppose that $E(T + Q) \neq \emptyset$ and let $\lambda \in E(T + Q)$. Then $\alpha(T + Q - \lambda I) > 0$ so there exists $x \neq 0$ such that $(T + Q - \lambda I)x = 0$. Since Q commutes with $T + Q - \lambda I$ then by Lemma 2.11 of [7] it follows that $\alpha(T + Q - \lambda I) = 0$, a contradiction. ∎

Theorem 2.19. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ is polaroid, $N \in \mathbf{B}(\mathcal{X})$ a nilpotent operator commuting with T.*
(i) *If T has SVEP then $T^* + N^*$ satisfies property* (gw)*, or equivalently generalized a-Weyls theorem holds for $T^* + N^*$.*
(ii) *If T^* has SVEP then $T + N$ satisfies property* (gw)*, or equivalently generalized a-Weyls theorem holds for $T + N$.*

Proof. (i) If T has SVEP then $T + N$ has SVEP, see Corollary 2.12 of [3]. Moreover, by Theorem 2.10 of [9] $T + N$ is polaroid. By Theorem 2.10 of [16] it then follows that property (gw) holds for $T^* + N^*$, or equivalently, since $T + N$ has SVEP, generalized a-Weyls theorem holds for $T^* + N^*$.
(ii) If T is polaroid then by Theorem 2.5 of [9] T^* is polaroid. Clearly, N^* is nilpotent, since $(N^*)^n = (N^n)^*$ for some $n \in \mathbb{N}$. Therefore, $T^* + N^*$ is polaroid, by Theorem 2.10 of [9]. Since $T^* + N^*$ has SVEP, by Corollary 2.12 of [3], it then follows, by Theorem 2.10 of [16], that $T + N$ satisfies property (gw), or equivalently generalized a-Weyls theorem holds for $T + N$. ∎

Theorem 2.20. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ is polaroid, $N \in \mathbf{B}(\mathcal{X})$ a nilpotent operator commuting with T. If T^* has SVEP and $f \in Hol(\sigma(T))$ then property* (gw) *holds for $f(T) + N$, or equivalently generalized a-Weyls theorem holds for $f(T) + N$.*

Proof. By Theorem 2.10 of [16], T satisfies property (gw), or equivalently, by Theorem 2.7 of [16] generalized a-Weyls theorem holds for T. The SVEP for T^* implies that $\sigma(T) = \sigma_a(T)$, so every isolated point of $\sigma_a(T)$ is a pole of the resolvent of T. It follows from [16, Theorem 2.11] that property (gw) holds for $f(T)$. Finally, by Theorem 2.16 $f(T) + N$ satisfies property (gw). Since $f(T^*) = f(T)^*$ has the SVEP, see [3, Theorem 2.40], by Theorem 2.7 of [16] it then follows that property (gw) and generalized a-Weyls theorem are equivalent. ■

Remark A. It is somewhat meaningful to ask what we can say about the operators $f(T+N)$, always under the assumptions of Theorem 2.20. Now, if T is polaroid then $T+N$ is polaroid, by Theorem 2.10 of [9]. Moreover, by $T^* + N^* = (T+N)^*$ has SVEP by Corollary 2.12 of [3]. Hence by [16, Thoeorem 2.11] $f(T+N)$ satisfies property (gw) for every $f \in Hol(\sigma(T))$.

Theorem 2.21. *Suppose that* $iso\sigma_a(T) = \emptyset$. *If* T *satisfies property* (gw) *and* F *is a finite rank operator commuting with* T, *then* $T + F$ *satisfies property* (gw).

Proof. By Theorem 2.3 T satisfies generalized a-Browder's theorem, it follows from [37, Theorem 2.1] that $T + F$ satisfies generalized a-Browder's theorem. By Lemma 2.6 of [8], $\sigma_a(T+F) = \sigma_a(T)$, by Lemma 2.13 we have $\pi^a(T+F) \subseteq E(T+F)$.
It is easily seen that $E(T+F)$ is empty. Indeed, suppose that $E(T+F) \neq \emptyset$. Let $\lambda \in E(T+F)$. By assumption $\lambda \in iso\sigma(T+F)$ and $\alpha(T+F-\lambda I) > 0$. Clearly, λ is an isolated of $\sigma_a(T+F) = \sigma_a(T)$, and this is impossible since $iso\sigma_a(T) = \emptyset$. Therefore, $E(T+F) = \pi^a(T+F) = \emptyset$, so by Theorem 2.3 $T+F$ satisfies property (gw). ■

Theorem 2.22. *Suppose that* $T \in \mathbf{B}(\mathcal{X})$ *is isoloid and* F *is a finite rank operator commuting with* T.
(i)*If* T^* *has SVEP and* T *satisfies property* (gw), *then* $T + F$ *satisfies property* (gw).
(ii) *If* T *has SVEP and* T^* *satisfies property* (gw), *then* $T^* + F^*$ *satisfies property* (gw).

