

Systems of Submodules and an Isomorphism Problem for Auslander-Reiten Quivers

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Abstract

Fix a poset \mathcal{P} and a natural number n . For various commutative local rings Λ , each of Loewy length n , consider the category $\text{sub}_\Lambda \mathcal{P}$ of Λ -linear submodule representations of \mathcal{P} . We give a criterion for when the underlying translation quiver of a connected component of the Auslander-Reiten quiver of $\text{sub}_\Lambda \mathcal{P}$ is independent of the choice of the base ring Λ . If \mathcal{P} is the one-point poset and $\Lambda = \mathbb{Z}/p^n$, then $\text{sub}_\Lambda \mathcal{P}$ consists of all pairs $(B; A)$ where B is a finite abelian p^n -bounded group and $A \subset B$ a subgroup. We can respond to a remark by M. C. R. Butler concerning the first occurrence of parametrized families of such subgroup embeddings.

1 Introduction.

Let Λ be a commutative (local artinian) uniserial ring, for example $\Lambda = k[x]/x^n$ where k is a field or $\Lambda = \mathbb{Z}/p^n$ where p is a prime number, and let \mathcal{P} be a finite partially ordered set (poset). We consider systems $(M_*, (M_i)_{i \in \mathcal{P}})$ where M_* is a finitely generated Λ -module and each $M_i \subset M_*$ is a submodule such that $M_i \subset M_j$ holds for $i < j$ in \mathcal{P} . The systems $(M_*, (M_i)_{i \in \mathcal{P}})$ form the objects of a category $\text{sub}_\Lambda \mathcal{P}$; here homomorphisms from $(M_*, (M_i)_{i \in \mathcal{P}})$ to $(N_*, (N_i)_{i \in \mathcal{P}})$ are given by Λ -linear maps $f : M_* \rightarrow N_*$ which satisfy $f(M_i) \subset N_i$ for each $i \in \mathcal{P}$. This defines an additive category which has the Krull-Remak-Schmidt property, so any object has a unique decomposition into indecomposable ones. Thus, the focus is on the classification of the indecomposable objects and a description of the homomorphisms between them.

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Given two commutative uniserial rings Λ and Δ of the same Loewy length n , then one may ask how the categories $\text{sub}_\Lambda \mathcal{P}$ and $\text{sub}_\Delta \mathcal{P}$ are related: Do they have the same number of isomorphism classes of indecomposable objects? If so, is there a bijection between the isomorphism classes which preserves (A) combinatorial data that describe the objects and (B) combinatorial data which describe the homomorphisms between them? We will describe objects in terms of their *type*: For a Λ -module $M_* = \Lambda/p^{m_1} \oplus \cdots \oplus \Lambda/p^{m_t}$, p a radical generator for Λ , the type is the partition $t(M_*) = (m_1, \dots, m_t)$; then for a system $M = (M_*, (M_i))$, the type is given by the tuple $t(M) = (t(M_*), t(M_i))$. For the description of homomorphisms we consider the combinatorial data given by the Auslander-Reiten quiver of the category.

The categories of type $\text{sub}_\Lambda \mathcal{P}$ studied for example in [12], [14] and [17] have finite representation type; it is shown there that the classification of the indecomposable objects does not depend on the choice of the base ring Λ . On the other hand covering theory, which is used for example in [16] and [18] for certain categories of finite and tame infinite representation type delivers both the full list of the indecomposable modules and the structure of the Auslander-Reiten quiver. However, this method requires that the uniserial base ring Λ is an algebra over a field, hence of type $\Lambda = k[x]/x^n$.

If Δ is another commutative uniserial ring of length n , then we can use the main result in this manuscript as a criterion for the existence of a bijection between the indecomposable objects in $\text{sub}_\Lambda \mathcal{P}$ and $\text{sub}_\Delta \mathcal{P}$ which preserves (A) and (B).

THEOREM 1.1. *Suppose Λ, Δ are commutative uniserial rings of the same length n , \mathcal{P} is a finite poset, Γ_Λ and Γ_Δ are connected components of the Auslander-Reiten quivers of $\text{sub}_\Lambda \mathcal{P}$ and $\text{sub}_\Delta \mathcal{P}$, and \mathcal{L}_Λ and \mathcal{L}_Δ are slices in Γ_Λ and Γ_Δ , respectively, such that the following conditions are satisfied:*

1. *The slices \mathcal{L}_Λ and \mathcal{L}_Δ are isomorphic as valued graphs.*
2. *Points in \mathcal{L}_Λ and \mathcal{L}_Δ which correspond to each other under this isomorphism represent indecomposable objects of the same type.*
3. *Each indecomposable object represented by a point in \mathcal{L}_Λ or in \mathcal{L}_Δ is determined uniquely, up to isomorphism, by its type.*

Then the components Γ_Λ and Γ_Δ are isomorphic as valued translation quivers, and points which correspond to each other under this isomorphism represent indecomposable objects of the same type.

Subgroups of abelian groups. Our research was motivated by a remark by M.C.R. Butler after the authors talk on categories of type $\mathcal{S}_m(k[x]/x^n)$ at the ICRA XI in 2004 in Patzcuaro, Mexico. For Λ a commutative uniserial ring of length n and $m \leq n$, the category $\mathcal{S}_m(\Lambda)$ consists of all pairs $(A \subset B)$ where B is a (finitely generated) Λ -module and A a submodule of B which is annihilated by $\text{rad}^m \Lambda$. Thus $\mathcal{S}_n(\Lambda) = \text{sub}_\Lambda \mathcal{P}$ for \mathcal{P} the one point poset, and for each $m < n$, $\mathcal{S}_m(\Lambda)$ is a full subcategory. Using coverings, it has been shown in [18] for which pairs (m, n) the category $\mathcal{S}_m(\Lambda)$ where $\Lambda = k[x]/x^n$ has finite, tame or wild representation type. It turns out that the two pairs where $(m, n) = (4, 6)$ or $(3, 7)$ mark the first occurrences of parametrized families of indecomposable objects.

Since Birkhoff [8] the subgroup categories $\mathcal{S}_m(\Lambda)$ where $\Lambda = \mathbb{Z}/p^n$ for p a prime number have attracted a lot of interest. Here we are dealing with the possible embeddings of p^m -bounded subgroups in p^n -bounded finite abelian groups. It is of particular interest, as Butler points out, to detect the first occurrences of parametrized families of indecomposable objects in the case of subgroup embeddings. It is shown in [8] and in [18] that parametrized families of indecomposable objects do occur in $\mathcal{S}_4(\mathbb{Z}/p^6)$ and in $\mathcal{S}_3(\mathbb{Z}/p^7)$, respectively. But are these the *first* occurrences (as in the case of modules over the polynomial ring)? In the group case we know from [14] that the category $\mathcal{S}_5(\mathbb{Z}/p^5)$ has finite type, so there are no parametrized families of indecomposables in any of the categories $\mathcal{S}_m(\mathbb{Z}/p^n)$ for $m \leq n \leq 5$. Also if $m \leq 2$ then it is known in group theory [7] that for any n , each indecomposable object ($A \subset B$) in $\mathcal{S}_m(\mathbb{Z}/p^n)$ is determined uniquely by its type — again there are no families. So there remains a single case to decide, namely where $m = 3$ and $n = 6$, and this will be our main example.

Related results. For previous results in the representation theory of a Λ -category where the base ring Λ is not necessarily a field or a finite dimensional algebra, see in particular the article [12] by Plahotnik from 1976 which classifies the posets of finite representation type over a commutative uniserial ring Λ ; [21] for the case where \mathcal{P} is a chain; [6] for modules over the group ring $\mathbb{Z}/p^n C_p$; and [17] for lattices over tiled orders. In [13] it is shown that for suitable categorical ideals \mathcal{I} and \mathcal{J} , the categories $\mathcal{S}_2(k[x]/x^n)/\mathcal{I}$ and $\mathcal{S}_2(\mathbb{Z}/p^n)/\mathcal{J}$ are equivalent categories. Moreover, according to [15], the category $\mathcal{S}_m(\Lambda)$ is controlled wild whenever the base ring Λ has length at least 7 and $m \geq 4$.

The contents of each section. In order to reconstruct the underlying (valued) translation quiver of an Auslander-Reiten quiver from a given slice, it is essential that projective objects, injective objects, the summands of the first term in a sink map for a projective object, and the summands of the last term of a source map for an injective object can be detected by their types. This is the case, as we see in the next section if Λ is a commutative uniserial ring and we are dealing with a category of type $\text{sub}_\Lambda \mathcal{P}$ or $\mathcal{S}_m(\Lambda)$.

Given a short split exact sequence, then clearly, the type of the middle term is the union of the types of the end terms. This is also true for almost split sequences, with finitely many exceptions. In Section 3 we describe these exceptional sequences. It turns out that the starting and end terms of these sequences are indecomposable objects determined uniquely by their types.

As a consequence if Γ is a connected component of the Auslander-Reiten quiver and \mathcal{L} a slice in Γ , then the type of each module in Γ is determined uniquely by the types of the modules in \mathcal{L} . This implies, as we show in Section 4 that whenever the conditions in Theorem 1.1 apply, the combinatorial structure of the component of the Auslander-Reiten quiver does not depend on the base ring chosen. For illustration we consider the category of all chains of length 3 of Λ -modules where $\Lambda = k[x]/x^2$ or $\Lambda = \mathbb{Z}/p^2$. This case is particularly straightforward as the slice is obtained from the sequences studied in Section 3.

In general it may require a lot of legwork to verify that a particular short exact sequence is almost split. A test for Auslander-Reiten sequences from [4] applies also

to our situation, as we show in Section 5. With the help of this test we obtain the slice needed in Theorem 1.1 to demonstrate that the structure of the Auslander-Reiten quiver for $\mathcal{S}_3(\Lambda)$ is independent of the choice of the commutative uniserial ring Λ of length 6.

Notation and Remarks. For terminology related to Auslander-Reiten sequences, relative homological algebra and coverings we refer the reader to [4], [5], and [9]. For classical results in the representation theory of a poset with coefficients in a field we refer to the monographs [20], [11], and [2]. Some of the results in this manuscript have been presented at the 2005 Oberwolfach meeting on representation theory of finite dimensional algebras [19], and at the 2005 regional meeting of the AMS in Santa Barbara.

We are grateful for the recommendations by the referee which have improved the exposition of the results. He has also pointed our attention to [3] which contains a version of our Theorem 1.1. However, it appears to us that the functor \tilde{G} in [3] which should induce the isomorphism of the Auslander-Reiten quivers ([3, Corollary 2.5]) does not have the properties claimed. In the Appendix we demonstrate that \tilde{G} does not necessarily preserve indecomposable objects, thus providing a counter example to [3, Theorem 2.4].

