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PSEUDOCONVEXITY AND THE PROBLEM OF LEVI

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The Levi problem is a very old problem in the theory of several complex variables and in its original form was solved long ago. However, over the years various extensions and generalizations of the Levi problem were proposed and investigated. Some of the more general forms of the Levi problem still remain unsolved. In the past few years there has been a lot of activity in this area. The purpose of this lecture is to give a survey of the developments in the theory of several complex variables which arise from the Levi problem. We will trace the developments from their historical roots and indicate the key ideas used in the proofs of these results wherever this can be done intelligibly without involving a lot of technical details. For the first couple of sections of this survey practically no knowledge of the theory of several complex variables is assumed on the part of the reader. However, as the survey progresses, an increasing amount of knowledge of the theory of several complex variables is assumed.

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1. Domains of holomorphy.

(1.1) One of the great differences between one complex variable and several complex variables is the concept of a domain of holomorphy. On any open subset G of $\mathbf{C}$ there is a holomorphic function which cannot be extended across any boundary point of G. This is not the case in several complex variables, as was first pointed out by Hartogs [43]. The simplest example is the domain

$$\Omega = (\Delta_1 \times \Delta_{1/2}) \cup \left((\Delta_1 - \bar{\Delta}_{1/2}) \times \Delta_1\right)$$

where Δ_r is the open disc in $\mathbf{C}$ with center 0 and radius r. Any holomorphic

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function f on Ω can be extended to a holomorphic function on $\Delta_1 \times \Delta_1$, because the function

$$\frac{1}{2\pi i} \int_{|\zeta_1| = r} \frac{f(\zeta_1, z_2)\,d\zeta_1}{\zeta_1 - z_1} \qquad \left(\operatorname{Max}\left(\frac{1}{2}, |z_1|\right) < r < 1\right)$$

is holomorphic on $\Delta_1 \times \Delta_1$ and agrees with f on $\Delta_1 \times \Delta_{1/2}$ by virtue of Cauchy's formula.

Such an example calls for the introduction of the concept of a domain of holomorphy. A *domain of holomorphy* is a domain on which there exists a holomorphic function which cannot be extended to a larger domain.

Hartogs [43] obtained the following necessary condition for a special kind of domain (now called Hartogs' domains) to be a domain of holomorphy.

(1.2) THEOREM. *Let D be a domain in $\mathbf{C}$ and R be a positive function on D such that the set Ω in $\mathbf{C}^2$ defined by $z_1 \in D$ and $|z_2| < R(z_1)$ is a domain of holomorphy. Then $-\log R(z_1)$ is a subharmonic function on D.*

We will indicate the idea of its proof a little bit later. The domain Ω in the above example is a domain of the type described in Theorem (1.2), where $R(z_1) = 1$ for $\frac{1}{2} < |z_1| < 1$ and $R(z_1) = \frac{1}{2}$ for $|z_1| \leq \frac{1}{2}$. In this case $-\log R$ is not subharmonic, because $z_1 = 0$ is a maximum. So the theorem gives an explanation for the example.

The most obvious way to get a domain of holomorphy is to start with a holomorphic function on a domain and then use analytic continuation to continue the function to its maximum domain of definition, which, of course, is a domain of holomorphy. However, in this way one gets in general only a domain spread over $\mathbf{C}^n$ instead of a domain in $\mathbf{C}^n$. Such a domain is also called a *Riemann domain*. More precisely, a Riemann domain is a complex manifold together with a locally biholomorphic holomorphic map into some $\mathbf{C}^n$. When one considers domains of holomorphy, it is natural to consider Riemann domains instead of just domains in $\mathbf{C}^n$.

Cartan and Thullen [13] gave the following characterization of domains of holomorphy.

(1.3) THEOREM. *The following conditions for a Riemann domain $\pi: \Omega \to \mathbf{C}^n$ are equivalent.*

(i) *Ω is a domain of holomorphy.*

(ii) *Ω is holomorphically convex in the sense that, for every compact subset K of Ω the holomorphically convex hull $\hat{K}$ of K is compact, where $\hat{K}$ is defined as the set of all points x of Ω such that $|f(x)| \leq$ the supremum $\|f\|_K$ of $|f|$ on K for every holomorphic function f on Ω.*

(iii) *For every compact subset K of Ω, the infimum of d on K equals the infimum of d on $\hat{K}$, where for $x \in \Omega$, $d(x)$ is the largest positive number such that π maps an open neighborhood of x biholomorphically onto the ball in $\mathbf{C}^n$ with center $\pi(x)$ and radius $d(x)$.*

We indicate the idea of its proof only for the case $\Omega \subset \mathbf{C}^n$.

