Inclusion relations and convolution properties of certain subclasses of analytic functions defined by generalized Sălăgean operator

J. Patel

Abstract

In the present paper, we obtain some inclusion relations and convolution properties for certain subclasses of analytic functions in the unit disk which are defined by the generalized Sălăgean operator. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

1 Introduction, definitions and preliminaries

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}^\star(\alpha)$ and $\mathcal{K}(\alpha)$ $(0 \leqq \alpha < 1)$ denote the subclasses of functions in $\mathcal{A}$ which are respectively starlike of order α and convex of order α in $\mathbb{U}$. We denote $\mathcal{S}^\star(0) \equiv \mathcal{S}^\star$ and $\mathcal{K}(0) \equiv \mathcal{K}$.

If f and g are analytic in $\mathbb{U}$, we say that f is subordinate to g, written symbolically as $f \prec g$ or $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$.

Received by the editors December 2006.
Communicated by F. Brackx.
2000 *Mathematics Subject Classification :* 30C45.
Key words and phrases : Analytic functions; Differential subordination; Sălăgean operator; Gauss hypergeometric function.

The Hadamard product (or convolution) of two power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is defined as the power series $(f \star g)(z) = f(z) \star g(z) = \sum_{k=0}^{\infty} a_k\, b_k\, z^k$.

For $n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$, $\lambda \geqq 0$ and f given by (1.1), we consider the generalized Sălăgean operator defined as follows:

$$I_\lambda^n f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n\, a_k z^k \quad (z \in \mathbb{U}). \tag{1.2}$$

It follows from (1.2) that

$$\begin{aligned} I_\lambda^0 f(z) &= f(z), \\ I_\lambda^1 f(z) = I_\lambda f(z) &= (1-\lambda) f(z) + \lambda\, z f'(z), \\ I_\lambda^n f(z) &= I_\lambda \left(I_\lambda^{n-1} f(z) \right),\ n = 2, 3, \cdots . \end{aligned}$$

For $n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and $\lambda \geqq 0$, the operator I_λ^n was introduced and studied by Al-Oboudi [1] which reduces to the Sălăgean differential operator [14] for $\lambda = 1$. Furthermore, for negative integral values of n and $\lambda > 0$, we have

$$\begin{aligned} I_\lambda^{-1} f(z) &= \frac{z^{1-(1/\lambda)}}{\lambda} \int_0^z t^{(1/\lambda)-2} f(t)\, dt \quad (z \in \mathbb{U}), \\ I_\lambda^{-2} f(z) &= \frac{z^{1-(1/\lambda)}}{\lambda} \int_0^z t^{(1/\lambda)-2} I_\lambda^{-1} f(t)\, dt \quad (z \in \mathbb{U}) \end{aligned}$$

and, in general

$$\begin{aligned} I_\lambda^{-m} f(z) &= \frac{z^{1-(1/\lambda)}}{\lambda} \int_0^z t^{(1/\lambda)-2}\ I_\lambda^{-m+1} f(t)\, dt \\ &= \underbrace{I_\lambda^{-1}\left(\frac{z}{1-z}\right) \star I_\lambda^{-1}\left(\frac{z}{1-z}\right) \star \cdots \star I_\lambda^{-1}\left(\frac{z}{1-z}\right)}_{m-\text{times}} \star f(z) \\ &\qquad\qquad (m \in \mathbb{N} = \{1, 2, \cdots\};\ z \in \mathbb{U}). \end{aligned}$$

We now introduce the subclasses $\mathcal{S}_\lambda^n (A, B)$ and $\mathcal{R}_\lambda^n(\delta; A, B)$ of $\mathcal{A}$ as follows:

Definition 1. Let $n \in \mathbb{Z}$, A, B and λ be arbitrary fixed real numbers such that $-1 \leqq B < A \leqq 1$ and $\lambda \geqq 0$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\lambda^n (A, B)$ if it satisfies

$$\frac{z \left(I_\lambda^n f\right)'(z)}{I_\lambda^n f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}). \tag{1.3}$$

It is seen that

$$\mathcal{S}_0^1(1-2\alpha, -1) = \mathcal{S}_\lambda^0(1-2\alpha, -1) \equiv \mathcal{S}^\star(\alpha) \text{ and } \mathcal{S}_1^1(1-2\alpha, -1) \equiv \mathcal{K}(\alpha) \quad (0 \leqq \alpha < 1).$$

We denote $\mathcal{S}_\lambda^n (1, -1) \equiv \mathcal{S}_\lambda^n$. Similarly, let

$$\mathcal{K}_\lambda^n(A, B) = \left\{ f \in \mathcal{A} : 1 + \frac{z(I_\lambda^n f)''(z)}{(I_\lambda^n f)'(z)} \prec \frac{1 + Az}{1 + Bz},\ z \in \mathbb{U} \right\}$$

and we write $\mathcal{K}_\lambda^n(1,-1)=\mathcal{K}_\lambda^n$. Using the fact that

$$(I_\lambda^n\, zf')(z)=z(I_\lambda^n f)'(z)\quad (z\in\mathbb{U})$$

for any $f\in\mathcal{A}$, it is readily seen that

$$f\in\mathcal{K}_\lambda^n(A,B)\Longleftrightarrow zf'\in\mathcal{S}_\lambda^n\,(A,B)\,.$$

Definition 2. Let $n\in\mathbb{Z}$, A, B, δ and λ be arbitrary fixed real numbers such that $-1\leqq B<A\leqq 1$, $\delta\geqq 0$ and $\lambda\geqq 0$. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}_\lambda^n(\delta;A,B)$ if it satisfies the following subordination:

$$(I_\lambda^n f)'\,(z)+\delta\;(I_\lambda^n f)''\,(z)\prec\frac{1+Az}{1+Bz}\quad (z\in\mathbb{U}). \tag{1.4}$$