2 Projectives and injectives.

We introduce the submodule category $\text{sub}_\Lambda \mathcal{P}$ and determine the indecomposable projective and injective objects and their respective sink and source maps. In case Λ is a field, then $\text{sub}_\Lambda \mathcal{P}$ is just the category \mathcal{P} -sp of \mathcal{P} spaces. We are interested in the situation where Λ is a commutative uniserial ring. Then the projective and the injective indecomposable objects, and the last and the first term of each source and sink map, respectively, are indecomposable modules (or zero) determined uniquely by their type.

For \mathcal{P} a finite poset, denote by \mathcal{P}^* and by \mathcal{P}^0 the poset obtained from \mathcal{P} by adding a largest and a smallest element, respectively. The incidence algebra $\Lambda \mathcal{P}^*$ of the poset \mathcal{P}^* has free Λ -basis $\{q_{ij} \mid i, j \in \mathcal{P}^*, i \leq j\}$ and multiplication is given by the formula $q_{ij}q_{kl} = \delta_{jk}q_{il}$. We identify $\text{sub}_\Lambda \mathcal{P}$ with the full subcategory of $\text{mod } \Lambda \mathcal{P}^*$ of all (right) modules M such that for each pair $i, j \in \mathcal{P}^*, i \leq j$, the multiplication map $Mq_{ii} \rightarrow Mq_{jj}, m \mapsto mq_{ij}$, is monic. Each indecomposable projective $\Lambda \mathcal{P}^*$ -module has the form $q_{ii}\Lambda \mathcal{P}^*$ for some $i \in \mathcal{P}^*$ and hence the module itself and its radical both lie in the subcategory $\text{sub}_\Lambda \mathcal{P}$.

Similarly we identify $\text{fac}_\Lambda \mathcal{P}$, the category of all finitely generated Λ -linear factor space representations of \mathcal{P} , with the full subcategory of $\text{mod } \Lambda \mathcal{P}^0$ of all modules for which each of the maps $Mq_{ii} \rightarrow Mq_{jj}, m \mapsto mq_{ij}$, where $i, j \in \mathcal{P}^0, i \leq j$, is onto. The self duality $\text{Hom}_\Lambda(-, I) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda$ given by the injective envelope $I = E(\Lambda/\text{rad } \Lambda)$ [1, 30.6] gives rise to a duality $\text{mod } \Lambda(\mathcal{P}^0)^{\text{op}} \rightarrow \text{mod } \Lambda \mathcal{P}^0$. Hence the injective $\Lambda \mathcal{P}^0$ -modules are under control; they and their socle factors all lie in the subcategory $\text{fac}_\Lambda \mathcal{P}$.

Neither $\text{sub}_\Lambda \mathcal{P}$ nor $\text{fac}_\Lambda \mathcal{P}$ is an abelian category, unless \mathcal{P} is the empty poset. However, both $\text{sub}_\Lambda \mathcal{P}$ and $\text{fac}_\Lambda \mathcal{P}$ are full exact subcategories of the module cate-

gories $\text{mod } \Lambda \mathcal{P}^*$ and $\text{mod } \Lambda \mathcal{P}^0$, respectively.

LEMMA 2.1. 1a) Every projective $\Lambda \mathcal{P}^*$ -module is a projective subspace representation of \mathcal{P} .

1b) Every injective $\Lambda \mathcal{P}^0$ -module is an injective factorspace representation of \mathcal{P} .

2a) For each indecomposable projective object P in $\text{sub}_\Lambda \mathcal{P}$, the sink map has the form $\text{rad } P \rightarrow P$.

2b) For each indecomposable injective object I in $\text{fac}_\Lambda \mathcal{P}$, the source map has the form $I \rightarrow I/\text{soc } I$.

3a) There are sufficiently many projective objects in $\text{sub}_\Lambda \mathcal{P}$.

3b) There are sufficiently many injective objects in $\text{fac}_\Lambda \mathcal{P}$. ■

In order to obtain information about the injective objects in $\text{sub}_\Lambda \mathcal{P}$ and the projective objects in $\text{fac}_\Lambda \mathcal{P}$ we use that the two categories are in fact equivalent. In the case of \mathcal{P} -vector spaces, this equivalence is just the reflection duality studied in [20, Section 5.2].

LEMMA 2.2. There is an equivalence of categories between $\text{sub}_\Lambda \mathcal{P}$ and $\text{fac}_\Lambda \mathcal{P}$.

Proof. The equivalence is given by functors

$$E : \text{sub}_\Lambda \mathcal{P} \rightarrow \text{fac}_\Lambda \mathcal{P} \quad \text{and} \quad E' : \text{fac}_\Lambda \mathcal{P} \rightarrow \text{sub}_\Lambda \mathcal{P}$$

where E maps a system $(M_*, (M_i)_{i \in \mathcal{P}})$ of submodules of M_* to the system

$$(M_0, (M_0 \rightarrow M_0/M_i)_{i \in \mathcal{P}})$$

of the cokernel maps where $M_0 = M_*$. Conversely, E' sends a system $(M_0, (M_0 \rightarrow M_i)_{i \in \mathcal{P}})$ to the object $(M_*, \ker(M_0 \rightarrow M_i)_{i \in \mathcal{P}})$ given by the kernel maps where $M_* = M_0$. ■

Using this equivalence, we obtain the structure of the injective objects in $\text{sub}_\Lambda \mathcal{P}$. We omit the corresponding assertions about the projective objects in $\text{fac}_\Lambda \mathcal{P}$.

COROLLARY 2.3. 1. Each injective object in $\text{sub}_\Lambda \mathcal{P}$ is isomorphic to $E'(I)$ for some injective $\Lambda \mathcal{P}^0$ -module I .

2. Let I be an indecomposable injective $\Lambda \mathcal{P}^0$ -module. The morphism $E'(I) \rightarrow E'(I/\text{soc } I)$ is a source map in the category $\text{sub}_\Lambda \mathcal{P}^*$.

3. There are sufficiently many injective objects in $\text{sub}_\Lambda \mathcal{P}$.

Here is an even more explicit description of the indecomposable projective and the indecomposable injective representations in $\text{sub}_\Lambda \mathcal{P}$, and of their respective sink and source maps.

Notation. For M a Λ -module and S a convex subset of \mathcal{P}^* , define the $\Lambda\mathcal{P}^*$ -module M^S by

$$M_i^S = \begin{cases} M & \text{if } i \in S \\ 0 & \text{otherwise,} \end{cases}$$

and let all multiplication maps $M_i^S \rightarrow M_j^S, m \mapsto mq_{ij}$ be zero unless both $i, j \in S$ and the map can be taken to be 1_M . In particular if $\rho \subset \mathcal{P}^* \times \mathcal{P}^*$ is a binary relation and $x \in \mathcal{P}^*$ then $M^{\rho x} = M^S$ where $S = \{i : (i, x) \in \rho\}$. Thus, for example $M^{\geq x}$ given by

$$M_i^{\geq x} = \begin{cases} M & \text{if } i \geq x \\ 0 & \text{otherwise} \end{cases}$$

is a submodule representation of \mathcal{P} .

PROPOSITION 2.4. *1. If P is an indecomposable projective Λ -module and $x \in \mathcal{P}^*$ then $P^{\geq x}$ is an indecomposable projective object in $\text{sub}_\Lambda \mathcal{P}$. The corresponding sink map is the inclusion $(\text{rad } P)^{\geq x} + P^{>x} \subset P^{\geq x}$. Each indecomposable projective object in $\text{sub}_\Lambda \mathcal{P}$ has this form.*

2a) If I is an indecomposable injective Λ -module and $x \in \mathcal{P}$, then $I^{\leq x}$ is an indecomposable injective object in $\text{sub}_\Lambda \mathcal{P}$. The corresponding source map is the inclusion $I^{\leq x} \subset (\text{soc } I)^{\leq x} + I^{\leq x}$.

*2b) For I an indecomposable injective Λ -module, the representation $I^{\leq *}$ is indecomposable injective in $\text{sub}_\Lambda \mathcal{P}$ and has as source map the canonical map $I^{\leq *} \rightarrow (I/\text{soc } I)^{\leq *}$.*

2c) Each indecomposable injective object in $\text{sub}_\Lambda \mathcal{P}$ is isomorphic to one listed above. ■

Observation 2.5. In each case the starting term A of a sink map for an indecomposable projective object in $\text{sub}_\Lambda \mathcal{P}$, and the end term C of the source map for an indecomposable injective object in $\text{sub}_\Lambda \mathcal{P}$, is indecomposable or zero. Either module A or C is determined uniquely by its type.

Example 2.6 (Chains of Subgroups, I). We compute the projective and the injective indecomposables in for the category $\mathcal{C}_3(\Lambda) = \text{sub}_\Lambda \mathcal{P}$ of submodule representations of the linear poset $\mathcal{P} = \begin{matrix} & & 3 \\ & \bullet & \\ & \vdots & \\ & 2 \\ & \bullet & \\ & \vdots & \\ & 1 \\ & \bullet & \end{matrix}$ where Λ is any commutative uniserial ring. Thus we are dealing with chains $A = (A_1 \subset A_2 \subset A_3 \subset A_*)$ of submodules of Λ -modules. The peak module A_* is a direct sum of cyclic Λ -modules and hence is given by the partition which lists the lengths of the summands. We picture A by rotating the Young diagram of the partition for A_* by 90° and by using the symbols \square , \square , and \square to indicate in which of the radical layers in A_* the generators of A_1, A_2 , and A_3 , respectively, can be found. For example if $\Lambda = \mathbb{Z}/p^2, A_3 = A_* = \Lambda$ and $A_1 = A_2 = p\mathbb{Z}/p^2$, then we picture A as follows.

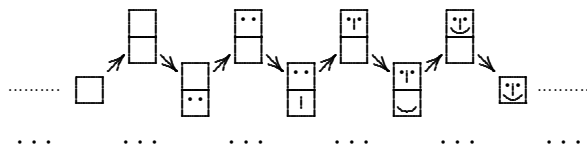
$$A : \begin{matrix} \square \\ \square \\ \square \end{matrix}$$

For this particular poset, but independent of the length n of Λ , the projective representation $\Lambda^{\geq x}$ corresponding to a point $x \in \mathcal{P}^*$ and the injective representation

$\Lambda^{\not\leq x}$ corresponding to this point coincide. We assume that in addition the length of Λ is $n = 2$; then the four indecomposable projective-injective objects can be pictured as follows.

$$\Lambda^{\geq *}: \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \quad \Lambda^{\geq 3}: \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad \Lambda^{\geq 2}: \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \quad \Lambda^{\geq 1}: \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$$

In this case for $x \in \mathcal{P}$, the radical of the projective presentation $\Lambda^{\geq x}$ coincides with the last term of the source map for the injective presentation $\Lambda^{\leq x}$. We obtain the following part of the Auslander-Reiten quiver for $\text{sub}_\Lambda \mathcal{P}$.