Proof. (i) The SVEP of T^* implies that $\sigma(T) = \sigma_a(T)$. Since T satisfies property (gw) then T satisfies generalized Weyl's theorem, so it follows from Lemma 3.2 of [23] that T is polaroid. By Lemma 2.9 of [23], $T+F$ is polaroid. Since $T^* + F^* = (T+F)^*$ has SVEP by Theorem 2.14 of [9]. Therefore, property (gw) holds for $T+F$ by Theorem 2.10 of [16].
(ii)The argument is analogous to that of part (i). The SVEP of T implies that $\sigma(T^*) = \sigma_a(T^*)$. Since T^* satisfies property (gw) then T^* satisfies generalized Weyl's theorem, so it follows from Lemma 3.2 of [23] that T^* is polaroid. By Lemma 2.9 of [23], $T^* + F^*$ is polaroid. Since $(T+F)$ has SVEP by Theorem 2.14 of [9]. Therefore, property (gw) holds for $= (T+F)^* = T^* + F^*$ by Theorem 2.10 of [16]. ■

Theorem 2.23. *Suppose that* $T \in \mathbf{B}(\mathcal{X})$ *is polaroid and* K *is a finite rank operator commuting with* T.
(i) *If* T^* *has SVEP then* $f(T) + K$ *satisfies property* (gw) *for every* $f \in Hol(\sigma(T))$.
(ii) *If* T *has SVEP then* $f(T^*) + K^*$ *satisfies property* (gw) *for every* $f \in Hol(\sigma(T))$.

Proof. (i) By [3, Corollary 2.45] the SVEP of T^* implies $\sigma(T) = \sigma_a(T)$. Since T is polaroid, by Theorem 2.11 of [16] it then follows that $f(T)$ has property (gw) for every $f \in Hol(\sigma(T))$. Now, by Theorem 2.40 of [3] $f(T^*) = f(T)^*$ has SVEP, so that, by Theorem 2.7 of [16] generalized a-Weyl's theorem holds for $f(T)$. Since $f(T)$ and K commutes, T is a-polaroid, by Theorem 3.2 of [10] and Corollary 3.10 of [23] we then obtain $f(T) + K$ satisfies generalized a-Weyl's theorem. By Lemma 2.8 of [8] $f(T^*) + K^* = (f(T) + K)^*$ has SVEP. This implies that property (gw) and generalized a-Weyl's theorem for $f(T) + K$ are equivalent, again by Theorem 2.7 of [16], so the proof is achieved.
(ii) The argument is analogous to that of part (i). Just observe that $\sigma(T^*) = \sigma_a(T^*)$ by [3, Corollary 2.45], so that T^* is a-polaroid. Moreover, by Theorem 2.11 of [16] it then follows that $f(T^*)$ has property (gw) for every $f \in Hol(\sigma(T))$. By Theorem 2.40 of [3] $f(T)$ has SVEP, so that, by Theorem 2.7 of [16] generalized a-Weyl's theorem holds for $f(T^*)$. Since $f(T^*)$ and K^* commutes, by Theorem 3.2 of [10] and Corollary 3.10 of [23] we then obtain $f(T) + K$ satisfies generalized a-Weyl's theorem. By Lemma 2.8 of [8] $f(T) + K$ has SVEP, so that property (gw) and generalized a-Weyl's theorem for $f(T^*) + K^*$ are equivalent, by Theorem 2.7 of [16]. ■

A bounded linear operator T on a Hilbert space $\mathcal{H}$ is said to be quasi-class A if

$$T^*|T^2|T \geq T^*|T|^2T.$$

The *quasi-class A* operators were introduced , and their properties were studied in [34]. (see also [30, 43, 44]). In particular, it was shown in [34] that the class of *quasi-class A* operators contains properly classes of *class A* and *p-quasihyponormal* operators. *Quasi-class A* operators were independently introduced by Jeon and Kim [34]. They gave an example of a *quasi-class A* operator which is not *paranormal* nor *normaloid*. Jeon and Kim example show that neither the class *paranormal* operators nor the class of *quasi-class A* contains the other. A bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to be algebraically quasi-class A if there exists a non-trivial polynomial h such that $h(T)$ is quasi-class A, see [17]. It is shown in [17] operators of algebraically quasi-class A are polaroid and has SVEP.

Corollary 2.24. *Suppose that $T \in \mathbf{B}(\mathcal{H})$, $\mathcal{H}$ is a Hilbert space and K is a finite rank operator commuting with T.*
(i) *If T^* is an algebraically quasi-class A then $f(T) + K$ satisfies property* (gw) *for every $f \in Hol(\sigma(T))$.*
(ii) *If T is an algebraically quasi-class A then $f(T^*) + K^*$ satisfies property* (gw) *for every $f \in Hol(\sigma(T))$.*

In general, property (gw) is not transmitted under commuting finite rank perturbation.