The class $\mathcal{R}_\lambda^n(\delta;A,B)$ generalizes a number of function classes studied earlier by several authors (see, e.g., MacGregor [4], Ponnusamy [12] and Al-Oboudi [1]). We write $\mathcal{R}_\lambda^n(0;1-2\alpha,-1)\equiv\mathcal{R}^n(\lambda,\alpha)$, the class of functions $f\in\mathcal{A}$ which satisfy the condition

$$\Re\,(I_\lambda^n f)'\,(z)>\alpha\quad (0\leqq\alpha<1;\; z\in\mathbb{U}).$$

The class $\mathcal{R}^n(\lambda,\alpha)$ for $n\in\mathbb{N}_0$ is recently studied by Al-Oboudi [1]. We, further denote

$$\mathcal{R}^1(\lambda,\alpha)\equiv\mathcal{R}(\lambda,\alpha)=\{f\in\mathcal{A}:\Re\,(f'(z)+\lambda\, zf''(z))>\alpha,\; 0\leqq\alpha<1;\; z\in\mathbb{U}\}\,.$$

The object of the present paper is to obtain several inclusion relationships and other interesting convolution properties of functions belonging to the subclasses $\mathcal{S}_\lambda^n\,(A,B)$ and $\mathcal{R}_\lambda^n(\delta;A,B)$ of $\mathcal{A}$ by using the method of differential subordination. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

In order to derive our main results, we need the following lemmas.

Lemma 1. *Let the function h be analytic and convex (univalent) in $\mathbb{U}$ with $h(0)=1$. Suppose also that the function ϕ given by*

$$\phi(z)=1+c_1z+c_2z^2+\cdots \tag{1.5}$$

is analytic in $\mathbb{U}$. If

$$\phi(z)+\frac{z\phi'(z)}{\gamma}\prec h(z)\quad (\Re(\gamma)\geqq 0,\;\gamma\neq 0;\; z\in\mathbb{U})\,,$$

then

$$\phi(z)\prec\psi(z)=\frac{\gamma}{z^\gamma}\int_0^z t^{\gamma-1}h(t)\,dt\prec h(z)\quad (z\in\mathbb{U})$$

and ψ is the best dominant.

Lemma 1 is due to Miller and Mocanu [6, 8] (see also Hallenbeck and Ruscheweyh [2]).

With a view to stating a well-known result, we denote by $\mathcal{P}(\alpha)$, the class of functions ϕ of the form (1.5) which are also analytic in $\mathbb{U}$ and satisfy the following inequality:

$$\Re\,(\phi(z)) > \alpha \quad (0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

It is known [16] that if $\phi_j \in \mathcal{P}(\alpha_j)$ $(0 \leqq \alpha_j < 1;\ j = 1, 2)$, then

$$(\phi_1 \star \phi_2) \in \mathcal{P}(\alpha_3) \quad (\alpha_3 = 1 - 2(1-\alpha_1)(1-\alpha_2)) \tag{1.6}$$

and the bound α_3 is the best possible.

Lemma 2 (cf., e.g., Pashkouleva [10]). *Let the function ϕ, given by* (1.5) *be in the class $\mathcal{P}(\alpha)$. Then*

$$\Re\,\{\phi(z)\} \geqq 2\alpha - 1 + \frac{2(1-\alpha)}{1+|z|} \quad (0 \leqq \alpha < 1;\ z \in \mathbb{U}).$$

For real or complex numbers a, b, $c\,(c \notin \mathbb{Z}_0^- = \{0, -1, -2, \cdots\})$, the Gauss hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(a,\, b;\, c;\, z) = 1 + \frac{a\,b}{c}\frac{z}{1!} + \frac{a(a+1)\,b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots .$$

We note that the above series converges absolutely for $z \in \mathbb{U}$ and hence represents an analytic function in the unit disk $\mathbb{U}$ (see, for details, [17, Chapter 14]).

The following identities are well known (cf., e.g., [17, Chapter 14]).

Lemma 3. *For real or complex numbers a, b, c $(c \notin \mathbb{Z}_0^-)$, we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,\, b;\, c;\, z) \quad (\Re(c) > \Re(b) > 0)\,; \tag{1.7}$$

$${}_2F_1(a,\, b;\, c;\, z) = {}_2F_1(b,\, a;\, c;\, z); \tag{1.8}$$

$${}_2F_1(a,\, b;\, c;\, z) = (1-z)^{-a}\,{}_2F_1\left(a,\, c-b;\, c;\, \frac{z}{1-z}\right); \tag{1.9}$$

$$(a+1)\,{}_2F_1(1,\, a;\, a+1;\, z) = (a+1) + az\,{}_2F_1(1,\, a+1;\, a+2;\, z). \tag{1.10}$$

We now state a result obtained by Singh and Singh [15].

Lemma 4. *Let ϕ be analytic in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re\,(\phi(z)) > 1/2$ in $\mathbb{U}$. Then for any function F analytic in $\mathbb{U}$, the function $\phi \star F$ takes values in the convex hull of the image of $\mathbb{U}$ under F.*

Lemma 5 (Miller and Mocanu [8, p. 35]). *Suppose that the function $\Psi : \mathbb{C}^2 \times \mathbb{C} \longrightarrow \mathbb{C}$ satisfies the condition*

$$\Re\,(\Psi(ix,\, y;\, z)) \leqq \varepsilon$$

for $\varepsilon > 0$, real $x, y \leqq -(1+x^2)/2$ and for all $z \in \mathbb{U}$. If ϕ, given by (1.5) *is analytic in $\mathbb{U}$ and*

$$\Re\,(\Psi(\phi(z),\, z\phi'(z);\, z)) > \varepsilon,$$

then $\Re\,(\phi(z)) > 0$ in $\mathbb{U}$.