3 Auslander-Reiten sequences which are not split exact in each component

Suppose that $\mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an Auslander-Reiten sequence in the submodule category $\text{sub}_\Lambda \mathcal{P}$. For $x \in \mathcal{P}^*$ one can consider the short exact sequence $\mathcal{E}_x : 0 \rightarrow A_x \rightarrow B_x \rightarrow C_x \rightarrow 0$ consisting of the x -components. It turns out that \mathcal{E}_x is either split exact or almost split. We show that given \mathcal{E} , then at most one of the sequences \mathcal{E}_x is not split exact. Moreover, almost all Auslander-Reiten sequences are split exact in every component.

Clearly, for representations of a poset with coefficients in a field, all Auslander-Reiten sequences are split exact in every component and we refer to [20, Chapters 11 and 12] for a description of the Auslander-Reiten quiver in this case.

The purpose of this section is to determine exactly which Auslander-Reiten sequences in $\text{sub}_\Lambda \mathcal{P}$ are not split exact in every component. Let

$$\mathcal{T} : 0 \rightarrow U \xrightarrow{r} V \xrightarrow{s} W \rightarrow 0$$

be an Auslander-Reiten sequence in the category of modules over the base ring Λ , and let $x \in \mathcal{P}^*$. Define the sequence

$$\mathcal{E} = \mathcal{E}(\mathcal{T}, x) : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

as follows. Let $i : U \rightarrow E$ be the inclusion of U in its injective envelope, and let $j : V \rightarrow E$ be a lifting of i over the monomorphism $r : U \rightarrow V$, so $jr = i$ holds. Then there are modules $A, B, C \in \text{sub}_\Lambda \mathcal{P}$ given by the following Λ -modules where $y \in \mathcal{P}^*$.

$$A_y = \begin{cases} U, & y \leq x \\ E, & y \not\leq x \end{cases}, \quad B_y = \begin{cases} V, & y = x \\ A_y \oplus C_y, & y \neq x \end{cases}, \quad C_y = \begin{cases} W, & y \geq x \\ 0, & y \not\geq x \end{cases}$$

It is clear which inclusion maps make up the modules A and C ; for B , the subspaces E, U, V are embedded in $E \oplus W$ via the maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : E \rightarrow E \oplus W$, $i : U \rightarrow E$, $r : U \rightarrow V$, and $\begin{pmatrix} j \\ s \end{pmatrix} : V \rightarrow E \oplus W$.

The maps $f : A \rightarrow B$ and $g : B \rightarrow C$ are given as follows. For $y \neq x$, f_y and g_y are the canonical inclusions and projections, respectively, while $f_x = r$ and $g_x = s$.

Note that there are at most 4 possibilities for the components \mathcal{E}_y of \mathcal{E} , depending on whether $y = x$, $y < x$, $y > x$ or y and x are unrelated. The following example is such that all 4 possibilities are realized.

Example 3.1. If the poset is $\mathcal{P}^* = 2 \begin{matrix} \nearrow^* \\ \searrow_1 \end{matrix} 3$, and $x = 2$, then the Auslander-Reiten sequence $\mathcal{T} : 0 \rightarrow U \xrightarrow{r} V \xrightarrow{s} W \rightarrow 0$ in $\text{mod } \Lambda$ gives rise to the Auslander-Reiten sequence \mathcal{E} in $\text{sub}_\Lambda \mathcal{P}$:

$$\mathcal{E} : 0 \longrightarrow \begin{array}{ccc} & E & \\ i \nearrow & & \searrow 1 \\ U & & V \\ 1 \nwarrow & & \nearrow E \\ & U & i \end{array} \xrightarrow{r \begin{pmatrix} 1 \\ 0 \end{pmatrix} 1} \begin{array}{ccc} & E \oplus W & \\ \begin{pmatrix} j \\ s \end{pmatrix} \nearrow & & \searrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ V & & W \\ r \nwarrow & & \nearrow E \\ & U & i \end{array} \xrightarrow{s \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \begin{array}{ccc} & W & \\ 1 \nearrow & & \searrow 0 \\ W & & 0 \\ 1 \nwarrow & & \nearrow 0 \\ & W & 0 \end{array} \longrightarrow 0$$

THEOREM 3.2. *1. Let \mathcal{T} be an almost split sequence in $\text{mod } \Lambda$ and $x \in \mathcal{P}^*$. Then the sequence $\mathcal{E} = \mathcal{E}(\mathcal{T}, x)$ is an Auslander-Reiten sequence in $\text{sub}_\Lambda \mathcal{P}$ which is split exact in every component except in the x -component.*

2. Conversely if $\mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an Auslander-Reiten sequence in $\text{sub}_\Lambda \mathcal{P}$, and $x \in \mathcal{P}^$ is such that the sequence in $\text{mod } \Lambda$ of the x -components $\mathcal{T} : 0 \rightarrow A_x \rightarrow B_x \rightarrow C_x \rightarrow 0$ is not split exact, then \mathcal{T} is an Auslander-Reiten sequence in $\text{mod } \Lambda$ and the sequences \mathcal{E} and $\mathcal{E}(\mathcal{T}, x)$ are equivalent.*

Proof. Clearly, the sequence $\mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in the first assertion is not split exact and the modules A and C are indecomposable. We show that \mathcal{E} is right almost split.

Let $T \in \text{sub}_\Lambda \mathcal{P}$ and let $t : T \rightarrow C$ be a morphism which is not a split epimorphism. Then $t_x : T_x \rightarrow C_x$ is not a split epimorphism in $\text{mod } \Lambda$ and hence factors over $g_x : V \rightarrow W$: There is $t'_x : T_x \rightarrow B_x$ such that $g_x t'_x = t_x$, as illustrated below in the situation of the example.

$$\begin{array}{ccc} & & \begin{array}{ccc} & T_* & \\ T_x \nearrow & & \searrow T_3 \\ & T_1 & \end{array} \\ & \swarrow^{t'_x \ t'_1 \ t'_3} & \downarrow^{t_x \ t_1 \ t_3} \\ \begin{array}{ccc} \begin{pmatrix} j \\ s \end{pmatrix} \nearrow E \oplus W \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ V & & W \\ r \nwarrow & & \nearrow E \\ & U & i \end{array} & \xrightarrow{s \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} & \begin{array}{ccc} & W & \\ 1 \nearrow & & \searrow 0 \\ W & & 0 \\ 1 \nwarrow & & \nearrow 0 \\ & W & 0 \end{array} \end{array}$$

If $x \neq *$, define $t'_* : T_* \rightarrow E \oplus W$ as follows. Since E is injective and $T_x \subseteq T_*$, the map $jt'_x : T_x \rightarrow E$ extends to a map $t_+^* : T_* \rightarrow E$, hence $t_+^*|_{T_x} = jt'_x$ holds. Put $t'_* = \begin{pmatrix} t_+^* \\ t_* \end{pmatrix}$. As a consequence, for $y \geq x$, the map $t'_y : T_y \rightarrow E \oplus W$ is defined uniquely as the restriction of t'_* to T_y .

Next for $y \in \mathcal{P}$ which is not in relation with x , define the map $t'_y : T_y \rightarrow E$ as follows. Since $t_*|_{T_y} = 0$, the image $t'_*(T_y)$ is contained in $E \oplus 0$ and hence there is a unique map $t'_y : T_y \rightarrow E$ such that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} t'_y = t'_*|_{T_y} : T_y \rightarrow E \oplus W$.

Finally consider $y \in \mathcal{P}$ such that $y < x$. Since $t_x|_{T_y} = 0$, also $st'_x|_{T_y} = 0$ and hence the restriction of t'_x to T_y factors over $r : U \rightarrow V$: There exists a unique $t'_y : T_y \rightarrow U$ such that $t'_x|_{T_y} = rt'_y$.

In conclusion, $t' = (t'_y)_{y \in \mathcal{P}} : T \rightarrow B$ is a morphism which satisfies $gt' = t$.

It remains to show the second assertion. Note that all the sequences constructed have the property that the last term has the form $W^{\geq x}$ where W is an indecomposable Λ -module and $x \in \mathcal{P}^*$. We show that any Auslander-Reiten sequence $\mathcal{E} : 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ for which the last term is not of type $W^{\geq x}$ for a Λ -module W , is split exact in each component. For $y \in \mathcal{P}^*$ consider the sequence of the y -components

$$0 \rightarrow A_y \xrightarrow{f_y} B_y \xrightarrow{g_y} C_y \rightarrow 0$$

in \mathcal{E} . Define the \mathcal{P} -representation $T = (C_y)^{\geq y}$ and consider the map $t : T \rightarrow C$ given by the identity map on the y -component. By assumption, C has not the form $W^{\geq x}$ so t is not an epimorphism and hence not a split epimorphism. Since t factors over g , the map $g_y : B_y \rightarrow C_y$ must be a split epimorphism in $\text{mod } \Lambda$. This finishes the proof. ■

Example 3.3 (Chains of subgroups, II). We use the notation from Example 2.6 where $\mathcal{P} = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix}$ and Λ is a commutative uniserial ring of length 2. Denote the only Auslander-Reiten sequence in $\text{mod } \Lambda$ by $\mathcal{T} : 0 \rightarrow k \xrightarrow{\iota} \Lambda \xrightarrow{\pi} k \rightarrow 0$ where $k = \Lambda/\text{rad } \Lambda$. For $x = *$ we obtain the Auslander-Reiten sequence in $\text{sub}_\Lambda \mathcal{P}$:

$$\mathcal{E}(\mathcal{T}, *) : 0 \rightarrow (k=k=k=k) \rightarrow (k=k=k \xrightarrow{\iota} \Lambda) \rightarrow (0=0=0 \subset k) \rightarrow 0$$

For $x \in \mathcal{P}$, the sequence $\mathcal{E}(\mathcal{T}, x)$ which we picture here for $x = 2$

