Gorenstein homological dimension and Ext-depth of modules*

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Abstract

Let $(R, \mathfrak{m}, k)$ be a commutative Noetherian local ring. It is well-known that R is regular if and only if the flat dimension of k is finite. In this paper, we show that R is Gorenstein if and only if the Gorenstein flat dimension of k is finite. Also, we will show that if R is a Cohen-Macaulay ring and M is a Tor-finite R-module of finite Gorenstein flat dimension, then the depth of the ring is equal to the sum of the Gorenstein flat dimension and Ext-depth of M. As a consequence, we get that this formula holds for every syzygy of a finitely generated R-module over a Gorenstein local ring.

1 Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring with non-zero identity. In **[16]**, Sharif and Yassemi have introduced Tor-finite R-modules. The R-module M is called Tor-finite if for any finitely generated R-module N, each $\mathrm{Tor}_i^R(N, M)$ for all $i \geq 1$ is finitely generated. Obviously every finitely generated R-module is Tor-finite and it is easy to see that every syzygy of a Tor-finite module is also Tor-finite. Enochs, Jenda and Torrecillas **[9]** defined and studied Gorenstein flat modules. Now recall that an R-module M is said to be Gorenstein flat if there exists an exact sequence

$$\ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \ldots$$

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of flat R-modules with $M = \ker(F^0 \longrightarrow F^1)$ such that for any injective R-module E, $E \otimes_R -$ leaves the sequence exact. We say that an R-module M has Gorenstein flat dimension at most t, denoted $\operatorname{Gfd} M \leq t$, if there is an exact sequence

$$0 \longrightarrow T_t \longrightarrow T_{t-1} \longrightarrow \ldots \longrightarrow T_1 \longrightarrow T_0 \longrightarrow M \longrightarrow 0$$

with each T_i Gorenstein flat. If there is no shorter such sequence, we set $\operatorname{Gfd} M = t$. Also, if there is no such a t, we set $\operatorname{Gfd} M = \infty$. We note that the notion of Gorenstein flat generalizes flat and so Gorenstein flat dimension generalizes flat dimension.

We start in section 2 by studying the Gorenstein flat, Gorenstein flat dimension, cotorsion and cotorsion flat modules. Recall that an R-module C is called cotorsion if $\operatorname{Ext}^1_R(F, C) = 0$ for all flat modules F. If F is flat and cotorsion, then it was proved in [**6**] that F can be written uniquely in the form $F \cong \sqcap_{\mathfrak{p} \in \operatorname{Spec}(R)} T_{\mathfrak{p}}$, where $T_{\mathfrak{p}} \cong \operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)})$ for some set X. This result is similar to the Matlis theorem for injective modules. Also, we characterize the Gorenstein local rings by Gorenstein flat dimension of modules. We then proceed in section 3 to study Tor-finite Gorenstein flat R-modules and we show that if $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring and M is a non-flat Tor-finite R-module, then M Gorenstein flat implies that $\operatorname{Ext-depth} M = \operatorname{depth} R$ (see Proposition 3.5), where $\operatorname{Ext-depth}(M) = \inf\{i : \operatorname{Ext}^i_R(k, M) \neq 0\}$. Ext-depth is called E-depth in [**17**, Definition 5.3.6]. Also, by the above hypothesis we prove that if $\operatorname{Gfd} M < \infty$, then $\operatorname{Gfd} M + \operatorname{Ext-depth} M = \operatorname{depth} R$. This result is an improvement of the Auslander-Bridger formula (see [**1**]). To obtain these results, we will repetitively make use of the following. If $(R, \mathfrak{m}, k)$ is a local ring and $x \in \mathfrak{m}$ is an R-regular element, then we have an exact sequence $0 \longrightarrow \operatorname{Tor}^R_1(R/xR, M) \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$. If M is a Gorenstein flat R-module, then $\operatorname{Tor}^R_1(R/xR, M) = 0$ since $\operatorname{fd} R/xR \leq 1$ (see [**4**, Lemma 3.3]). So if x is R-regular, then x is M-regular for any Gorenstein flat module.