For (ii) $\Rightarrow$ (i), we exhaust Ω by an increasing sequence of $\hat{K}_\nu$ with compact K_ν. Take a discrete sequence $\{x_\nu\}$ in Ω with $x_\nu \notin \hat{K}_\nu$ which has every boundary point of Ω as an accumulation point. Construct $f = \prod_\nu (1 - f_\nu^{m_\nu})^\nu$ for suitable positive integers m_ν, where f_ν is a holomorphic function on Ω with $f_\nu(x_\nu) = 1$ and $|f_\nu| < 1$ on $\hat{K}_\nu$. Then f has too high a vanishing order near the boundary of Ω to be extended across any boundary point of Ω.

For (iii) $\Rightarrow$ (ii), one simply notes that $\hat{K}$ is bounded because the coordinate functions of $\mathbf{C}^n$ are holomorphic.

(i) implies (iii), because, if f is a holomorphic function on Ω which cannot be holomorphically extended across any boundary point of Ω and if there is a boundary point x of Ω whose distance η from a point y of $\hat{K}$ is $<$ the infimum σ of d on K, then for any n-tuple k the kth partial derivative $D^k f$ of f, being a holomorphic function on Ω, satisfies

$$|(D^k f)(y)| < \|D^k f\|_K < k!\varepsilon^{-|k|}\|f\|_{K_\varepsilon}$$

where $\eta < \varepsilon < \sigma$ and K_ε is the set of points having distance $< \varepsilon$ from K. By forming the power series of f at y, one concludes that f extends holomorphically across x, which is a contradiction.

(1.4) Now we want to indicate the idea of the proof of Theorem (1.2). First we introduce a definition. A complex manifold M is said to *satisfy the Kontinuitätssatz* if the following holds. For any sequence of maps $\varphi_\nu: \bar{\Delta} \to M$ (where Δ is the open unit disc in $\mathbf{C}$) which are holomorphic on Δ and continuous on $\bar{\Delta}$, if $\bigcup_\nu \varphi_\nu(\partial\Delta)$ is relatively compact in M, then $\bigcup_\nu \varphi_\nu(\bar{\Delta})$ is relatively compact in M. In the literature a manifold satisfying the Kontinuitätssatz is more commonly called *pseudoconvex*. We use our present terminology, because the adjective "pseudoconvex" is sometimes used to mean other things too in the literature. Because of the maximum modulus principle for holomorphic functions, it is obvious that a complex manifold which is holomorphically convex satisfies the Kontinuitätssatz.

We prove Theorem (1.2) by absurdity. If $-\log R$ is not subharmonic, then for some $z_1^0 \in D$ and some positive number r there exists a holomorphic polynomial $p(z_1)$ such that $-\log R < \operatorname{Re} p$ on $|z_1 - z_1^0| = r$ and $-\log R(z_1^0) = \operatorname{Re} p(z_1^0)$. Then the Kontinuitätssatz property of Ω is contradicted by the following sequence of discs indexed by ν:

$$z_1 \mapsto \left(z_1, \left(1 - \frac{1}{\nu}\right)e^{-p(z_1)}\right) \qquad \left(|z_1 - z_1^0| < r\right).$$

2. The original Levi problem.

(2.1) Suppose a domain Ω is given by $r < 0$, where r is a C^2 function whose gradient is nowhere zero on the boundary of Ω. Ω is said to be *pseudoconvex* (respectively *strictly pseudoconvex*) at a boundary point x if the complex Hessian $(\partial^2 r(x)/\partial z_i \partial \bar{z}_j)$ is positive semidefinite (respectively positive definite) when restricted to the complex tangent space of $\partial\Omega$. This property is independent of the choice of r. Ω is said to be pseudoconvex (respectively strictly pseudoconvex) if it is pseudoconvex (strictly pseudoconvex) at its every boundary point. When one wants to emphasize that a domain is only pseudoconvex and not necessarily strictly pseudoconvex, one also says that it is weakly pseudoconvex.