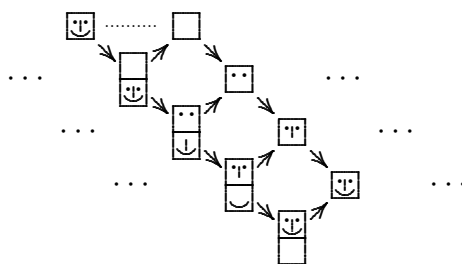
$$\mathcal{E}(\mathcal{T}, 2) :$$

$$0 \rightarrow (k=k \xrightarrow{\iota} \Lambda = \Lambda) \xrightarrow{1, \iota, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}} (k \xrightarrow{\iota} \Lambda \xrightarrow{\begin{pmatrix} 1 \\ \pi \end{pmatrix}} (\Lambda \oplus k) = (\Lambda \oplus k)) \xrightarrow{0, \pi, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}} (0 \subset k = k = k) \rightarrow 0$$

has decomposable middle term:

$$(k \xrightarrow{\iota} \Lambda \xrightarrow{\begin{pmatrix} 1 \\ \pi \end{pmatrix}} (\Lambda \oplus k) = (\Lambda \oplus k)) \cong (k \xrightarrow{\iota} \Lambda = \Lambda = \Lambda) \oplus (0 = 0 \subset k = k).$$

Note that the irreducible map onto the second summand of the middle term of $\mathcal{E}(\mathcal{T}, x)$ coincides with the irreducible map starting at the first summand of the middle term in the sequence $\mathcal{E}(\mathcal{T}, x + 1)$ (in case $x = 3$, take the right almost split map in $\mathcal{E}(\mathcal{T}, *)$). Thus the Auslander-Reiten sequences $\mathcal{E}(\mathcal{T}, x)$ for $x = *, 3, 2, 1$ line up to form the following part of the Auslander-Reiten quiver for $\text{sub}_\Lambda \mathcal{P}$:



There is also the following corresponding result for the category $\text{fac}_\Lambda \mathcal{P}$. Again, let $0 \rightarrow U \xrightarrow{r} V \xrightarrow{s} W \rightarrow 0$ be an Auslander-Reiten sequence in the category $\text{mod } \Lambda$ and $x \in \mathcal{P}^0$. Let $p : P \rightarrow W$ be a projective cover and $q : P \rightarrow V$ a lifting of p over the epimorphism $s : V \rightarrow W$. Define the representations A, B, C as follows

$$A_y = \begin{cases} U, & y \leq x \\ 0, & y \not\leq x \end{cases}, \quad B_y = \begin{cases} V, & y = x \\ A_y \oplus C_y, & y \neq x \end{cases}, \quad C_y = \begin{cases} W, & y \geq x \\ P, & y \not\geq x \end{cases},$$

where $y \in \mathcal{P}^0$.

Example 3.4. We illustrate the linear maps which make up $A, B, C, f,$ and g in the example where $\mathcal{P}^0 = \mathbf{1} \begin{matrix} \nearrow 3 \\ \bullet \\ \searrow 0 \end{matrix} \mathbf{2}$, and $x = 1$. The Auslander-Reiten sequence in $\text{fac}_\Lambda \mathcal{P}$ is as follows.

$$\mathcal{E} : 0 \longrightarrow \begin{matrix} & 0 & \\ U \nearrow & & \searrow U \\ 1 & & 0 \end{matrix} \xrightarrow{r \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \begin{matrix} & W & \\ s \nearrow & & \searrow p \\ V \nearrow & & \searrow P \\ (r \ q) \nearrow & U \oplus P & \searrow (0 \ 1) \end{matrix} \xrightarrow{s \begin{pmatrix} 1 \\ 0 \ 1 \end{pmatrix}} \begin{matrix} & W & \\ 1 \nearrow & & \searrow p \\ W \nearrow & & \searrow P \\ p \nearrow & P & \searrow 1 \end{matrix} \longrightarrow 0$$

THEOREM 3.5. 1. Let \mathcal{T} be an almost split sequence in $\text{mod } \Lambda$ and $x \in \mathcal{P}^0$. Then the sequence $\mathcal{E} = \mathcal{E}(\mathcal{T}, x)$ is an Auslander-Reiten sequence in $\text{fac}_\Lambda \mathcal{P}$ which is split exact in every component except in the x -component.

2. Conversely if $\mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an Auslander-Reiten sequence in $\text{fac}_\Lambda \mathcal{P}$, and $x \in \mathcal{P}^0$ is such that the sequence in $\text{mod } \Lambda$ of the x -components $\mathcal{T} : 0 \rightarrow A_x \rightarrow B_x \rightarrow C_x \rightarrow 0$ is not split exact, then \mathcal{T} is an Auslander-Reiten sequence in $\text{mod } \Lambda$ and the sequences \mathcal{E} and $\mathcal{E}(\mathcal{T}, x)$ are equivalent. ■

Example 3.6 (Bounded submodules of modules, I). In this example we address the remark by M.C.R. Butler. For $\mathcal{P} = \bullet_1$ the one point poset, write $\mathcal{S}(\Lambda) = \text{sub}_\Lambda \mathcal{P}$ and let for $m \leq n$ the full subcategory $\mathcal{S}_m(\Lambda)$ of $\mathcal{S}(\Lambda)$ consist of all pairs $(A_1 \subset A_*)$ where $\text{rad}^m A_1 = 0$ holds. Let t be a radical generator for Λ .

The category $\mathcal{S}_m(\Lambda)$ is an exact Krull-Remak-Schmidt category with Auslander-Reiten sequences [5]. Namely, each object $(A_1 \subset A_*)$ in $\mathcal{S}(\Lambda)$ has a minimal right and a minimal left approximation in $\mathcal{S}_m(\Lambda)$, given as follows.

$$\begin{aligned} & (\text{soc}^m A_1 \subset A_*) \rightarrow (A_1 \subset A_*); \\ & (A_1 \subset A_*) \rightarrow (A_1 / \text{rad}^m A_1 \subset A_* / \text{rad}^m A_1) \end{aligned}$$

We determine each Auslander-Reiten sequence for which one of the short exact sequences given by either (1) the submodules, (2) the total spaces, or (3) the factor modules is not split exact.

1. The sequence of submodules is not split exact. Let $\bar{\Lambda}$ be the factor ring $\Lambda / \text{rad}^m \Lambda$ and suppose that $0 \rightarrow A_1 \rightarrow B_1 \rightarrow C_1 \rightarrow 0$ is an Auslander-Reiten sequence in $\text{mod } \bar{\Lambda}$. Let $u : A_1 \rightarrow E$ be an injective envelope in $\text{mod } \Lambda$ and choose an extension $v : B_1 \rightarrow E$ of u . According to Theorem 3.2, the sequence

$$0 \rightarrow (A_1 \rightarrow E) \rightarrow (B_1 \rightarrow E \oplus C_1) \rightarrow (C_1 = C_1) \rightarrow 0$$

is an Auslander-Reiten sequence in $\mathcal{S}(\Lambda)$; since all objects are in $\mathcal{S}_m(\Lambda)$, the sequence is an Auslander-Reiten sequence in $\mathcal{S}_m(\Lambda)$, too. For example if $n = 6$ and $m \geq 3$ then the Auslander-Reiten sequence in $\mathcal{S}_m(\Lambda)$ ending at C where $C_1 = C_* = \Lambda/\text{rad}^2\Lambda$ is as follows.

$$0 \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \longrightarrow 0$$

The objects have type $t_A = (2; 6)$, $t_B = (31; 62)$, and $t_C = (2; 2)$. As a consequence, the middle term of this sequence is an indecomposable object. It is given by the inclusion $(B_1 \subset B_*)$ where B_* is generated by two elements y_1 and y_2 , say, bounded by t^6 and t^2 , respectively. The submodule B_1 also has two generators, for example $x_1 = t^3y_1 + y_2$ (indicated in the diagram by the connected dots) and $x_2 = ty_2$ (given by the isolated point).

2. The sequence of the total spaces is not split exact. Let $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ be an Auslander-Reiten sequence in $\text{mod } \Lambda$. According to Theorem 3.2,

$$0 \rightarrow (A_* = A_*) \rightarrow (A_* \subset B_*) \rightarrow (0 \subset C_*) \rightarrow 0$$

is an Auslander-Reiten sequence in $\mathcal{S}(\Lambda)$ and, using right approximations,

$$0 \rightarrow (\text{soc}^m A_* \subset A_*) \rightarrow (\text{soc}^m A_* \subset B_*) \rightarrow (0 \subset C_*) \rightarrow 0$$

is an Auslander-Reiten sequence in $\mathcal{S}_m(\Lambda)$. For example if $n \geq 5$, and $m = 3$, then the Auslander-Reiten sequence in $\mathcal{S}_3(\Lambda)$ starting at A where $A_* = \Lambda/\text{rad}^4\Lambda$ and $A_1 = \text{soc}^3 A_*$ is as follows.

$$0 \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \longrightarrow 0$$

The modules have type $t_A = (3; 4)$, $t_B = (3; 53)$, and $t_C = (-; 4)$; the middle term B is an indecomposable object.