2 Gorenstein flat dimension

We start this section with the following lemma.

Lemma 2.1. *Let M be an R-module with finite Gorenstein flat dimension. Then the following are equivalent:*
(i) M is Gorenstein flat;
(ii) $\operatorname{Ext}^i_R(M, F) = 0$ *for all cotorsion flat modules F and all $i \geq 1$;*
(iii) $\operatorname{Ext}^1_R(M, F) = 0$ *for all cotorsion flat modules F;*
(iv) $\operatorname{Ext}^i_R(M, L) = 0$ *for all cotorsion modules L with finite flat dimension and all $i \geq 1$.*

Proof. $(i) \Longrightarrow (ii)$. Let F be a cotorsion flat module. Then F is a summand of a module $\operatorname{Hom}_R(E, E')$ where E and E' are injective (see [**6**, Lemma 2.3]). Hence it is enough to prove that $\operatorname{Ext}^i_R(M, \operatorname{Hom}_R(E, E')) = 0$ for all $i \geq 1$. By the following isomorphisms $\operatorname{Ext}^i_R(M, \operatorname{Hom}_R(E, E')) \cong \operatorname{Hom}_R(\operatorname{Tor}^R_i(M, E), E')$ and by using [**12**, Theorem 3.6], we have $\operatorname{Ext}^i_R(M, \operatorname{Hom}_R(E, E')) = 0$ for all $i \geq 1$.
$(ii) \Longrightarrow (iii)$ is trivial.

$(iii) \Longrightarrow (i)$. Let $I = \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$. Then I is injective cogenerator for all R-modules and hence, by [**6**, Lemma 2.1], $\operatorname{Hom}_R(E, I)$ is a cotorsion flat module for all injective R-modules E. Therefore, by the isomorphism $\operatorname{Ext}^1_R(M, \operatorname{Hom}_R(E, I)) \cong \operatorname{Hom}_R(\operatorname{Tor}^R_1(M, E), I)$, we have $\operatorname{Tor}^R_1(M, E) = 0$ for all injective R-modules E. Hence, by [**12**, Theorem 3.14], M is Gorenstein flat.
$(iv) \Longrightarrow (ii)$ is trivial.
$(ii) \Longrightarrow (iv)$. Let L be a cotorsion module with $\operatorname{fd} L = n$. We use induction on n. If $n = 0$, then L is cotorsion flat and there is nothing to prove. Now, we assume that $\operatorname{fd} L = n > 0$. Let F be a flat cover of L with kernel K such that F is flat and cotorsion (see [**6**, Corollary 2.2]). Then, by the exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow L \longrightarrow 0$$

where K is cotorsion and of flat dimension $n-1$, we have the following exact sequence

$$\operatorname{Ext}^i_R(M, K) \longrightarrow \operatorname{Ext}^i_R(M, F) \longrightarrow \operatorname{Ext}^i_R(M, L) \longrightarrow \operatorname{Ext}^{i+1}_R(M, K)$$

for all $i \geq 1$. Hence, by induction hypothesis, $\operatorname{Ext}^i_R(M, L) = 0$ for all $i \geq 1$. ∎

Theorem 2.2. *Let M be an R-module with finite Gorenstein flat dimension. Let n be a non-negative integer. Then the following are equivalent:*
(i) $\operatorname{Gfd} M \leq n$*;*
(ii) $\operatorname{Ext}^i_R(M, L) = 0$ *for all $i > n$ and all cotorsion modules L with finite flat dimension;*
(iii) $\operatorname{Ext}^i_R(M, F) = 0$ *for all $i > n$ and all cotorsion flat R-modules F;*
(iv) $\operatorname{Ext}^{n+1}_R(M, F) = 0$ *for all cotorsion flat R-modules F.*