It is very easy to see that if Ω is strictly pseudoconvex at x then there exists an open neighborhood U of x in $\mathbf{C}^n$ and a holomorphic local coordinate system on U so that with respect to the new coordinate system $U \cap \Omega$ is strictly Euclidean convex at x.

(2.2) THEOREM (E. E. LEVI [56]). *A domain of holomorphy Ω with smooth boundary is pseudoconvex at every boundary point.*

The idea of the proof is as follows. Suppose the domain Ω is not pseudoconvex at a boundary point x. Then one can find a plane H of complex dimension 2 containing x such that $H \cap \Omega$ has smooth boundary at x and $H - \Omega$ is *strictly* pseudoconvex at x as a domain in H. There is an open neighborhood U of x in H such that $H - \Omega$ is strictly Euclidean convex at x with respect to some coordinate system S of U. Hence in $\Omega \cap U$ one can find a sequence of one-dimensional closed discs D_ν with respect to S such that $\bigcup_\nu \partial D_\nu$ is relatively compact in $\Omega \cap U$ but $\bigcup_\nu D_\nu$ is not relatively compact in $U^- \cap \Omega$, contradicting the Kontinuitätssatz for Ω.

(2.3) The original problem of Levi is to prove the converse that every domain Ω with smooth pseudoconvex boundary is a domain of holomorphy.

The Levi problem was first solved by Oka. He did the case $n = 2$ in [67] and the general case in [68]. The case of a general n was also solved at the same time independently by Bremermann [8] and Norguet [66].

Before we state Oka's result in its general form, let us first observe that for the Ω in the Levi problem, $-\log d$ is a plurisubharmonic function on Ω, where $d(x)$ is the distance from x to the boundary of Ω. To prove the observation, we assume the contrary. Then for some $x \in \Omega$ and some complex line L through x the Laplacian of the restriction of $-\log d$ to L is negative at x. We can assume that x is the origin and L is given by $z_2 = \cdots = z_n = 0$. From the power series expansion of $-\log d|L$ at x, it follows that for some $\varepsilon > 0$ and $r > 0$ there exists a holomorphic function $f(z_1)$ on $|z_1| < r$ such that $-\log d(x) = \operatorname{Re} f(0)$ and

$$-\log d(z_1, 0, \ldots, 0) < \operatorname{Re} f(z_1) - \varepsilon |z_1|^2$$

for $|z_1| < r$. Let y be a point of $\partial\Omega$ such that $|x - y| = d(x)$. Consider the disc

$$z_1 \to (z_1, 0, \ldots, 0) + (y - x)e^{-f(z_1)}$$

for $|z_1| < r$. This disc is tangential to $\partial\Omega$ at y and it is easy to verify that the restriction of the complex Hessian of r to the tangent space of this disc is negative, which is a contradiction.

(2.4) THEOREM (OKA [68]). *For a domain Ω spread over $\mathbf{C}^n$, the following conditions are equivalent.*

(i) *Ω is a domain of holomorphy.*

(ii) *Ω satisfies the Kontinuitätssatz.*

(iii) *$-\log d$ is plurisubharmonic, where d is as defined in* (1.3).

It is easy to see the equivalence of (ii) and (iii). (iii) $\Rightarrow$ (ii) is simply a consequence of the maximum principle for subharmonic functions. (ii) $\Rightarrow$ (iii) can be proved in more or less the same way as (1.2). To get (i) from (ii), Oka used the Cauchy-Weil integral formula to obtain a solution of the Cousin problem.

3. Stein manifolds.

(3.1) For the domain Ω in the original Levi problem, when Ω is bounded, $-\log d$ is a plurisubharmonic function on Ω which is at the same time an exhaustion function on Ω in the sense that for any $c \in \mathbf{R}$, $\{-\log d \leqslant c\}$ is a compact subset of Ω. In general, $-\log d + |z|^2$ is an exhaustion function on Ω. Moreover, $-\log d + |z|^2$ is strictly plurisubharmonic, in the sense that locally when one adds to it any C^2 function with sufficiently small second order partial derivatives, the result is still plurisubharmonic. From this point of view, a stronger version of the original Levi problem is to prove that every domain with a strictly plurisubharmonic exhaustion function is a domain of holomorphy. This was solved by Grauert [30] whose result is actually more general than this and deals with a general manifold instead of a domain. In the case of a general manifold, as an analog to a domain of holomorphy we have the concept of a Stein manifold.