3. The sequence of the factor modules is not split exact.

In this section we consider sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in $\mathcal{S}(\Lambda)$ which are split exact in every component, but for which the (short exact) sequence of factors $0 \rightarrow A_*/A_1 \rightarrow B_*/B_1 \rightarrow C_*/C_1 \rightarrow 0$ is almost split. Let $0 \rightarrow \bar{A} \xrightarrow{f} \bar{B} \xrightarrow{g} \bar{C} \rightarrow 0$ be an Auslander-Reiten sequence in $\text{mod } \Lambda$, $u : P \rightarrow \bar{C}$ a projective cover with kernel $v : \bar{K} \rightarrow P$ and $w : P \rightarrow \bar{B}$ a lifting of u . Let $z : K \rightarrow \bar{A} \oplus P$ be the kernel of (f, w) . By Theorem 3.5, we have the Auslander-Reiten sequence in $\text{fac}_\Lambda \mathcal{P}$.

$$0 \rightarrow (\bar{A} = \bar{A}) \rightarrow (\bar{A} \oplus P \rightarrow \bar{B}) \rightarrow (P \rightarrow \bar{C}) \rightarrow 0$$

Using the equivalence between $\text{sub}_\Lambda \mathcal{P}$ and $\text{fac}_\Lambda \mathcal{P}$ given by the kernel functor (Lemma 2.2), we obtain the following Auslander-Reiten sequence in $\mathcal{S}(\Lambda)$.

$$0 \rightarrow (0 \subset \bar{A}) \rightarrow (K \rightarrow \bar{A} \oplus P) \rightarrow (\bar{K} \rightarrow P) \rightarrow 0$$

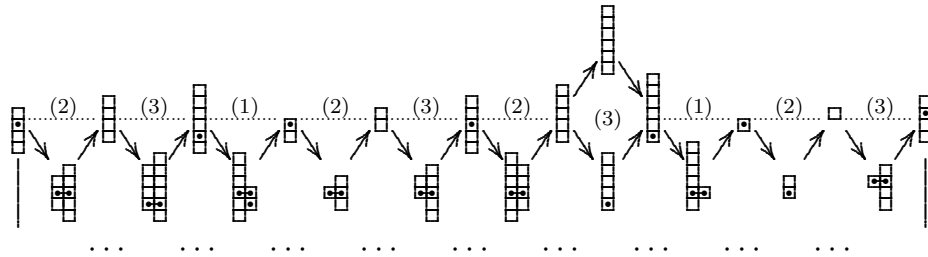
(By the snake lemma, the two kernels K and \bar{K} are isomorphic.) Using left approximations, we arrive at the Auslander-Reiten sequence in $\mathcal{S}_m(\Lambda)$.

$$0 \rightarrow (0 \subset \bar{A}) \rightarrow (K/\text{rad}^m K \rightarrow (\bar{A} \oplus P)/z(\text{rad}^m K)) \rightarrow (\bar{K}/\text{rad}^m \bar{K} \rightarrow P/v(\text{rad}^m \bar{K})) \rightarrow 0$$

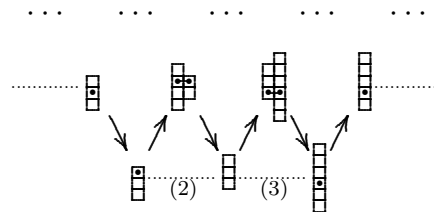
For example if $n \geq 6$, $m = 3$, and $\bar{A} = \Lambda/\text{rad}^2 \Lambda$ we obtain the following Auslander-Reiten sequence in the category $\mathcal{S}_3(\Lambda)$. If the *cotype* of an object A is the type of the factor A_*/A_1 then the modules have cotypes $\text{cot}_A = (2)$, $\text{cot}_B = (31)$, and $\text{cot}_C = (2)$. In particular, B is an indecomposable object.

$$0 \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \longrightarrow \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} \longrightarrow 0$$

It turns out that in the case $m = 3$, $n = 6$, the first and last modules in the above Auslander-Reiten sequences form one stable orbit under the translation to which the projective injective module is attached,



and also an orbit of length three from the projective noninjective indecomposable to the injective nonprojective indecomposable, as pictured below.



The number in parantheses describes if in the Auslander-Reiten sequence labelled, (1) the sequence of submodules, (2) the sequence of peak modules, or (3) the sequence of the factors is not split exact.

4 Isomorphy of Auslander-Reiten quivers.

We demonstrate the result in the introduction about the independence of the Auslander-Reiten quiver from the commutative uniserial base ring. As an immediate application we obtain the Auslander-Reiten quiver for the category in Example 2.6 (chains of subgroups). We use the terminology related to translation quivers and coverings from [9]. In particular, we recall that given a translation quiver Γ , there is a universal covering $\pi : \tilde{\Gamma} \rightarrow \Gamma$.

Definition. A full connected finite nonempty subquiver \mathcal{L} of a (valued) translation quiver Γ is a *slice* provided whenever $x \in \mathcal{L}$ is a noninjective point and $f : x \rightarrow y$ an arrow into a nonprojective point y , then exactly one of the arrows f or $\sigma(f) : \tau y \rightarrow x$ is in \mathcal{L} . Here, σ denotes the semitranslation in Γ .

PROPOSITION 4.1. *Let Λ be a commutative local uniserial ring, \mathcal{P} a finite poset, Γ a connected component of the Auslander-Reiten quiver of $\text{sub}_\Lambda \mathcal{P}$, and \mathcal{L} a slice in Γ . Let $\tilde{\Gamma} \xrightarrow{\pi} \Gamma$ be the universal covering for Γ and define the type of a point x in $\tilde{\Gamma}$ as the type of $\pi(x)$ in Γ . The structure of the (valued) translation quiver $\tilde{\Gamma}$ and the types assigned to its points are uniquely determined by the structure of \mathcal{L} as a (valued) quiver and by the types of the points in \mathcal{L} .*

Proof. A point $x \in \tilde{\Gamma}$ such that $\pi(x) \in \mathcal{L}$ defines a unique subquiver $\tilde{\mathcal{L}} \subset \tilde{\Gamma}$ which corresponds to \mathcal{L} under π ; this quiver is a slice in $\tilde{\Gamma}$. We define a sequence \mathcal{L}_i of slices shifted against $\tilde{\mathcal{L}}$ such that every point $y \in \tilde{\Gamma}$ with a path $y \rightarrow \tilde{\mathcal{L}}$ (or a path $\tilde{\mathcal{L}} \rightarrow y$) is contained in one of the \mathcal{L}_i , $i \geq 0$ (or in one of the \mathcal{L}_i , $i \leq 0$). We show that the types of the points in each of the \mathcal{L}_i , $i > 0$, are determined uniquely by the types of the points in $\tilde{\mathcal{L}}$ and omit the proof of the corresponding result for \mathcal{L}_i , $i < 0$. Given $i \geq 0$ and $z \in \mathcal{L}_i$ a sink, obtain \mathcal{L}_{i+1} by performing the first possible of the following operations. a) First we deal with the case that z is a projective vertex in $\tilde{\Gamma}$. If z is an isolated point then Λ is a field, $z = [(\Lambda; (0)_{j \in \mathcal{P}})]$, and we are done with all points x which have a path into $\tilde{\mathcal{L}}$. Otherwise there is a unique arrow $y \rightarrow z$ in $\tilde{\Gamma}$ ending in z where y is the class of the radical of the projective object corresponding to z as noted in Observation 2.5. This arrow is in \mathcal{L}_i since \mathcal{L}_i is connected. Let \mathcal{L}_{i+1} be obtained from \mathcal{L}_i by deleting the point z and the arrow $y \rightarrow z$. The graph \mathcal{L}_{i+1} is connected, hence a slice.

b) Here we assume that z is the end term of a source map $y \rightarrow z$ where y is some injective object not in \mathcal{L}_i . We have seen in Observation 2.5 that the endterm z of the source map for y is an indecomposable object which is determined uniquely by its type. By adding to \mathcal{L}_i the point y and the arrow $y \rightarrow z$ we arrive at a slice \mathcal{L}_{i+1} in which the types of all points are determined uniquely by the types of the points in \mathcal{L}_i .

c) If neither of the above two operations can be performed then we are in the situation that z is the class of an endterm of an Auslander-Reiten sequence \mathcal{E} and that for each indecomposable summand y of the middle term there is an arrow $y \rightarrow z$ in \mathcal{L}_i . In this case we replace z by τz and each arrow $y \rightarrow z$ by a corresponding arrow $\tau z \rightarrow y$ to obtain a new slice \mathcal{L}_{i+1} . The type of z decides whether the sequence \mathcal{E} is split exact in each component (in which case additivity of types yields the type of τz) or if \mathcal{E} is one of the sequences studied in the previous section (with the type of τz known). ■

The fact used in the above proof namely that the first term in the source map for a projective module, and the last term in the sink map for an injective module are indecomposable modules (or zero), has the following consequence for the shape of the Auslander-Reiten quiver of a submodule category.

PROPOSITION 4.2. *Let Λ be a commutative uniserial ring, \mathcal{P} a finite poset and Γ^s the stable part of a connected component Γ of the Auslander-Reiten quiver of $\text{sub}_\Lambda \mathcal{P}$. If Γ^s is non empty, then Γ^s itself is connected and Γ is obtained from Γ^s by successively attaching non stable orbits.* ■

We can now give the proof of Theorem 1.1.

Proof. We use the notation in the theorem. Since the slices \mathcal{L}_Λ and \mathcal{L}_Δ are isomorphic as quivers and consist of objects of the same types, Proposition 4.1 yields universal coverings $\pi_\Lambda : \tilde{\Gamma}_\Lambda \rightarrow \Gamma_\Lambda$, $\pi_\Delta : \tilde{\Gamma}_\Delta \rightarrow \Gamma_\Delta$ and a type preserving isomorphism of translation quivers $\tilde{\varphi}$ which maps \mathcal{L}_Λ onto \mathcal{L}_Δ .

$$\begin{array}{ccc} \tilde{\Gamma}_\Lambda & \xrightarrow{\tilde{\varphi}} & \tilde{\Gamma}_\Delta \\ \pi_\Lambda \downarrow & & \downarrow \pi_\Delta \\ \Gamma_\Lambda & & \Gamma_\Delta \end{array}$$

We show that there is a type preserving isomorphism $\varphi : \Gamma_\Lambda \rightarrow \Gamma_\Delta$ of translation quivers which makes the diagram commutative.

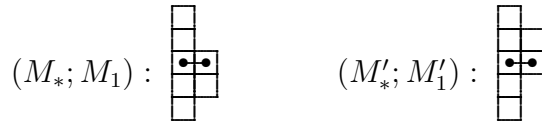
Suppose that $x_\Lambda \in \Gamma_\Lambda$ and $x_\Delta \in \Gamma_\Delta$ are objects which have the same type and which are determined uniquely by their type. For example, projective objects, injective objects, and the objects in \mathcal{L} have this property. Then the fibers $\pi_\Lambda^{-1}(x_\Lambda)$ and $\pi_\Delta^{-1}(x_\Delta)$ coincide and we can define $\varphi(x_\Lambda) = x_\Delta$. Since π_Λ, π_Δ define coverings, the maps commute with the translation and the definition of φ can be extended to the preprojective, to the preinjective, and to the stable objects. Thus, φ is a bijection of the points in Γ_Λ and Γ_Δ which satisfies $\varphi \circ \pi_\Lambda = \pi_\Delta \circ \tilde{\varphi}$ and which extends to an isomorphism of translation quivers. ■

Remark 4.3. Note that we cannot expect in Theorem 1.1 that the type determines an object uniquely, up to isomorphism, not even in the representation finite case, as the following example shows. Thus, condition (3) may not hold for all objects in the component \mathcal{C} .