Proof. $(i) \Longrightarrow (ii)$. By [**12**, Theorem 3.14], we have the following exact sequence

$$0 \longrightarrow K_n \longrightarrow G_{n-1} \longrightarrow \ldots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

such that $G_0, \ldots, G_{n-1}$ and K_n are Gorenstein flats. By Lemma 2.1, it is easy to see that $\operatorname{Ext}^i_R(M, L) \cong \operatorname{Ext}^1_R(K_n, L) = 0$ for all $i > n$.
$(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (iv)$ are trivial.
$(iv) \Longrightarrow (i)$ follows by [**12**, Theorem 3.14] and by using the same proof as Lemma 2.1($(iii) \Longrightarrow (i)$). ∎

Corollary 2.3. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let M be an R-module of finite Gorenstein flat dimension such that $\operatorname{Ext}^i_R(M, T_{\mathfrak{p}}) = 0$ for all $i \geq 1$ and all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Then*

$$\operatorname{Gfd} M = \sup\{i : \operatorname{Ext}^i_R(M, T_{\mathfrak{m}}) \neq 0\}.$$

Proof. Let $\operatorname{Gfd} M = t$ and K_t be a t-th syzygy of M. Hence K_t is Gorenstein flat and so $\operatorname{Ext}^i_R(M, T_{\mathfrak{m}}) \cong \operatorname{Ext}^{i-t}_R(K_t, T_{\mathfrak{m}}) = 0$ for all $i > t$. It therefore follows that $\sup\{i : \operatorname{Ext}^i_R(M, T_{\mathfrak{m}}) \neq 0\} \leq t$. We shall prove the assertion of the corollary by assuming that $\sup\{i : \operatorname{Ext}^i_R(M, T_{\mathfrak{m}}) \neq 0\} < t$ and deriving a contradiction. Let K_{t-1} be a $(t-1)$-th syzygy of M. Then $\operatorname{Ext}^i_R(K_{t-1}, T_{\mathfrak{m}}) \cong \operatorname{Ext}^{i+(t-1)}_R(M, T_{\mathfrak{m}}) = 0$ for all $i \geq 1$. Since $\operatorname{Ext}^i_R(K_{t-1}, F) = 0$ for all $i \geq 1$ and all cotorsion flat modules F and K_{t-1} has finite Gorenstein flat dimension, we have K_{t-1} Gorenstein flat and so $\operatorname{Gfd} M \leq t-1$. But this contradicts with $\operatorname{Gfd} M = t$, and so we must have $\sup\{i : \operatorname{Ext}^i_R(M, T_{\mathfrak{m}}) \neq 0\} = t$. ■

The following theorem is an improvement of [**3**, Theorem 5.2.10].

Theorem 2.4. *Let $(R, \mathfrak{m}, k)$ be a local ring. Then the following are equivalent:*
(i) R is Gorenstein;
(ii) $\operatorname{Gfd} k$ *is finite;*
(iii) $\operatorname{Gfd} M$ *is finite for any finitely generated R-module M;*
(iv) $\operatorname{Gfd} M$ *is finite for any R-module M.*

Proof. $(iv) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (ii)$ are trivial.
$(ii) \Longrightarrow (i)$. Let $t \geq 0$. One has the following isomorphism
$\operatorname{Tor}^R_t(k, \mathrm{E}(k)) \cong \operatorname{Tor}^R_t(k, \mathrm{D}\,\mathrm{D}(\mathrm{E}(k))) \cong \mathrm{D}(\operatorname{Ext}^t_R(k, \mathrm{D}(\mathrm{E}(k))) \cong \mathrm{D}(\operatorname{Ext}^t_{\hat{R}}(k, \hat{R}))$,
where $\hat{R}$ is the completion of R in $\mathfrak{m}$-adic topology and $\mathrm{D}(-) = \operatorname{Hom}_R(-, \mathrm{E}(k))$. Now, by using [**5**, P. 178], we have $\operatorname{id}_{\hat{R}} \hat{R} = \operatorname{Gfd} k < \infty$. Hence $\hat{R}$ and so R are Gorenstein.
$(i) \Longrightarrow (iv)$ follows from [**3**, Theorem 5.2.10]. ■

In the following theorem we use the notion of Gorenstein injective dimension and the notion of Gorenstein projective dimension. The reader is referred to [**8**] for more results in this direction.