A complex manifold is said to be *Stein* if it is holomorphically convex and its global holomorphic functions separate points and give local coordinates at every point. A result of Bishop-Narasimhan-Remmert [3], [63], [71] says that a complex manifold is Stein if and only if it is a (closed) complex submanifold of some $\mathbf{C}^N$.

(3.2) THEOREM (GRAUERT [30]). *A complex manifold which admits a smooth strictly plurisubharmonic exhaustion function is Stein.*

Grauert's method is to use the bumping technique to prove the finite-dimensionality of the first cohomology group of a sublevel set of the exhaustion function with coefficients in a coherent sheaf.

Narasimhan [64] generalized Grauert's result to the case of a complex space.

(3.3) THEOREM (NARASIMHAN). *A complex space which admits a continuous strictly plurisubharmonic exhaustion function is Stein.*

A complex space is the generalization of a complex manifold to allow singularities. More precisely, it is defined as follows. A subvariety of an open subset of a complex Euclidean space is a closed subset which locally is the set of common zeros of a finite number of holomorphic functions. A holomorphic or a (strictly) plurisubharmonic function on a subvariety is a function which locally is the restriction of such a function on some open subset of the Euclidean space. A holomorphic map from a subvariety to another subvariety is locally the restriction of a holomorphic map from an open subset of a Euclidean space to another Euclidean space. A complex space is constructed by using biholomorphic maps to piece together subvarieties. A complex space is Stein if it is holomorphically convex and its global holomorphic functions separate points and give local embeddings at every point.

A by-product of the results of Grauert and Narasimhan is the following.

(3.4) THEOREM. *If X is a Stein space, p is a continuous plurisubharmonic function on X and $Y = \{p < c\}$ for some real number c, then the pair (X, Y) is* *a Runge pair in the sense that every holomorphic function on Y can be approximated uniformly on compact subsets of Y by holomorphic functions on X.*

Grauert's characterization of Stein manifolds by the existence of smooth strictly plurisubharmonic functions can also be proved by using the L^2 estimates of $\bar{\partial}$. This approach is due to Kohn [50], Andreotti-Vesentini [2], and Hörmander [48]. Roughly speaking, this $\bar{\partial}$ method is a generalization of Kodaira's vanishing theorem for compact manifolds to the case of manifolds with boundaries. Like the proof of Kodaira's vanishing theorem it depends on Bochner's formula for the Laplacian.

4. Locally Stein open subsets.

(4.1) Because of Oka's characterization of domains of holomorphy by the plurisubharmonicity of $-\log d$, a domain Ω of $\mathbf{C}^n$ is Stein if and only if it is locally Stein in the sense that for every $x \in \partial\Omega$ there exists an open neighborhood U of x in $\mathbf{C}^n$ such that $U \cap \Omega$ is Stein.

A natural question to raise is the relationship between Steinness and local Steinness for open subsets of a general complex space. For example, we have the following problem.

(4.2) *Problem.* Is a locally Stein open subset of a Stein space Stein?

This problem still remains unsolved. The main difficulty is, of course, the lack of an analog of $-\log d$ for the case of a complex space. In the manifold case, Docquier-Grauert [18] proved the following.

(4.3) THEOREM. *Every locally Stein open subset G of a Stein manifold M is Stein.*

Their proof consists in finding a holomorphic retraction π: $U \to M$ from a Stein open neighborhood U of M in some $\mathbf{C}^N$ in which M is an embedded closed complex submanifold. Since $\pi^{-1}(G)$ is locally Stein, it follows that $\pi^{-1}(G)$ is Stein and G, being a submanifold of $\pi^{-1}(G)$, is Stein. Such a holomorphic retraction cannot exist whenever there is any singularity. So this technique cannot be applied to the general case of a complex space.

For complex spaces Andreotti-Narasimhan [1] proved the following partial result.