2 Inclusion relationships

Unless otherwise mentioned, we shall assume throughout this paper that $-1 \leqq B < A \leqq 1, \lambda > 0, \delta > 0$ and $n \in \mathbb{Z}$.

Theorem 1. *Let* $f \in \mathcal{S}_\lambda^{n+1}(A,B)$ *and*

$$1 - B - \lambda(A-B) \geqq 0. \tag{2.1}$$

(i) *Then*

$$\mathcal{S}_\lambda^{n+1}(A,B) \subset \mathcal{S}_\lambda^{n}(A,B).$$

Further for $f \in \mathcal{S}_\lambda^{n+1}(A,B)$, *we also have*

$$\frac{z\left(I_\lambda^n f\right)'(z)}{I_\lambda^n f(z)} \prec \frac{1}{Q(z)} + 1 - \frac{1}{\lambda} = q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}), \tag{2.2}$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^z t^{(1/\lambda)-1}\left(\frac{1+Btz}{1+Bz}\right)^{(A-B)/B} dt & (B \neq 0) \\ \displaystyle\int_0^z t^{(1/\lambda)-1}\exp\left((t-1)Az\right)\, dt & (B = 0) \end{cases} \tag{2.3}$$

and q *is the best dominant of* (2.2).

(ii) *If in addition to* (2.1) *one has* $A \leqq -\dfrac{B}{\lambda}$ $(-1 \leqq B < 0)$, *then*

$$\mathcal{S}_\lambda^{n+1}(A,B) \subset \mathcal{S}_\lambda^{n}(1-2\,\rho,-1), \tag{2.4}$$

where $\rho = \left[\left\{{}_2F_1\left(1, \dfrac{B-A}{B}; 1+\dfrac{1}{\lambda}; \dfrac{B}{B-1}\right)\right\}^{-1} + (\lambda-1)\right]/\lambda$. *The result is the best possible.*

Proof. Let $f \in \mathcal{S}_\lambda^{n+1}(A,B)$ and

$$\varphi(z) = \frac{z\left(I_\lambda^n f\right)'(z)}{I_\lambda^n f(z)} \quad (z \in \mathbb{U}). \tag{2.5}$$

Then φ is of the form (1.5) and is analytic in $\mathbb{U}$. Using the identity

$$z\left(I_\lambda^n f\right)'(z) = \frac{1}{\lambda}\, I_\lambda^{n+1} f(z) + \left(1 - \frac{1}{\lambda}\right) I_\lambda^n f(z) \quad (z \in \mathbb{U}) \tag{2.6}$$

in (2.5) and carrying out logarithmic differentiation in the resulting equation, we deduce that

$$\varphi(z) + \frac{z\varphi'(z)}{\varphi(z) + (1/\lambda) - 1} = \frac{z\left(I_\lambda^{n+1} f\right)'(z)}{I_\lambda^{n+1} f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Hence by applying a result [7, Corollary 3.2] (with $\beta = 1$ and $\gamma = (1/\lambda) - 1$), we find that

$$\varphi(z) \prec \frac{1}{Q(z)} + 1 - \frac{1}{\lambda} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}),$$

where q is the best dominant of (2.2) and Q is defined by (2.3). This proves part (i) of the theorem.

To establish (2.4), we need to show that

$$\inf_{z\in\mathbb{U}} \{\Re\,(q(z))\} = q(-1). \tag{2.7}$$

The proof of the assertion (2.7) can be deduced on the same lines as in [11, Theorem 1].

The result is the best possible as q is the best dominant of (2.2). This completes the proof of Theorem 1.

Taking $A = 1 - 2\alpha,\ B = -1,\ n = 0$ and $\lambda = 1$ in Theorem 1, we get the following result due to MacGregor [5].

Corollary 1. *For $0 \leqq \alpha < 1$, we have*

$$\mathcal{K}(\alpha) \subset \mathcal{S}^\star(\rho_1),$$

where

$$\rho_1 = \left[{}_2F_1\left(1, 2(1-\alpha); 2; \frac{1}{2}\right)\right]^{-1} = \begin{cases} \dfrac{1-2\alpha}{2^{2(1-\alpha)}(1-2^{2\alpha-1})} & \left(\alpha \neq \dfrac{1}{2}\right) \\ \dfrac{1}{2\ln 2} & \left(\alpha = \dfrac{1}{2}\right). \end{cases}$$

The result is the best possible.