Theorem 2.5. *Let R be a ring. Then the following are equivalent:*
(i) R is n-Gorenstein;
(ii) $\operatorname{Gid} M \leq n$ *for all R-modules M;*
(iii) $\operatorname{Gfd} M \leq n$ *for all R-modules M;*
(iv) $\operatorname{Gfd} M \leq n$ *for all finitely generated R-modules M;*
(v) $\operatorname{Gpd} M \leq n$ *for all R-modules M;*
(vi) $\operatorname{Gpd} M \leq n$ *for all finitely generated R-modules M.*

Proof. $(i) \Longrightarrow (ii)$. Let R be n-Gorenstein. Then, by [**8**, Theorem 10.1.13], $\operatorname{Gid} M \leq n$ for all R-modules M.
$(ii) \Longrightarrow (i)$ follows by [**13**, Theorem 2.1].
$(iii) \Longrightarrow (iv)$ and $(v) \Longrightarrow (vi)$ are trivial.
$(iii) \Longleftrightarrow (v)$ and $(iv) \Longleftrightarrow (vi)$ conclude by [**11**, Theorem B].
$(i) \Longrightarrow (iii)$ follows by [**8**, Theorem 10.3.13].
$(iii) \Longrightarrow (ii)$. By [**11**, Theorem B], [**13**, Theorem 2.6] and [**12**, Theorem 2.28] we have $\operatorname{fd} E(R/\mathfrak{m}) \leq n$ for every maximal ideal $\mathfrak{m}$ of R. Then by [**19**, Theorem 5.1.2] and [**8**, Theorem 10.1.13] the result follows.
$(iv) \Longrightarrow (i)$. Let $\operatorname{Gfd} M \leq n$ for all finitely generated modules M. Then, by [**3**, Lemma 5.1.3], $\operatorname{Gfd} M_{\mathfrak{p}} \leq n$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and all finitely generated modules M. Hence, by Theorem 2.4, $R_{\mathfrak{p}}$ and so R are n-Gorenstein. ■

3 Gorenstein flat dimension and Ext-depth

Lemma 3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring and let x be an R-regular element of $\mathfrak{m}$. If M is a Gorenstein flat R-module, then M/xM is a Gorenstein flat R/xR-module.*

Proof. Set $\bar{R} = R/xR$ and $\bar{X} = X \otimes_R \bar{R}$. Let

$$\ldots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F^0 \longrightarrow F^1 \longrightarrow \ldots$$

be a complete flat resolution of M. Then, by [**4**, Lemma 3.3],

$$\ldots \longrightarrow \bar{F}_1 \longrightarrow \bar{F}_0 \longrightarrow \bar{F}^0 \longrightarrow \bar{F}^1 \longrightarrow \ldots$$

is a complete flat resolution of the $\bar{R}$-module $\bar{M}$ since $\operatorname{fd} \bar{R} \leq 1$. We only need to show that this sequence exact when $E \otimes_{\bar{R}} -$ is applied to it for any injective $\bar{R}$-module E. But $E \otimes_{\bar{R}} \bar{F} \cong E \otimes_R F$ and $\operatorname{id}_R(E) = 1$ by [**18**, Exercise 4.3.3]. So, by [**4**, Lemma 3.3], the result follows. ∎

Lemma 3.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a Tor-finite R-module. Then* $\operatorname{fd} M = \sup\{i : \operatorname{Tor}_i^R(k, M) \neq 0\}$.