Example 4.4. Let $\mathcal{P} = \bullet_1$ and $\Lambda = \mathbb{Z}/p^5$. The category $\text{sub}_\Lambda \mathcal{P}$ has finite representation type, in fact all 50 indecomposables have been determined in [14]. The Auslander-Reiten quiver Γ has a slice which consists of points which are determined uniquely by their type. But not all points in Γ are determined by their type: The two indecomposable objects in $\text{sub}_\Lambda \mathcal{P}$,

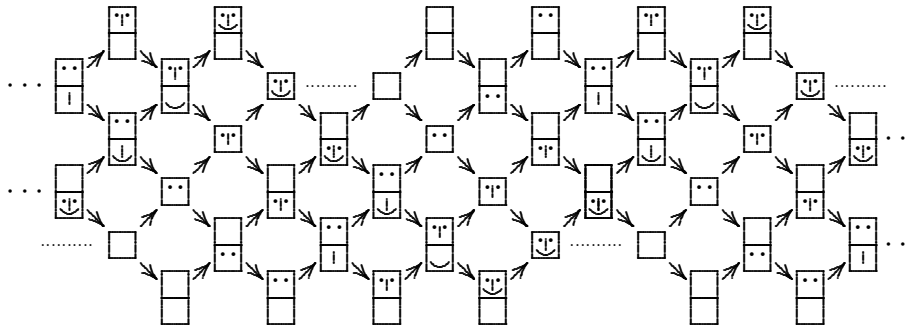
$$\begin{aligned} (M_*; M_1) &= (\mathbb{Z}/p^5 \oplus \mathbb{Z}/p^2; \mathbb{Z}/p^3([p^2], [1])) \quad \text{and} \\ (M'_*; M'_1) &= (\mathbb{Z}/p^5 \oplus \mathbb{Z}/p^2; \mathbb{Z}/p^3([p^2], [p])), \end{aligned}$$

are pictured as follows.



Both objects have the same type $(52; 3)$ but since their quotients $M_*/M_1 \cong \mathbb{Z}/p^4$ and $M'_*/M'_1 \cong \mathbb{Z}/p^3 \oplus \mathbb{Z}/p^1$ are not isomorphic in $\text{mod } \Lambda$, the objects M and M' cannot be isomorphic in $\text{sub}_\Lambda \mathcal{P}$.

Example 4.5 (Chains of subgroups, III). We continue Example 3.3 where we have seen that the sequence of irreducible maps $\square \rightarrow \square \rightarrow \square \rightarrow \square$ forms a slice in the Auslander-Reiten quiver for $\text{sub}_\Lambda \mathcal{P}$. Moreover, as endterms of Auslander-Reiten sequences which are not split exact in each component, the modules are uniquely determined by their type. In fact, every indecomposable object is determined uniquely by its type and the computation of the Auslander-Reiten quiver from the slice is a straightforward task. The example serves as an illustration for Theorem 1.1 as for any two commutative uniserial rings Λ and Δ of length two, there is a type preserving isomorphism of translation quivers between the Auslander-Reiten quivers for $\text{sub}_\Lambda \mathcal{P}$ and $\text{sub}_\Delta \mathcal{P}$. Each quiver is as follows.



5 Submodule categories of type $\mathcal{S}_3(\Lambda)$ where Λ has length 6.

For Λ a commutative uniserial ring of length 6 we show that the category $\mathcal{S}_3(\Lambda)$ is representation finite. We compute its Auslander-Reiten quiver and demonstrate that this quiver is independent of the choice of Λ . In particular, there are no parametrized families of indecomposable objects in $\mathcal{S}_3(\mathbb{Z}/p^6)$. This finishes our demonstration that the first occurrences of parametrized families of subgroup embeddings are in the categories $\mathcal{S}_4(\mathbb{Z}/p^6)$ and $\mathcal{S}_3(\mathbb{Z}/p^7)$.

We proceed as follows. The corresponding category $\mathcal{S}_3(k[x]/x^6)$ where $k = \Lambda/\text{rad } \Lambda$ has 84 indecomposables and the Auslander-Reiten quiver is as pictured on page 542. We have seen in Example 3.6 that the modules in the top orbit and in the non stable orbit occur as endterms of Auslander-Reiten sequences which are not split exact in each component; thus, also the corresponding modules in the category $\mathcal{S}_3(\Lambda)$ form orbits under the Auslander-Reiten translation. Starting from these modules, we construct a slice in the Auslander-Reiten quiver for $\mathcal{S}_3(\Lambda)$; as a quiver, the slice is isomorphic to a corresponding slice for $\mathcal{S}_3(k[x]/x^6)$. We deduce that there is a type preserving isomorphism between the components of the two Auslander-Reiten

quivers. Thus we have detected a finite component, and the Harada-Sai lemma, see [4, VI, Corollary 1.3], implies that we have computed the complete Auslander-Reiten quiver for $\mathcal{S}_3(\Lambda)$.

First we adapt Theorem 1.1 to the case of bounded subgroups.

COROLLARY 5.1. *Suppose Λ, Δ are commutative uniserial rings of the same length n , $m \leq n$ is a natural number, Γ_Λ and Γ_Δ are connected components of the Auslander-Reiten quivers of $\mathcal{S}_m(\Lambda)$ and $\mathcal{S}_m(\Delta)$, and \mathcal{L}_Λ and \mathcal{L}_Δ are slices in Γ_Λ and Γ_Δ , respectively, such that the following conditions are satisfied:*

1. *The slices \mathcal{L}_Λ and \mathcal{L}_Δ are isomorphic as valued graphs.*
2. *Points in \mathcal{L}_Λ and \mathcal{L}_Δ which correspond to each other under this isomorphism represent indecomposable objects of the same type.*
3. *Each indecomposable object represented by a point in \mathcal{L}_Λ or in \mathcal{L}_Δ is determined uniquely, up to isomorphism, by its type.*

Then the components Γ_Λ and Γ_Δ are isomorphic as valued translation quivers, and points which correspond to each other under this isomorphism represent indecomposable objects of the same type.

Proof. The proof of Theorem 1.1 in Section 4 applies also in this situation: In [18] the projective and the injective indecomposables in categories of type $\mathcal{S}_m(\Lambda)$ and their respective sink and source maps have been computed. As in the case of submodule representations of a poset, each source or target in such a map is zero or an indecomposable module which is determined uniquely by its type. The second fact needed in the proof is the structure of those Auslander-Reiten sequences which are not split exact in each component. These sequences have been constructed in Example 3.6, (1) and (2). ■

Next we provide us with a tool to verify that given short exact sequences are in fact Auslander-Reiten sequences. Recall from [4, Proposition V.2.2] that in a category of modules over an artin algebra, a nonsplit short exact sequence $0 \rightarrow D\text{Tr } C \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ where C is an indecomposable nonprojective module is an Auslander-Reiten sequence provided only that every endomorphism of C which is not an automorphism factors over g . We adapt this result to our situation where we are dealing with a full subcategory \mathcal{S} of a module category which is closed under extensions.

We first recall the result corresponding [4, Proposition V.2.1].

LEMMA 5.2. *Let C be an indecomposable object in \mathcal{S} which is not Ext-projective, and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an Auslander-Reiten sequence in \mathcal{S} . Then $\text{Ext}_R^1(C, A)$ has a simple socle, both as $\text{End}(C)$ - and as $\text{End}(A)$ -module. These socles coincide and each nonzero element in the socle is an Auslander-Reiten sequence in \mathcal{S} . ■*

The following result is a minor adaption of [4, Proposition V.2.2] to the case of full subcategories.

PROPOSITION 5.3. *Let \mathcal{S} be a full extension closed subcategory of a category of modules over an artin algebra. For C an indecomposable object in \mathcal{S} which is not Ext-projective, let $0 \rightarrow A \rightarrow B' \rightarrow C \rightarrow 0$ be an Auslander-Reiten sequence in \mathcal{S} . The following assertions are equivalent for a nonsplit short exact sequence*

$$\mathcal{E} : 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

1. *The sequence \mathcal{E} is an Auslander-Reiten sequence in \mathcal{S} .*
2. *Every nonautomorphism $h : C \rightarrow C$ factors through C .*
3. *The image of the map $\text{Hom}(C, g) : \text{Hom}(C, B) \rightarrow \text{Hom}(C, C)$ is $\text{Rad End}(C)$.*
4. *The contravariant defect $\mathcal{E}^*(C) = \text{End}(C)/\text{Im Hom}(C, g)$ is a simple module over $\text{End}(C)$.*
5. *The socle of the $\text{End}(C)$ -module $\text{Ext}_R^1(C, A)$ is generated by the class of \mathcal{E} .*
- 2'. *Every nonautomorphism $k : A \rightarrow A$ factors through f .*
- 3'. *The the map $\text{Hom}(f, A) : \text{Hom}(B, A) \rightarrow \text{Hom}(A, A)$ has image $\text{Rad End}(A)$.*
- 4'. *The covariant defect $\mathcal{E}_*(A) = \text{End}(A)/\text{Im Hom}(f, A)$ is a simple module over $\text{End}(A)$.*
- 5'. *The socle of the $\text{End}(A)$ -module $\text{Ext}_R^1(C, A)$ is generated by the class of \mathcal{E} .*

Proof. The implications 1. \Rightarrow 2. and 2. \Leftrightarrow 3. \Leftrightarrow 4. are obvious and 1. \Leftrightarrow 5. follows from Lemma 4.1. We only show that 4. \Rightarrow 5. Consider the long exact sequence

$$0 \rightarrow \text{Hom}(C, A) \rightarrow \text{Hom}(C, B) \rightarrow \text{Hom}(C, C) \xrightarrow{\delta} \text{Ext}_R^1(C, A) \rightarrow \dots$$

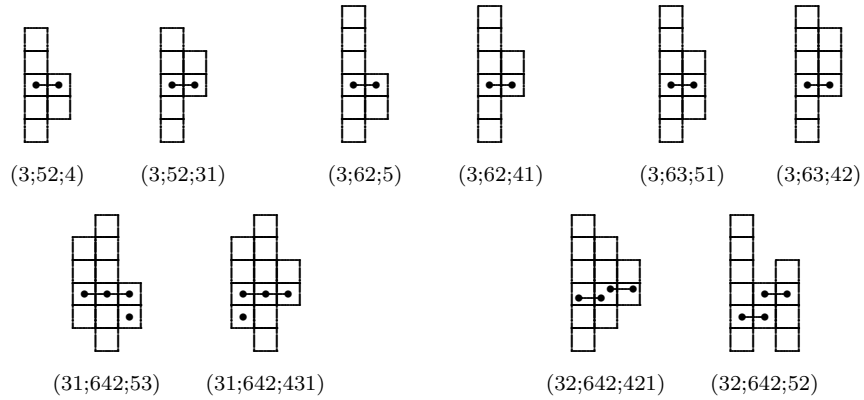
Recall that δ is given by sending the identity map 1_C to the class of \mathcal{E} in $\text{Ext}_R^1(C, A)$ and that the image of δ is the contravariant defect $\mathcal{E}^*(C)$. Thus, $\mathcal{E}^*(C)$ is the submodule of $\text{Ext}_R^1(C, A)$ generated by $\bar{\mathcal{E}}$. If $\mathcal{E}^*(C)_{\text{End}(C)}$ is a simple module then $\bar{\mathcal{E}}$ is a socle generator by Lemma 4.1. The statements 2', 3', 4'. and 5'. follow by duality. ■

We return to the setup from Example 3.6 and compute the Auslander-Reiten quiver for $\mathcal{S}_3(\mathbb{Z}/p^6)$.