For $A = 1 - 2\alpha\ (0 \leqq \alpha < 1),\ B = -1,\ n = -1$ and $\lambda = 1/(\mu+1)$, Theorem 1 yields

Corollary 2. *If $f \in \mathcal{S}^\star(\alpha),\ \max\{-\mu, -\mu/2\} \leqq \alpha < 1$ and $\mu + 1 > 0$, then*

$$\Re\left\{\frac{z^\mu f(z)}{\displaystyle\int_0^z t^{\mu-1} f(t)\,dt}\right\} > \rho_2 \quad (z \in \mathbb{U}),$$

where $\rho_2 = (\mu+1)\left[{}_2F_1\left(1, 2(1-\alpha); \mu+2; \frac{1}{2}\right)\right]^{-1}$. *The result is the best possible.*

Using the well known result

$$f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^\star(\alpha),$$

in Corollary 2, we have

Corollary 3. *If $f \in \mathcal{K}(\alpha),\ \max\{-\mu, -\mu/2\} \leqq \alpha < 1$ and $\mu + 1 > 0$, then*

$$\Re\left\{\frac{zf'(z)}{f(z) - \dfrac{\mu}{z^\mu}\displaystyle\int_0^z t^{\mu-1} f(t)\,dt}\right\} > \rho_2 \quad (z \in \mathbb{U}),$$

where ρ_2 is given as in Corollary 2. *The result is the best possible.*

Remark 1. We note that Corollaries 2 and 3 improve the results obtained by Obradović [9, Theorems 2 and 3].

Theorem 2. *We have*

$$\mathcal{R}_\lambda^{n+1}(A,B) \subset \mathcal{R}_\lambda^{n}(1-2\rho_3,-1),$$

where ρ_3 is given by

$$\rho_3 = \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1-B)^{-1}\,{}_2F_1\left(1,\dfrac{B-A}{B};1+\dfrac{1}{\lambda};\dfrac{B}{B-1}\right) & (B \neq 0) \\ 1-\dfrac{A}{1+\lambda} & (B=0). \end{cases}$$

The result is the best possible.
Proof. Setting

$$\varphi(z) = \left(I_\lambda^n f\right)'(z) \quad (z \in \mathbb{U}), \tag{2.8}$$

we note that φ is of the form (1.5) and is analytic in $\mathbb{U}$. By making use of the identity (2.6) in (2.8) and differentiating both sides of the resulting equation, we obtain

$$\varphi(z) + \frac{z\varphi'(z)}{1/\lambda} = \left(I_\lambda^{n+1} f\right)'(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}).$$

Thus by Lemma 1, we deduce that

$$\begin{aligned} \left(I_\lambda^n f\right)'(z) \prec q_1(z) &= \frac{z^{-1/\lambda}}{\lambda}\int_0^z t^{(1/\lambda)-1}\left(\frac{1+At}{1+Bt}\right)dt \\ &= \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1+Bz)^{-1}\,{}_2F_1\left(1,1;1+\dfrac{1}{\lambda};\dfrac{Bz}{Bz+1}\right) & (B \neq 0) \\ 1+\dfrac{1}{1+\lambda}Az & (B=0), \end{cases} \end{aligned}$$

where we have also made a change of variables followed by the use of the identities (1.7), (1.9) and (1.10).

Next we show that

$$\inf_{z \in \mathbb{U}} \{\Re(q_1(z))\} = q_1(-1). \tag{2.9}$$

Following the same lines as in our demonstration of Theorem 4 [11], we can prove the assertion (2.9).

The result is the best possible as q_1 is the best dominant. The proof of Theorem 2 is thus completed.

Putting $A = 1-2\alpha$ $(0 \leqq \alpha < 1)$ and $B = -1$ in Theorem 2, we obtain the following result which improves the corresponding work of Al-Oboudi [1, Theorem 2.4] for $n \in \mathbb{N}_0$.

Corollary 4. *We have*

$$\mathcal{R}^{n+1}(\lambda,\,\alpha)\subset\mathcal{R}^{n}(\lambda,\,\rho_4),$$

where $\rho_4=\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;1+\frac{1}{\lambda};\frac{1}{2}\right)-1\right]$. *The result is the best possible.*

Using Theorem 2 ($(n-m)$ times) we get, after some calculations, the following interesting result.

Corollary 5. *Let* $n,m\in\mathbb{N}_0$ *be such that* $n>m\geqq 0$. *Then*

$$f\in\mathcal{R}^{n}(\lambda,\,\alpha)\Longrightarrow f\in\mathcal{R}^{m}(\lambda,\,,\eta_m),$$

where

$$\eta_m=\alpha+(1-\alpha)\left[{}_2F_1\left(1,1;1+\frac{1}{\lambda};\frac{1}{2}\right)-1\right]\sum_{j=0}^{n-m-1}\left[2-{}_2F_1\left(1,1;1+\frac{1}{\lambda};\frac{1}{2}\right)\right]^{j}. \tag{2.10}$$

The result is the best possible.

Remark 2. (i) If we put $m=0$ in Corollary 5, we obtain the following best possible result.

$$f\in\mathcal{R}^{n}(\lambda,\,\alpha)\Longrightarrow\Re\left(f'(z)\right)>\eta_0\quad(z\in\mathbb{U}),$$

where η_0 is given by (2.10) with $m=0$. Thus the function f is close-to-convex and hence univalent in $\mathbb{U}$.

(ii) We note that Corollary 5 improves a result due to Al-Oboudi [1, Theorem 2.6].

3 Convolution properties

Theorem 3. Let $-1\leqq B_j<A_j\leqq 1$ $(j=1,2)$. If the functions $f_j\in\mathcal{R}_{\lambda}^{n}(\delta;A_j,B_j)$ $(j=1,2)$, then the function $h\in\mathcal{A}$ defined by

$$h(z)=I_{\lambda}^{n}(f_1\star f_2)(z)\quad(z\in\mathbb{U}) \tag{3.1}$$

belongs to the class $\mathcal{R}_{\lambda}^{n}(\delta;1-2\varkappa,-1)$, where

$$\varkappa=(2\sigma_3-1)+(1-\sigma_3)[\delta+2(1-\delta)\ln 2],\quad \sigma_3=1-2(1-\sigma_1)(1-\sigma_2)$$

and

$$\sigma_j=\begin{cases}\dfrac{A_j}{B_j}+\left(1-\dfrac{A_j}{B_j}\right)(1-B_j)^{-1}\,{}_2F_1\left(1,1;1+\dfrac{1}{\delta};\dfrac{B_j}{B_j-1}\right) & (B_j\neq 0)\\ 1-\dfrac{A_j}{1+\delta} & (B_j=0)\end{cases}$$

for $j=1,2$.