Proof. This is clear by [**16**, Theorem 2.6 and Proposition 2.5]. ∎

Lemma 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a Tor-finite R-module of finite flat dimension. Then* $\operatorname{fd} M \leq \operatorname{depth} R$.

Proof. We can assume that $\operatorname{fd} M = t \geq 1$. So, by Lemma 3.2, $\operatorname{Tor}_t^R(k, M) \neq 0$. Now, we assume that $x_1, \ldots, x_n \in \mathfrak{m}$ be a maximal R-sequence. Then $\mathfrak{m} \in \operatorname{Ass}(R/(x_1, \ldots, x_n)R)$ and so by the exact sequence

$$0 \longrightarrow k \longrightarrow R/(x_1, \ldots, x_n)R,$$

we have the exact sequence

$$0 \longrightarrow \operatorname{Tor}_t^R(k, M) \longrightarrow \operatorname{Tor}_t^R(R/(x_1, \ldots, x_n)R, M).$$

It therefore follows that $\operatorname{Tor}_t^R(R/(x_1, \ldots, x_n)R, M) \neq 0$. On the other hand $\operatorname{fd} R/(x_1, \ldots, x_n)R = n$. Then $n \geq t$ and the result follows. ∎

Theorem 3.4. *Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a Tor-finite R-module. Then* $\operatorname{Ext-depth} M \leq \dim R$.

Proof. $\operatorname{Tor-depth} M \leq \dim R$ by Lemma 3.3 and [**16**, Lemma 2.4]. Hence, by [**17**, Corollary 6.1.10], the result follows. ∎

Proposition 3.5. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let M be a Tor-finite Gorenstein flat R-module. Then* $\operatorname{Ext-depth} M = \operatorname{depth} R$.

Proof. Let $\operatorname{depth} R = n$. If $n = 0$, then the maximal ideal $\mathfrak{m}$ is nilpotent and since $\operatorname{Ass}(0 :_M \mathfrak{m}) = \operatorname{Ass}(0 :_M \mathfrak{m}^t)$ for all $t \geq 1$ we have $\operatorname{Hom}_R(R/\mathfrak{m}, M) \neq 0$. Hence $\operatorname{Ext-depth} M = 0$. Now, suppose that $n \geq 1$. Then there exists an R-regular element $x \in \mathfrak{m}$. By Lemma 3.1 and [**14**, P.140] M/xM is a Tor-finite Gorenstein flat R/xR-module. But R/xR is a Cohen-Macaulay ring of dimension $n - 1$. Hence, by induction hypothesis, $\operatorname{Ext-depth}_{R/xR}(M/xM) = n - 1$. Since x is a non-zero divisor on M, then $\operatorname{Ext-depth}_{R/xR}(M/xM) = \operatorname{Ext-depth}(M) - 1$. Therefore $\operatorname{Ext-depth}(M) = n$, as required. ∎

The following result is a dual of [**7**, Theorem 4.8].

Theorem 3.6. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let M be a non-flat Tor-finite R-module. If M is of finite Gorenstein flat dimension, then*

$$\operatorname{Gfd} M + \text{Ext-depth}\, M = \operatorname{depth} R.$$

Proof. We proceed by induction on $n = \operatorname{depth} R$. If $n = 0$, then R is complete and by [**3**, Corollary 5.2.15] $\operatorname{Gfd} M = 0$ and Ext-depth $M = 0$ by Proposition 3.5. Now, suppose that $n \geq 1$. Then there exists an R-regular element $x \in \mathfrak{m}$. By [**14**, P.140] and [**15**, Theorem 3.11] M/xM is a Tor-finite R/xR-module of finite Gorenstein flat dimension. Also, R/xR is Cohen-Macaulay ring of dimension $n-1$. Now, by induction hypothesis, we have $\operatorname{Gfd}_{R/xR}(M/xM) + \text{Ext-depth}_{R/xR}(M/xM) = \operatorname{depth} R/xR$. By using [**15**, Theorem 3.11] and [**14**, P.140], we have $\operatorname{Gfd} M + \text{Ext-depth}\, M = \operatorname{depth} R$, as required. ■

Proposition 3.7. *Let $(R, \mathfrak{m}, k)$ be a local ring and let M be a cotorsion flat R-module. Then* Ext-depth $M < \infty$ *if and only if $T_{\mathfrak{m}}$ is a summand of M. In this case,* Ext-depth $M =$ depth R.