THEOREM 5.4. *Let Λ be a commutative uniserial ring of length 6.*

1. *The category $\mathcal{S}_3(\Lambda)$ has 84 isomorphism classes of indecomposable objects. The stable part of the Auslander-Reiten quiver has type $\mathbb{Z}\mathbb{E}_8/\tau^{10}$ and there are two nonstable orbits of length 1 and 3 attached to the boundaries of the wings of width 5 and 3, respectively.*

2. Each indecomposable object in $\mathcal{S}_3(\Lambda)$ is determined uniquely, up to isomorphism, by its type with the following 10 exceptions for which we specify the type and the cotype.



Proof. Let k be any field. We show that there is a type preserving isomorphism of (valued) translation quivers between the Auslander-Reiten quiver for the category $\mathcal{S}_3(\Lambda)$ and the Auslander-Reiten quiver for $\mathcal{S}_3(k[x]/x^6)$, of which we include a copy on page 542. Both assertions in the theorem follow easily.

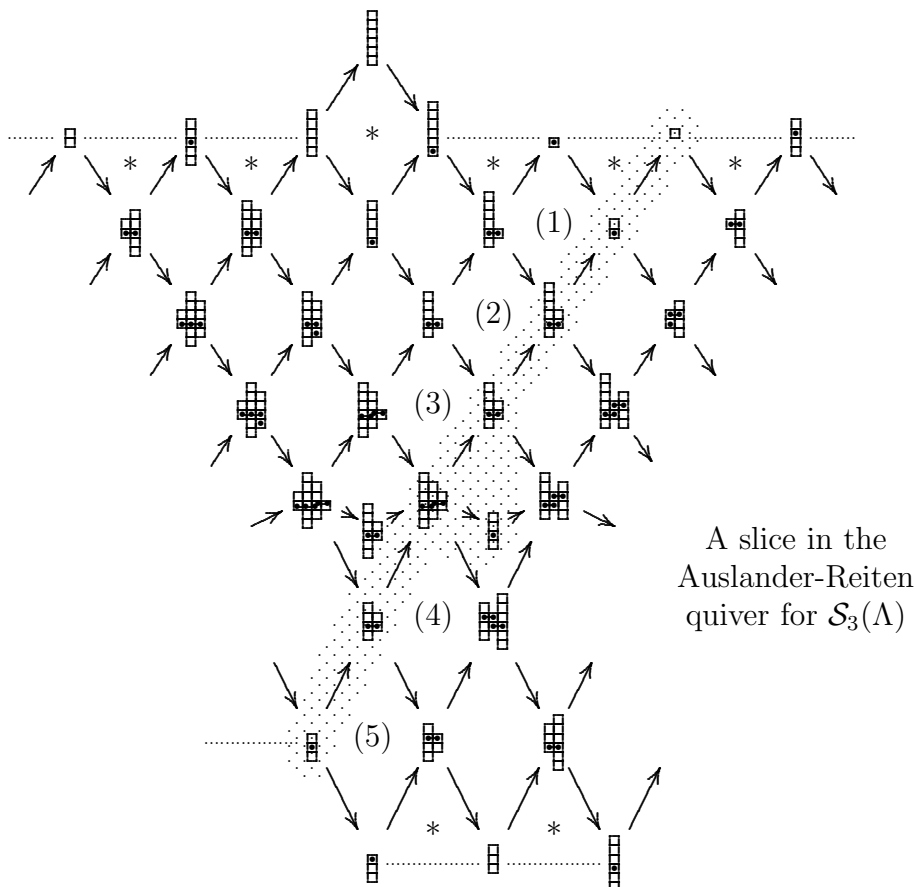
The diagram below is part of the Auslander-Reiten quiver for $\mathcal{S}_3(k[x]/x^6)$ and has been determined in [18] using covering theory. We show that the shaded region defines a slice also in the Auslander-Reiten quiver for $\mathcal{S}_3(\Lambda)$.

In Example 3.6 we have seen that the sequences labelled by a $*$ are Auslander-Reiten sequences. It is straightforward to verify that also the other meshes in the picture correspond to nonsplit short exact sequences. We show that each sequence is almost split. From the sequences labelled by a $*$ we obtain that the first term in a sequence given by one of the meshes (1) or (5) is the translate of the last term. Since every nonautomorphism of $C_1 = \begin{smallmatrix} \square \\ \square \end{smallmatrix}$ factors through $\begin{smallmatrix} \square \\ \square \end{smallmatrix} \rightarrow \begin{smallmatrix} \square \\ \square \end{smallmatrix}$, the sequence labelled (1) is an Auslander-Reiten sequence, by Proposition 5.3, (2. \Rightarrow 1.). As a consequence, the sequences ending at $C_1, \tau C_1, \tau^2 C_1, \tau^3 C_1$ and $\tau^{-1} C_1$ are all Auslander-Reiten sequences with the additional property that their middle term has two indecomposable summands which are as pictured. Similarly, every nonautomorphism of $A_5 = \begin{smallmatrix} \square \\ \square \end{smallmatrix}$ factors over the inclusion $\begin{smallmatrix} \square \\ \square \end{smallmatrix} \rightarrow \begin{smallmatrix} \square \\ \square \end{smallmatrix}$, hence it follows from Proposition 5.3 that the sequence labelled (5) is an Auslander-Reiten sequence with middle term a direct sum of two indecomposables, as in the picture. This holds also true for the sequence starting at $\tau^{-1} A_5$. In order to verify that the sequences labelled (2), (3), (4) are almost split, consider the modules

$$C_2 = \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}; \quad C_3 = \begin{smallmatrix} \square \\ \square \\ \square \\ \square \end{smallmatrix}; \quad A_4 = \begin{smallmatrix} \square \\ \square \\ \square \\ \square \end{smallmatrix}.$$

Let X be one of these modules, then $X = (U \subset V)$ where $V = V_1 \oplus V_2$ with $V_1 = \Lambda/\text{rad}^m \Lambda$ with $m = 4, 5, 6$ and $V_2 = \Lambda/\text{rad}^2 \Lambda$. Identifying $\text{End}_{\mathcal{S}} X$ as a subset of $\text{End}_{\Lambda} V$, it is easy to see that the radical of the endomorphism ring of the object X is given as follows.

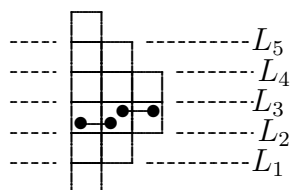
$$\text{rad End}_{\mathcal{S}} X = \text{rad End}_{\Lambda} V = \begin{pmatrix} \text{rad } \Lambda/\text{rad}^m \Lambda & \Lambda/\text{rad}^2 \Lambda \\ \Lambda/\text{rad}^2 \Lambda & \text{rad } \Lambda/\text{rad}^2 \Lambda \end{pmatrix}$$



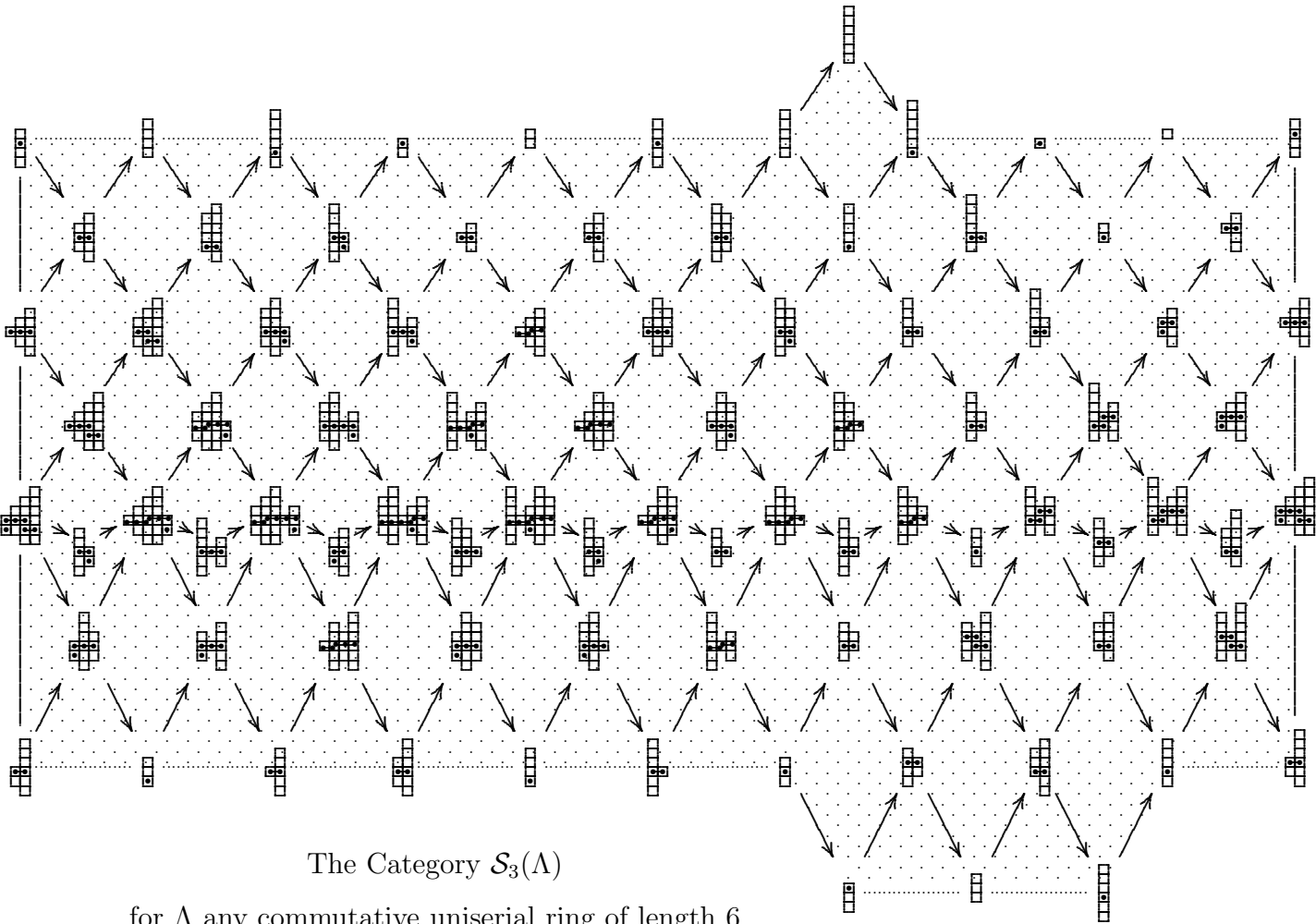
Thus the matrix units or radical generators form the four generators of $\text{rad End}_S X$ as a Λ -module. For each choice of X , each generator of $\text{rad End}_S X$ factors through the middle term of the sequence, thus condition 2. or 2'. in Proposition 5.3 is satisfied. As a consequence, the sequences labelled (2), (3), (4), and hence also the sequences ending at $\tau^2 C_2, \tau C_2, \tau^{-1} C_2, \tau C_3$ and $\tau^{-1} C_3$ are Auslander-Reiten sequences for which the middle term decomposes as indicated.