Proof. Letting

$$\varphi_j(z) = \left(I_\lambda^n f_j\right)'(z) \quad (z \in \mathbb{U}), \tag{3.2}$$

we note that φ_j is of the form (1.5) and is analytic in $\mathbb{U}$ for each $j = 1, 2$. Since $f_j \in \mathcal{R}_\lambda^n(\delta; A_j, B_j)$, we deduce from (3.2) that

$$\varphi_j(z) + \delta\, z\varphi_j'(z) = \left(I_\lambda^n f_j\right)'(z) + \delta\, \left(I_\lambda^n f_j\right)''(z) \prec \frac{1 + A_j z}{1 + B_j z} \quad (j = 1, 2;\ z \in \mathbb{U}).$$

Hence by using Lemma 1 (with $\gamma = 1/\delta$) and following the lines of proof of Theorem 2, we get

$$\left(I_\lambda^n f_j\right)' \in \mathcal{P}(\sigma_j), \tag{3.3}$$

where

$$\sigma_j = \begin{cases} \dfrac{A_j}{B_j} + \left(1 - \dfrac{A_j}{B_j}\right)(1 - B_j)^{-1}\, {}_2F_1\left(1, 1; 1 + \dfrac{1}{\delta}; \dfrac{B_j}{B_j - 1}\right) & (B_j \neq 0) \\ 1 - \dfrac{A_j}{1 + \delta} & (B_j = 0) \end{cases}$$

for $j = 1, 2$. So, for $h = I_\lambda^n(f_1 \star f_2)$, we have by (3.3) and (1.6)

$$\left(z\left(I_\lambda^n h\right)'\right)' = \left(I_\lambda^n f_1\right)' \star \left(I_\lambda^n f_2\right)' \in \mathcal{P}(\sigma_3) \quad \left(\sigma_3 = 1 - 2(1 - \sigma_1)(1 - \sigma_2)\right).$$

Now

$$\left(z\left(I_\lambda^n h\right)'(z)\right)' = \left(I_\lambda^n h\right)'(z) + z\left(I_\lambda^n h\right)''(z) \in \mathcal{P}(\sigma_3)$$

implies that

$$\left(I_\lambda^n h\right)' \in \mathcal{P}(\sigma_4) \quad \left(\sigma_4 = (2\sigma_3 - 1) + 2(1 - \sigma_3)\ln 2\right),$$

again by using Lemma 1 (with $\gamma = 1,\ A = 1 - 2\sigma_3$ and $B = -1$). Thus

$$\begin{aligned} \Re\left\{\left(I_\lambda^n h\right)'(z) + \delta\, z\left(I_\lambda^n h\right)''(z)\right\} &= (1 - \delta)\,\Re\left\{\left(I_\lambda^n h\right)'(z)\right\} + \delta\,\Re\left\{\left(z\left(I_\lambda^n h\right)'(z)\right)'\right\} \\ &> (1 - \delta)\,\sigma_4 + \delta\,\sigma_3 \\ &= (2\sigma_3 - 1) + (1 - \sigma_3)(\delta + 2(1 - \delta)\ln 2) \\ &= \varkappa \quad (z \in \mathbb{U}). \end{aligned}$$

This completes the proof of Theorem 3.

By putting $A_1 = A_2 = 1 - 2\alpha$ $(0 \leqq \alpha < 1)$, $B_1 = B_2 = -1$, $n = 0$ and $\delta = \lambda$ in Theorem 3, we obtain

Corollary 6. If $f_j \in \mathcal{R}(\lambda, \alpha)$ $(j = 1, 2)$, then the function $(f_1 \star f_2) \in \mathcal{R}(\lambda, \widetilde{\alpha})$, where

$$\widetilde{\alpha} = 1 - 2(1 - \alpha)^2\, [1 + (1 - \lambda)(1 - 2\ln 2)] \left[2 - {}_2F_1\left(1, 1; 1 + \frac{1}{\lambda}; \frac{1}{2}\right)\right]^2.$$

Theorem 4. *Let* $-1 \leqq B_j < A_j \leqq 1$ $(j = 1, 2)$. *If the functions* $f_j \in \mathcal{R}^n_\lambda(A_j, B_j)$ $(j = 1, 2)$, *then the function* h *defined by* (3.1) *belongs to the class* $\mathcal{R}^n_\lambda(1 - 2\kappa, -1)$, *where*

$$\kappa = 1 - 4\,\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\,(1 - \ln 2)\,.$$

The bound κ *is the best possible for* $B_1 = B_2 = -1$.