Proof. Since M is cotorsion flat, then we have $M \cong \sqcap_{\mathfrak{p} \in \operatorname{Spec}(R)} T_{\mathfrak{p}}$ in which $T_{\mathfrak{p}} \cong \operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)})$ for some set X. Hence, by [**2**, Theorem 3],

$$\begin{aligned}\operatorname{Ext}^i_R(k, M) &\cong \operatorname{Ext}^i_R(k, \sqcap_{\mathfrak{p}\in\operatorname{Spec}(R)} T_{\mathfrak{p}}) \cong \sqcap_{\mathfrak{p}\in\operatorname{Spec}(R)} \operatorname{Ext}^i_R(k, T_{\mathfrak{p}}) \\ &\cong \sqcap_{\mathfrak{p}\in\operatorname{Spec}(R)} \operatorname{Ext}^i_R(k, \operatorname{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{p})^{(X)})) \\ &\cong \sqcap_{\mathfrak{p}\in\operatorname{Spec}(R)} \operatorname{Hom}_R(\operatorname{Tor}^R_i(k, E(R/\mathfrak{p})), E(R/\mathfrak{p})^{(X)}).\end{aligned}$$

On the other hand, if $\mathfrak{p} \neq \mathfrak{m}$, then $\operatorname{Tor}^R_i(k, E(R/\mathfrak{p})) = 0$ for all $i \geq 0$. It therefore follows $\operatorname{Ext}^i_R(k, M) \cong \sqcap_{\mathfrak{p}\in\operatorname{Spec}(R)} \operatorname{Ext}^i_R(k, \hat{R}^{(X)})$ and so if $T_{\mathfrak{m}}$ is a summand of M, then Ext-depth $M = \operatorname{depth} \hat{R}^{(X)} = \operatorname{depth} \hat{R} = \operatorname{depth} R$. If $T_{\mathfrak{m}}$ is not a summand of M, then $\operatorname{Ext}^i_R(k, M) = 0$ for all $i \geq 0$ and so Ext-depth M is infinite. ■

Theorem 3.8. *Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring and let M be a syzygy of finitely generated R-module N. If M is of finite Gorenstein flat dimension, then*

$$\operatorname{Gfd} M + \text{Ext-depth}\, M = \operatorname{depth} R.$$

Proof. It is easy to see that M is Tor-finite and so by Theorem 3.6 the result follows. ■

Theorem 3.9. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring and let M be a syzygy of a finitely generated R-module N. Then*

$$\operatorname{Gfd} M + \text{Ext-depth}\, M = \operatorname{depth} R.$$

Proof. This is immediate by Theorems 2.4 and 3.8. ■

It is a natural to ask "is there a non-flat and a non-finitely generated Tor-finite module?" The answer is positive:

Example 3.10. Let $(R, \mathfrak{m})$ be a local Gorenstein integral domain with $\dim R = 1$. Set $M = E(R/\mathfrak{m})$. $\operatorname{Tor}^R_1(R/\mathfrak{p}, M) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \neq \mathfrak{m}$ and $\operatorname{Tor}^R_1(R/\mathfrak{m}, M)$ is finitely generated and hence by [**16**, Lemma 2.1] M is Tor-finite. Whereas M is not finitely generated and not flat since $\operatorname{fd}(M) = 1$ by [**8**, P. 238].

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