(4.4) THEOREM. *Let X be a complex space, S be its singular set, and G be a locally Stein open subset of X. If there exists an open neighborhood U of $S \cap \partial G$ in X such that $U \cap G$ is Stein, then G is Stein.*

The main idea of their proof is as follows. One can assume that X is of pure dimension n. Find a finite number of holomorphic maps π_ν: $X \to \mathbf{C}^n$ so that $S = \bigcap_\nu Z_\nu$, where Z_ν is the singular set of π_ν, i.e. the set of points of X where π_ν is not locally biholomorphic. For $z \in G - Z_\nu$, define $d_\nu(x)$ to be the largest positive number so that π_ν maps an open neighborhood of x biholomorphically onto the ball of radius $d_\nu(x)$ centered at $\pi_\nu(x)$. Then $-\log d_\nu$ is plurisubharmonic on $G - Z_\nu$, but approaches ∞ on Z_ν. To get a plurisubharmonic function on G, we define the following plurisubharmonic function

$$\varphi_\nu = -\log d_\nu + \log \sum_{\mu=1}^{k_\nu} |f_{\nu\mu}|^2$$

on G, where $f_{\nu\mu}$ $(1 \leq \mu \leq k_\nu)$ are suitable holomorphic functions on X whose common zero set is Z_ν (for example, if $v_1, \ldots, v_k$ are holomorphic vector fields on X generating the space of tangent vectors at every point of $X - S$, then one can take $f_{\nu\mu}$ to be the value of the Jacobian determinant of π_ν at $2v_{i_1} \wedge \cdots \wedge v_{i_n}$). Let p (respectively q) be a nonnegative smooth strictly plurisubharmonic exhaustion function on X (respectively $U \cap G$). Let

$$\psi = p + \max_\nu (\varphi_\nu, 0)$$

and let σ be a smooth function on U with compact support which is identically 1 on an open neighborhood of $S \cap \partial G$. Then one can find a smooth increasing convex function τ such that $\tau \circ \psi + \sigma q$ is a strictly plurisubharmonic exhaustion function on G.

(4.5) In both the proof of the theorem of Docquier-Grauert and that of Andreotti-Narasimhan $-\log$ of the Euclidean distance is used. Docquier-Grauert used it injectively, so to speak; and Andreotti-Narasimhan used it projectively.

It is natural to try to construct distance functions directly on Stein spaces to take the place of the Euclidean distance. There are two obvious choices. Unfortunately neither one works.

The first one can be described as follows. Suppose there is a proper map π from a Stein space X onto an open subset Ω of $\mathbf{C}^n$. Let G be a locally Stein open subset of X. For $x \in G$ let $d_1(x)$ be the largest positive number such that for some open neighborhood U of x in G, $\pi(U)$ is the open ball in $\mathbf{C}^n$ with center $\pi(x)$ and radius $d_1(x)$ and π maps U properly onto $\pi(U)$. In general $-\log d_1(x)$ is not plurisubharmonic even at the regular points of X. A simple counterexample is the following: $X = \mathbf{C}$ and π: $X \to \mathbf{C}$ is given by $\pi(z) = z^2$. $G =$ the complement of $(-\infty, -1] \cup [\sqrt{2}, \infty)$ in X. Then $-\log d_1$ assumes its maximum 0 at the point 1 of G. As a consequence, $-\log d_1$ cannot be plurisubharmonic on G.

The second obvious choice is the following. Suppose X is a subvariety of $\mathbf{C}^n$ and G is a locally Stein open subset of X. For $x \in G$, define $d_2(x)$ to be the largest positive number so that the intersection of X with the ball in $\mathbf{C}^n$ of center x and radius $d_2(x)$ is contained in G. Again, in general, $-\log d_2$ is not plurisubharmonic even at regular points of G. The following is a simple counterexample. $X = \mathbf{C}$ is embedded in $\mathbf{C}^2$ by $\varphi(z) = (z, z^2)$. $G = X - \{-\frac{3}{2}\}$. Then $-\log d_2$ assumes a local maximum $-\log(5\sqrt{5}/4)$ at the point 1 of G. As a consequence, $-\log d_2$ cannot be plurisubharmonic on G.