Proof. Consider the function φ_j defined by (3.2) for each $j = 1, 2$. Then each φ_j is of the form (1.5), is analytic in $\mathbb{U}$ and

$$\varphi_j \in \mathcal{P}(\gamma_j) \quad \left(\gamma_j = \frac{1 - A_j}{1 - B_j};\ j = 1, 2\right)$$

Hence by (1.6), we have

$$(\varphi_1 \star \varphi_2) \in \mathcal{P}(\gamma_3) \quad \left(\gamma_3 = 1 - 2\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\right). \tag{3.4}$$

For the function h given by (3.1), a simple calculation shows that

$$(I^n_\lambda h)'(z) = \int_0^1 (\varphi_1 \star \varphi_2)(uz)\, du \quad (z \in \mathbb{U}). \tag{3.5}$$

Now by using Lemma 2 in (3.5), we deduce that

$$\begin{aligned}
\Re\left\{(I^n_\lambda h)'(z)\right\} &\geqq \int_0^1 \Re\,(\varphi_1 \star \varphi_2)(uz)\, du \\
&\geqq \int_0^1 \left(2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u|z|}\right) du \\
&> \int_0^1 \left(2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u}\right) du \\
&= 1 - 4\,\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\,(1 - \ln 2) \\
&= \kappa \quad (z \in \mathbb{U}).
\end{aligned}$$

To see that the bound κ is the best possible for $B_1 = B_2 = -1$, we consider the functions $f_j \in \mathcal{A}$ defined by

$$(I^n_\lambda f_j)(z) = \int_0^z \frac{1 - A_j\, t}{1 - t}\, dt \quad (j = 1, 2;\ z \in \mathbb{U})$$

so that from (3.5)

$$\begin{aligned}
(I^n_\lambda h)'(z) &= \int_0^1 \left(1 - (1 + A_1)(1 + A_2) + \frac{(1 + A_1)(1 + A_2)}{1 - uz}\right) du \\
&= 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)\ln(1 - z) \\
&\to 1 - (1 + A_1)(1 + A_2) + (1 + A_1)(1 + A_2)\ln 2 \quad \text{as } z \to -1,
\end{aligned}$$

which evidently completes the proof of Theorem 4.

Theorem 5. *Let* $-1 \leqq B_j < A_j \leqq 1$ $(j = 1, 2)$. *If the functions* $f_j \in \mathcal{R}_\lambda^n(\delta; A_j, B_j)$ $(\delta \geqq 0;\ j = 1, 2)$, *then the function* $g \in \mathcal{A}$ *defined by*

$$\left(I_\lambda^n g\right)(z) = \int_0^z \left(\left(I_\lambda^n f_1\right)' \star \left(I_\lambda^n f_2\right)'\right)(t)\, dt \quad (z \in \mathbb{U}) \tag{3.6}$$

belongs to the class $\mathcal{R}_\lambda^n(\delta; 1 - 2\tau, -1)$, *where*

$$\tau = \begin{cases} 1 - 4\,\dfrac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left[1 - \dfrac{1}{2}\,{}_2F_1\left(1, 1; 1 + \dfrac{1}{\delta}; \dfrac{1}{2}\right)\right] & (\delta > 0) \\ 1 - 2\,\dfrac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} & (\delta = 0). \end{cases}$$

The bound τ *is the best possible when* $B_1 = B_2 = -1$.
Proof. Let us consider the case when $\delta > 0$. Upon setting

$$h_j(z) = \left(I_\lambda^n f_j\right)'(z) + \delta\, z \left(I_\lambda^n f_j\right)''(z) \quad (j = 1, 2;\ z \in \mathbb{U}), \tag{3.7}$$

for $f_j \in \mathcal{R}_\lambda^n(\delta; A_j, B_j)$, we find that

$$h_j \in \mathcal{P}(\gamma_j) \quad \left(\gamma_j = \frac{1 - A_j}{1 - B_j};\ j = 1, 2\right). \tag{3.8}$$

Also, by (3.8) and (1.6), we see that

$$(h_1 \star h_2) \in \mathcal{P}(\gamma_3) \quad \left(\gamma_3 = 1 - 2\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\right).$$

From (3.7), we have

$$\left(I_\lambda^n f_j\right)'(z) = \frac{1}{\delta}\, z^{-(1/\delta)} \int_0^z t^{(1/\delta) - 1} h_j(t)\, dt \quad (j = 1, 2) \tag{3.9}$$

so that by (3.6) and (3.9) followed by a simple calculation, we obtain

$$\begin{aligned} \left(I_\lambda^n g\right)'(z) &= \left(I_\lambda^n f_1)\right)' \star \left(I_\lambda^n f_2\right)'(z) \\ &= \left(\frac{1}{\delta}\, z^{-(1/\delta)} \int_0^z t^{(1/\delta)-1} h_1(t)\, dt\right) \star \left(\frac{1}{\delta}\, z^{-(1/\delta)} \int_0^z t^{(1/\delta)-1} h_2(t)\, dt\right) \\ &= \frac{1}{\delta} \int_0^1 u^{(1/\delta)-1} h_0(uz)\, du, \end{aligned}$$

where

$$h_0(z) = \left(I_\lambda^n g\right)'(z) + \delta\, z \left(I_\lambda^n g\right)''(z) = \frac{1}{\delta} \int_0^1 t^{(1/\delta)-1} (h_1 \star h_2)(t)\, dt. \tag{3.10}$$

Now it follows from (3.10) and Lemma 2 that

$$\begin{aligned} \Re\left(h_0(z)\right) &\geqq \frac{1}{\delta} \int_0^1 t^{(1/\delta)-1} \left(2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u|z|}\right) du \\ &> \frac{1}{\delta} \int_0^1 t^{(1/\delta)-1} \left(2\gamma_3 - 1 + \frac{2(1 - \gamma_3)}{1 + u}\right) du \\ &= 1 - 4\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 - \frac{1}{\delta} \int_0^1 \frac{t^{(1/\delta)-1}}{1 + u}\, du\right) \\ &= 1 - 4\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left[1 - \frac{1}{2}\, F\left(1, 1; 1 + \frac{1}{\delta}; \frac{1}{2}\right)\right] \\ &= \tau \quad (z \in \mathbb{U}), \end{aligned}$$

which proves that $g \in \mathcal{R}_\lambda^n(\delta; 1-2\tau, -1)$ for the function g defined by (3.6).