The module $A = (U \subset V)$ at the center of the slice where the summands of $V = V_1 \oplus V_2 \oplus V_3$ have length 6, 4, 2, is indecomposable: Consider the following submodules of V which are invariant under automorphisms of A . By ρ we denote the endomorphism of V given by a radical generator of Λ .

$$\begin{aligned}
 L_1(U \subset V) &= \text{rad}^4 V \cap \text{soc } V \\
 L_2(U \subset V) &= (\text{rad}^3 V \cap \text{soc}^2 V) \\
 L_i(U \subset V) &= \rho^{-(i-2)}(L_2(U \subset V)) \quad \text{for } i > 2
 \end{aligned}$$



As an isomorphism invariant for A we have the following system of vectorspaces and linear maps. If $k = \Lambda / \text{rad } \Lambda$ and $\frac{L_3}{L_2} = \frac{\text{rad}^3 V_1}{\text{rad}^4 V_1} \oplus \frac{\text{rad}^2 V_2}{\text{rad}^3 V_2} \oplus \frac{\text{rad} V_3}{0} = k \oplus k \oplus k$ then the



The Category $\mathcal{S}_3(\Lambda)$
 for Λ any commutative uniserial ring of length 6

map $u : \frac{(U \cap L_3) + L_2}{L_2} \rightarrow \frac{L_3}{L_2}$ given by the inclusion of the subspace $k(1, 1, 0) \oplus k(0, 1, 1)$ in $k \oplus k \oplus k$ makes the system, which has dimension type $\overset{2}{12321}$, indecomposable. Thus, A is indecomposable.

$$\begin{array}{ccccccc}
 & & & & \frac{(U \cap L_3) + L_2}{L_2} & & \\
 & & & & \downarrow u & & \\
 L_1 & \xleftarrow{\rho} & \frac{L_2}{L_1} & \xleftarrow{\rho} & \frac{L_3}{L_2} & \xleftarrow{\rho} & \frac{L_5}{L_4} \xleftarrow{\rho} \frac{V}{L_5}
 \end{array}$$

Finally consider the Auslander-Reiten sequence $\mathcal{E} : 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ starting at A . Our study of the sequences ending and starting at C_3 shows that C is as indicated and that C_3 occurs as a direct summand of B . Since A occurs as summand of the middle term of the sequence starting at A_4 , it follows that $\tau^{-1}A_4$ occurs as another summand of B . By Theorem 3.2, the sequence \mathcal{E} is split exact in each component, and hence B must have a third direct summand, say B' , of type $(4; 2)$. It remains to note that the arrow $A \rightarrow B'$ has trivial valuation; for this observe that the types of the summands of the middle term of the Auslander-Reiten sequence ending at A are as specified.

We have seen that the shaded region in the diagram forms a slice of the Auslander-Reiten quiver for $\mathcal{S}_3(\Lambda)$. According to Theorem 1.1, this translation quiver is independent of the choice of the base ring Λ which can be any commutative uniserial ring of length 6. ■

6 Appendix: A remark on gradings.

In the previous paragraph we demonstrated the indecomposability of the module $A = (V; U)$ at the center of the slice by introducing a filtration $(L_i)_i$ of V such that the associated graded module is indecomposable. For the grading to preserve indecomposability it is essential that the filtration be compatible with the submodule embedding, as we are going to show.

Let Λ be a commutative uniserial ring with radical generator p , radical factor field k and Loewy length n , and let V be a finitely generated Λ -module with filtration $0 = L_0 \subset L_1 \subset \dots \subset L_n = V$ such that the factors $V_i = L_i/L_{i-1}$, $1 \leq i \leq n$, are semisimple Λ -modules i.e. k -vector spaces. Then $G(V) = V_1 \oplus \dots \oplus V_n$ is a $k[t]/t^n$ -module with t acting on $(\bar{x}_1, \dots, \bar{x}_n) \in V_1 \oplus \dots \oplus V_n$ as $t \cdot (\bar{x}_1, \dots, \bar{x}_{n-1}, \bar{x}_n) = (\overline{px_2}, \dots, \overline{px_n}, 0)$.

Let also U be a finitely generated Λ -module, $(J_i)_i$ a filtration with semisimple factors, and $f : U \rightarrow V$ a Λ -homomorphism preserving the filtration, i.e. $f(J_i) \subset L_i$ holds for $1 \leq i \leq n$. Then

$$G(f) = f_1 \oplus \dots \oplus f_n : G(U) \rightarrow G(V)$$

is a homomorphism of $k[t]$ -modules where the maps $f_i : J_i \rightarrow L_i$ are induced by f . In particular if $U \subset V$ is a Λ -submodule, and $(J_i)_i$ is the induced filtration given by $J_i = U \cap L_i$, then the inclusion map $u : U \rightarrow V$ yields k -linear monomorphisms

$$u_i : \frac{U \cap L_i}{U \cap L_{i-1}} \cong \frac{(U \cap L_i) + L_{i-1}}{L_{i-1}} \rightarrow \frac{L_i}{L_{i-1}}$$

which make $G(u)$ an embedding of $k[t]$ -modules.

Thus if \mathcal{P} is a poset, $M = (V; (U_x)_{x \in \mathcal{P}}) \in \text{sub}_\Lambda \mathcal{P}$, and $(L_i)_{1 \leq i \leq n}$ a filtration of V , then $\tilde{G}(M) = (G(V); G(U_x)_x)$ is a corresponding object in $\text{sub}_{k[t]/t^n} \mathcal{P}$. Note that the isomorphism type of $\tilde{G}(M)$ in $\text{sub}_{k[t]/t^n} \mathcal{P}$ depends on the choice of the filtration of V :

Example 6.1. (Socle filtration) Let $\Lambda = \mathbb{Z}/p^3$, $\mathcal{P} = \bullet$ and $M = (V; U)$ where $V = \mathbb{Z}/p^3 \oplus \mathbb{Z}/p$ is the abelian group with generators a and b of order p^3 and p , respectively, and $U \subset V$ the subgroup generated by $pa + b$. Let $L_i = V[p^i] = \text{soc}^i V$, $1 \leq i \leq n$, be the layers of the socle filtration. Thus, the operation of t on $G(V)$ is as follows:

$$V_3 = k \xrightarrow{1} V_2 = k \xrightarrow{(1,0)} V_1 = k \oplus k$$

Hence, $G(V) \cong k[t]/t^3 \oplus k$, and similarly, $G(U) \cong k[t]/t^2$. The inclusion $u : U \rightarrow V$ gives rise to the linear maps $u_3 = 0$, $u_2 = 1 : k \rightarrow k$ and $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} : k \rightarrow k \oplus k$ and hence the image of $G(u)$ is contained in the first summand of $G(V)$. Thus, $\tilde{G}(M)$ decomposes as follows:

$$M : \begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \quad \tilde{G}(M) : \begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \square \\ \hline \end{array} \oplus \square$$

Example 6.2. (Filtration compatible with subgroup embedding) Let Λ , \mathcal{P} and $M = (V; U)$ be as above, and let the filtration $(L'_i)_i$ be given by $L'_1 = \text{soc } V \cap U$ and for $i \geq 2$ by $L'_i = p^{1-i}(L'_1)$ as in [13]. Then $G'(V)$ is the $k[t]$ -module with t acting as follows:

$$V_3 = k \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} V_2 = k \oplus k \xrightarrow{(1,0)} V_1 = k$$

Now the inclusion u gives rise to the maps $u'_3 = 0$, $u'_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : k \rightarrow k \oplus k$ and $u'_1 = 1$. Then $\tilde{G}'(M)$ is an indecomposable object in $\text{sub}_{k[t]/t^3} \mathcal{P}$:

$$\tilde{G}'(M) : \begin{array}{|c|} \hline \square \\ \hline \bullet \\ \hline \bullet \\ \hline \square \\ \hline \end{array}$$

There is a result in [3] related to our Theorem 1.1. The authors use the socle filtration to define a functor $\tilde{G} : \text{fspr}(\mathcal{P}^*, \Lambda) \rightarrow \text{fspr}(\mathcal{P}^*, k[t]/t^n)$ where $\text{fspr}(\mathcal{P}^*, \Lambda)$ is the full subcategory of $\text{sub}_\Lambda \mathcal{P}^*$ consisting of all objects with peak module injective. The triple $(E(V); V; U)$ in $\text{fspr}(\mathcal{P}^*, \Lambda)$ corresponding to $(V; U)$ in Example 6.1 is indecomposable where $E(V)$ denotes the injective envelope of V . However, the above decomposition of $\tilde{G}(M)$ gives rise to a decomposition

$$\tilde{G}(E(V); V; U) = (k[t]/t^3; k[t]/t^3; tk[t]/t^3) \oplus (k[t]/t^3; t^2k[t]/t^3; 0),$$

contradicting [3, Theorem 2.4(b)]. Hence the functor \tilde{G} cannot be expected to induce the correspondence between indecomposables in $\text{fspr}(\mathcal{P}^*, \Lambda)$ and in $\text{fspr}(\mathcal{P}^*, k[t]/t^n)$, or to preserve Auslander-Reiten sequences [3, Corollary 2.5].

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