(4.6) The recent work of Hirschowitz [45] sheds some light on the problem of finding distance functions whose $-\log$ is plurisubharmonic for locally Stein open subsets. He considered a complex manifold X on which there are enough global holomorphic vector fields to generate the tangent space of X at every point of X. Such a manifold is called *infinitesimally homogeneous*. For this kind of manifold X one can find a finite number of holomorphic vector fields $v_1, \ldots, v_N$ on X so that they generate the tangent space of X at every point of X. Suppose G is an open subset of X. For $x \in G$, we define $d(x)$ as follows. For $a = (a_1, \ldots, a_N) \in \mathbf{C}^N$, let $\varphi_{x,a}(t)$ be the trajectory for the vector field $\operatorname{Re} \sum_{k=1}^N a_k v_k$ whose initial point $\varphi_{x,a}(0)$ is x. Now $d(x)$ is defined as the largest positive number such that $\varphi_{x,a}(t) \in G$ for all $0 \leqslant t < d(x)$ and all a satisfying $\sum_{k=1}^{N}|a_k|^2 = 1$.

(4.7) THEOREM (HIRSCHOWITZ). *If G satisfies the Kontinuitätssatz, then $-\log d$ is plurisubharmonic on G.*

The main idea of his proof is as follows. The function $d(x)$ can be alternatively described in the following way. From the existence theorem for ordinary differential equations, one can construct a holomorphic map σ from an open subset Ω of $X \times \mathbf{C}^N$ into X such that

(i) σ maps $X \times 0$ biholomorphically onto X,

(ii) $(x \times \mathbf{C}^N) \cap \Omega$ is connected for every $x \in X$,

(iii) for every $x \in X$ and $a = (a_1, \ldots, a_N) \in \mathbf{C}^N$,

$$\frac{\partial}{\partial t}\,\sigma(x, ta) = \sum_{k=1}^{N} a_k v_k(\sigma(x, ta))$$

for $t \in \mathbf{C}$ with $(x, ta) \in \Omega$ (where the left-hand side means, of course, the image of $\partial/\partial t$ under the differential of the map $t \mapsto \sigma(x, ta)$).

We can assume that Ω is the maximum open subset of $X \times \Omega$ with these properties. Now $d(x)$ can be alternatively defined as the largest positive number such that $(x, a) \in \sigma^{-1}(G)$ for every a belonging to the ball in $\mathbf{C}^N$ of radius $d(x)$ and center 0. From this alternative definition one easily sees (as in the proof of (1.2)) that $-\log d$ is plurisubharmonic on G, because $\sigma^{-1}(G)$ satisfies the Kontinuitätssatz and we are measuring distance only along the Euclidean direction of Ω.

Hirschowitz's result was used by Brun to obtain the following result [9].

(4.8) THEOREM. *Let π: $X \to S$ be a holomorphic fiber bundle whose base S is a Stein manifold and whose fiber F is a compact homogeneous manifold. Let A be a locally Stein open subset of X such that $A \cap \pi^{-1}(s)$ is Stein for every $s \in S$. Then A is Stein.*

The main idea of his proof is as follows. By considering the holomorphic vector bundle over S whose fiber at $s \in S$ is the vector space of all holomorphic vector fields on $\pi^{-1}(s)$, we conclude that X is infinitesimally homogeneous and we can construct, by the method of Hirschowitz, a distance function d to the boundary of A. It suffices to prove that for every $c \in \mathbf{R}$, the open subset $A_c := \{-\log d < c\}$ is Stein. We need only produce a strictly plurisubharmonic function φ on A_c. Since $A \cap \pi^{-1}(s)$ is Stein for every $s \in S$, we can cover S by an open cover $\{U_i\}$ such that $A_c \cap \pi^{-1}(U_i)$ is a relatively compact open subset of a Stein open subset W_i of X. Take a strictly plurisubharmonic function ψ_i on W_i and take a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$. Then one can find a strictly plurisubharmonic function σ on S such that $\sigma \circ \pi + \Sigma(\rho_i \circ \pi)\psi_i$ is strictly plurisubharmonic on A_c.

The special case where F is a 1-dimensional torus was proved earlier by Matsugu [59].

(4.9) In general, $-\log$ of the distance function constructed by Hirschowitz is not strictly plurisubharmonic. So in general we cannot conclude that a locally Stein relatively compact open subset of an infinitesimally homogeneous manifold is Stein. The following counterexample is due to Grauert [33].