Example 2.25. Let $S : \ell^2 \longrightarrow \ell^2$ be an injective quasinilpotent operator which is not nilpotent and let $U : \ell^2 \longrightarrow \ell^2$ be defined by $U(x_1, x_2, \cdots) := (-x_1, 0, \cdots)$, $x_n \in \ell^2(\mathbb{N})$. Define on $\mathcal{X} := \ell^2 \oplus \ell^2$ the operators T and K by $T := I \oplus S$ where I is the identity on ℓ^2 and $K := U \oplus 0$.
It is easily that $\sigma(T) = \{0, 1\}$, $E(T) = \{1\}$ and $\sigma_{Bw}(T) = \{0\}$. Hence T satisfies

generalized Weyl's theorem. Now K is finite rank operator and $TK = KT$, and since T^* has a finite spectrum then T^* has SVEP and consequently property (gw) holds for T. Moreover, $\sigma(T+K) = \{0,1\}$ and $E(T+K) = \{0,1\}$. As $\sigma_{Bw}(T+K) = \sigma_{Bw}(T) = \{0\}$, Then $T+K$ does not satisfy generalized Weyl's theorem and hence $T+K$ does not has the property (gw) by Theorem 2.7 of [16].

Example 2.26. This example shows that the commutativity hypothesis in Theorem 2.18 is essential. Let $\mathcal{X} = \ell^2(\mathbb{N})$ and T and F be defined by

$$T(x_1, x_2, \cdots) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \cdots), \quad \{x_n\} \in \ell^2(\mathbb{N})$$

and

$$F(x_1, x_2, \cdots) := (0, \frac{-x_1}{2}, 0, \cdots), \quad \{x_n\} \in \ell^2(\mathbb{N})$$

Clearly, F is a nilpotent operator and hence of finite rank operator, and T is a quasi-nilpotent satisfying generalized Weyl's theorem since $\sigma(T) = \sigma_{Bw}(T) = \{0\}$ and $E(T) = \emptyset$. Now T and F do not commute, $\sigma(T+F) = \sigma_W(T+F) = E_0(T+F) = \{0\}$, and $T+F$ does not satisfy Weyl's theorem. So $T+F \notin gW$ and hence $T+F$ does not satisfy property (gw).

The basic role of SVEP arises in local spectral theory since for all decomposable operators both T and T^* have SVEP. Every generalized scalar operator on a Banach space is decomposable (see [39] for relevant definitions and results). In particular, every spectral operators of finite type is decomposable.

Corollary 2.27. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ is generalized scalar and K is a finite rank operator commuting with T. Then property* (gw) *holds for both $f(T)+K$ and $f(T^*)+K^*$. In particular, this is true for every spectral operator of finite type.*

Proof. Both T and T^* have SVEP. Moreover, every generalized scalar operator is polaroid. The second statement is clear: every spectral operators of finite type is generalized scalar. ∎

Recall that a bounded operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set. A nilpotent operator is a trivial example of an algebraic operator. Also finite rank operators K are algebraic; more generally, if K^n is a finite rank operator for some $n \in \mathbb{N}$ then K is algebraic. Clearly, if T is algebraic then its dual T^* is algebraic.

Theorem 2.28. *Suppose that $T \in \mathbf{B}(\mathcal{X})$ and $K \in \mathbf{B}(\mathcal{X})$ is an algebraic operator commuting with T.*
(i) *If $T \in P(\mathcal{X})$ then property* (gw) *holds for $T^* + K^*$.*
(ii) *If $T^* \in P(\mathcal{X})$ then property* (gw) *holds for $T+K$.*

Proof. (i) If $T \in P(\mathcal{X})$ then T has SVEP and hence $T+K$ has SVEP by Theorem 2.14 of [9]. Moreover, T is polaroid so also $T+K$ is polaroid by Theorem 2.14 of [9]. By Theorem 2.10 of [16], then property (gw) holds for T^*+K^*.
(ii) If $T^* \in P(\mathcal{X})$ then T^* has SVEP and hence T^*+K^* has SVEP by Theorem 2.14 of [9]. Moreover, T^* is polaroid so also T^*+K^* is polaroid by Theorem 2.14 of [9]. By Theorem 2.10 of [16], then property (gw) holds for $T+K$. ∎

A bounded linear operator T on a Banach space $\mathcal{X}$ is said to be paranormal if

$$\|Tx\|^2 \leq \left\|T^2x\right\| \|x\| \quad \text{holds for all } x \in \mathcal{X}.$$

The class of paranormal operators properly contains a relevant number of Hilbert space operators, among them p-hyponormal operators, log-hyponormal operators, and the class A operators. Note that, in general, paranormal operators do not satisfy property $H(p)$, see [13] for a counter-example. A bounded operator $T \in \mathbf{B}(\mathcal{X})$ is said to be *algebraically paranormal* if there exists a non-trivial polynomial h such that $h(T)$ is paranormal. Note that every paranormal operator on a Hilbert space $\mathcal{H}$ has SVEP, see [9, Page 1799]. Moreover, algebraically paranormal operators are polaroid.

Corollary 2.29. *Suppose that* $T \in \mathbf{B}(\mathcal{H})$*,* $\mathcal{H}$ *is a Hilbert space and* $K \in \mathbf{B}(\mathcal{X})$ *is an algebraic operator commuting with* T *.*
(i) *If* T *is algebraically paranormal then property* (gw) *holds for* $T^* + K^*$*.*
(ii) *If* T^* *is algebraically paranormal then property* (gw) *holds for* $T + K$*.*

Proof. Proceed as in the proof of Theorem 2.28. ∎

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