In order to show that the bound τ is sharp for $B_1 = B_2 = -1$, we take the functions $f_j \in \mathcal{A}$ defined by

$$(I_\lambda^n f_j)'(z) = \frac{1}{\delta} z^{-(1/\delta)} \int_0^z t^{(1/\delta)-1} \left(\frac{1+A_j t}{1-t}\right) dt \quad (j=1,2;\ z \in \mathbb{U}).$$

Then for g given by (3.6), we have

$$\begin{aligned}(I_\lambda^n g)'(z) &+ \delta z (I_\lambda^n g)''(z) \\ &= \frac{1}{\delta}\int_0^1 t^{(1/\delta)-1}\left(1-(1+A_1)(1+A_2)+\frac{(1+A_1)(1+A_2)}{1-uz}\right) du \\ &= 1-(1+A_1)(1+A_2)+(1+A_1)(1+A_2)(1-z)^{-1}\, F\left(1,1;1+\frac{1}{\delta};\frac{z}{z-1}\right) \\ &\to 1-(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2)\, F\left(1,1;1+\frac{1}{\delta};\frac{1}{2}\right) \text{ as } z \to -1.\end{aligned}$$

Finally, for the case $\delta = 0$, the proof is simple and so we choose to omit the details involved.

Theorem 6. *If $f \in \mathcal{R}_\lambda^n(\delta; A, B)$ and $\psi \in \mathcal{K}$, then $f \star \psi \in \mathcal{R}_\lambda^n(\delta; A, B)$.*

Proof. It is well known that

$$\psi \in \mathcal{K} \Longrightarrow \Re\left(\frac{\psi(z)}{z}\right) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

Letting

$$F(z) = (I_\lambda^n f)'(z) + \delta z (I_\lambda^n f)''(z), \quad \varphi(z) = \frac{\psi(z)}{z} \quad (z \in \mathbb{U})$$

and using convolution properties, we get

$$(I_\lambda^n(f\star\psi))'(z) + \delta z (I_\lambda^n(f\star\psi))''(z) = (F\star\varphi)(z) \quad (z \in \mathbb{U}).$$

Since F is subordinate to the convex univalent function $(1+Az)/(1+Bz)$ in $\mathbb{U}$, the result follows by an application of Lemma 4.

Remark 4. It is known [13] that the following functions

$$\psi_1(z) = \sum_{k=1}^{\infty} \frac{\mu+1}{\mu+k} z^k \ (\mu > -1),\ \psi_2(z) = \sum_{k=1}^{\infty} \frac{z^n}{n} = -\log(1-z)$$
$$\text{and } \psi_3(z) = \frac{1}{1-x}\log\left(\frac{1+xz}{1-z}\right) (|x| \leqq 1,\ x \neq 1)$$

are convex (univalent) in $\mathbb{U}$. So, in view of Theorem 6 the class $\mathcal{R}_\lambda^n(\delta; A, B)$ is invariant under the following integral operators:

$$(f\star\psi_1)(z) = \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)\, dt \quad (\mu > -1), \quad (f\star\psi_2)(z) = \int_0^z \frac{f(t)}{t}\, dt \quad \text{and}$$
$$(f\star\psi_3)(z) = \int_0^z \frac{f(t)-f(xt)}{t-tz}\, dt \quad (|x| \leqq 1,\ x \neq 1).$$

Theorem 7. *Let* $-1 \leqq B_j < A_j \leqq 1$ $(j = 1, 2)$*. If the functions* $f_j \in \mathcal{R}_\lambda^n(A_j, B_j)$ $(j = 1, 2)$*, then the function* h *defined by* (3.1) *belongs to the class* $\mathcal{S}_\lambda^n$*, provided*

$$\frac{(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} < \frac{3}{4\left[1 + 2\left(\ln 2 - 1\right)^2\right]}. \tag{3.11}$$

Proof. From (3.4) and (3.5), we deduce that

$$\Re\left\{(I_\lambda^n h)'(z) + z\,(I_\lambda^n h)''(z)\right\} = \Re\left\{(I_\lambda^n f_1)'(z) \star (I_\lambda^n f_2)'(z)\right\}$$
$$> 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \quad (z \in \mathbb{U}), \tag{3.12}$$

which in view of Lemma 1 for

$$\gamma = 1, \quad A = -1 + \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \quad \text{and} \quad B = -1$$

yields

$$\Re\left\{(I_\lambda^n h)'(z)\right\} > 1 + \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left(\ln 2 - 1\right) \quad (z \in \mathbb{U}). \tag{3.13}$$

Again, from (3.13) and Lemma 1 for

$$\gamma = 1, \quad A = -1 - \frac{8(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left(\ln 2 - 1\right) \quad \text{and} \quad B = -1,$$

we get

$$\Re\left\{\vartheta(z)\right\} > 1 - \frac{8(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)}\left(\ln 2 - 1\right)^2 \quad (z \in \mathbb{U}), \tag{3.14}$$

where $\vartheta(z) = (I_\lambda^n h)(z)/z$ $(z \in \mathbb{U})$. If we let

$$\varphi(z) = \frac{z\,(I_\lambda^n h)'(z)}{(I_\lambda^n h)(z)} \quad (z \in \mathbb{U}), \tag{3.15}$$

then φ is of the form (1.5) and is analytic in $\mathbb{U}$. From (3.15), we obtain

$$(I_\lambda^n h)'(z) + z\,(I_\lambda^n h)''(z) = \vartheta(z)\left\{\varphi^2(z) + z\varphi'(z)\right\} = \Psi(\varphi(z), z\varphi'(z); z) \quad (z \in \mathbb{U}), \tag{3.16}$$

where $\Psi(u, v; z) = \vartheta(z)(u^2 + v)$. Thus by using (3.12) in (3.16), we get

$$\Re\left\{\Psi(\varphi(z), z\varphi'(z); z)\right\} > 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \quad (z \in \mathbb{U}).$$

Now for all real $x, y \leqq -\frac{1}{2}(1 + x^2)$, we have

$$\Re\left\{\Psi(ix, y; z\right\} = (y - x^2)\Re\{\vartheta(z)\} \leqq -\frac{1}{2}(1 + 3x^2)\Re\{\vartheta(z)\}$$
$$\leqq -\frac{1}{2}\Re\{\vartheta(z)\} \leqq 1 - \frac{2(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \quad (z \in \mathbb{U}),$$

by (3.11) and (3.14). Thus by an application of Lemma 5, we conclude that $\Re\{\varphi(z)\} > 0$ in $\mathbb{U}$, i.e., $h \in \mathcal{S}^n_\lambda$. The proof of Theorem 7 is thus completed.

Corollary 7. *If the functions* $f_j \in \mathcal{R}^n_\lambda(A_j, B_j)$ $(j = 1, 2)$*, then*

$$\xi(z) = \int_0^z \frac{I^n_\lambda(f_1 \star f_2)(t)}{t}\, dt \in \mathcal{K}^n_\lambda,$$

provided (3.11) *is satisfied.*

The proof of the above corollary follows from Theorem 7 by using the fact that $f \in \mathcal{K}^n_\lambda \Longleftrightarrow zf' \in \mathcal{S}^n_\lambda$.

Finally, we prove

Theorem 8. *If* $f_j \in \mathcal{A}\,(j = 1, 2)$ *and*

$$\Re\left\{(I^n_\lambda(f_1 \star f_2))'(z)\right] > 1 - \frac{3}{2\,[1 + 2(\ln 2 - 1)^2]} \quad (z \in \mathbb{U}),$$

then the function

$$\zeta(z) = \int_0^z \frac{I^n_\lambda(f_1 \star f_2)(t)}{t}\, dt \in \mathcal{S}^\star.$$

Proof. From the definition of ζ, we see that

$$\Re\left\{(I^n_\lambda(f_1 \star f_2))'(z)\right\} = \Re\{\zeta'(z) + z\zeta''(z)\} > 1 - \frac{3}{2\,[1 + 2(\ln 2 - 1)^2]} \quad (z \in \mathbb{U})$$

and the proof is completed similar to Theorem 7.

Remark 5. Taking $n = 0,\ A_j = 1 - 2\alpha_j$ $(0 \leqq \alpha_j < 1)$ and $B_j = -1$ for $j = 1, 2$ in Theorem 7 and Corollary 7, we get the corresponding results obtained by Lashin [3]. Similarly, for $n = 0$ in Theorem 8, we obtain the result of Lashin [3, Theorem 3].

Acknowledgement. The present investigation was supported by the University Grants Commission of India under its DRS Financial Assistance Program.

References

[1] F.M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Int. J. Math. Math. Sci., 27 (2004), 1429-1436.

[2] D.J. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc., 52 (1975), 191-195.

[3] A.Y. Lashin, Some convolution properties of analytic functions, Appl. Math. Lett., 18 (2005), 135-138.

[4] T.H. MacGregor, Functions whose derivative has a positive real part, Trans. Amer. Math. Soc., 104 (1962), 532-537.

[5] T.H. MacGregor, A subordination for convex functions of order α, J. London Math. Soc., 9 (1975), 530-536.

[6] S.S. Miller and P.T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J., 28 (1981), 157-171.

[7] S.S. Miller and P.T. Mocanu, Univalent solutions of Briot-Bouquet differential subordinations, J. Differential Equations, 56 (1985), 297-309.

[8] S.S. Miller and P.T. Mocanu, Differential Subordinations: Theory and Applications, Monogr. Textbooks Pure Appl. Math., vol. 225, Marcel Dekker, New York, 2000.

[9] M. Obradović, On certain inequalities for some regular functions in $|z| < 1$, Int. J. Math. Math. Sci., 8 (1985), 677-681.

[10] D.Ž. Pashkouleva, The starlikeness and spiral-convexity of certain subclasses of analytic functions, in: H.M. Srivastava, S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 1992.

[11] J. Patel, N.E. Cho and H.M. Srivastava, Certain subclasses of multivalent functions associated with a family of linear operators, Math. Comput. Modelling, 43 (2006), 320-338.

[12] S. Ponnusamy, Differential subordination and starlike functions, Complex Variables Theory Appl., 19 (1992), 185-194.

[13] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49 (1975), 109-115.

[14] G.S. Sălăgean , Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part I (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer, Berlin, 1983, pp. 362-372.

[15] R. Singh and S. Singh, Convolution properties of a class of starlike functions, Proc. Amer. Math. Soc., 108 (1989), 145-152.

[16] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions, Ann. Univ. Mariae Curie-Sklodowska Sect. A 40 (1986), 251-265.

[17] E.T. Whittaker and G.N. Watson, A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions, 4th edition, Cambridge University Press, Cambridge, 1927.

Department of Mathematics, Utkal University, Vani Vihar
Bhubaneswar-751004, India
E-mail: jpatelmath@yahoo.co.in