Let $x_1, \ldots, x_{2n}$ be the coordinates of $\mathbf{R}^{2n}$ and let L be the lattice of $\mathbf{R}^{2n}$ generated by $e_\nu = (0, \ldots, 0, 1, 0, \ldots, 0)$, $1 \leq \nu \leq 2n$, where 1 is in the νth place. Let π: $\mathbf{R}^{2n} \to \mathbf{R}^{2n}/L$ be the natural projection. Choose an $\mathbf{R}$-linear map σ: $\mathbf{C}^n \to \mathbf{R}^{2n}$ such that for some $v \in \mathbf{C}^n$, $\sigma(\mathbf{C}v)$ is contained in $\{x_1 = 0\}$ and $\pi(\sigma(\mathbf{C}v))$ is dense in $\pi(\{x_1 = 0\})$. Let $X = \mathbf{C}^n/\sigma^{-1}(L)$ and τ: $\mathbf{C}^n \to X$ be the natural projection and $G = \tau(\sigma^{-1}(\{|x_1| < \frac{1}{2}\}))$. Then G is locally Stein, but every holomorphic function f on G is constant, because, by applying Liouville's theorem to the composite of $\mathbf{C} \to \tau(\mathbf{C}v)$ and f, we conclude that f is constant on the submanifold $\tau(\sigma^{-1}(\{x_1 = 0\}))$ of real codimension 1 in G, which is possible only when f is constant on G. In [57] Malgrange showed that one can construct Grauert's example in such a way that $H^1(G, \mathcal{O}_G)$ is not Hausdorff.

The key ingredient in the preceding example of Grauert is the existence of the relatively compact curve $\tau(\mathbf{C}v)$ in G. This holomorphic curve is a maximum integral curve for a holomorphic vector field on G. Hirschowitz [47] proved that under the assumption of the nonexistence of such a curve, a locally Stein relatively compact open subset of an infinitesimally homogeneous manifold is Stein. More precisely, we state his result as follows. Let X be an infinitesimally homogeneous manifold. An *interior integral curve* is a holomorphic map γ: $\mathbf{C} \to X$ with relatively compact image whose tangent vectors belong to some holomorphic vector field on X.

(4.10) THEOREM (HIRSCHOWITZ). *If an infinitesimally homogeneous complex manifold X admits a continuous plurisubharmonic exhaustion function φ and admits no interior integral curve, then X is Stein.*

The main idea of his proof goes as follows. Since it suffices to show that each $X_\alpha := \{\varphi < \alpha\}$ is Stein and since we can use local automorphisms defined by holomorphic vector fields to smooth out functions on compact subsets, we can assume without loss of generality that φ is C^∞. For any open subset Y of X, let $C(Y)$ be the set of all tangent vectors δ of Y such that the differential of every C^∞ plurisubharmonic function on Y is zero at δ. It suffices to show that $C(X_\alpha)$ is empty. For it follows from the emptiness of $CX_{\alpha+1}$ that there exist a finite number of C^∞ plurisubharmonic functions $\psi_1, \ldots, \psi_k$ on $X_{\alpha+1}$ such that $d\psi_1, \ldots, d\psi_k$ do not simultaneously vanish at any tangent vector of X_α. Then

$$(\alpha - \varphi)^{-1} + \sum_{j=1}^{k} \exp \psi_j$$

is a strictly plurisubharmonic exhaustion function on X_α. Now, suppose $\delta_0 \in C(X_\alpha)$. Let γ: $D \to X_\alpha$ (where D is a domain in $\mathbf{C}$) be the largest connected integral curve in X_α for a holomorphic vector field v on X such that δ_0 is a tangent vector to $\gamma(D)$. Let E be the set of all $z \in D$ such that the tangent vector of $\gamma(D)$ at $\gamma(x)$ belongs to $C(X_\alpha)$. Obviously E is closed. We want to show that E is open. Take $z \in E$. Let $x = \gamma(z)$ and δ be the tangent vector of $\gamma(D)$ at x. Let G be the 1-parameter group of local automorphisms defined by v. For $g \in G$ sufficiently close to the identity of G, g is defined on X_α and $gx \in X_\alpha$ and $g(X_\alpha) \subset X_{\alpha+1}$. The image $g\delta$ of δ under g belongs to

$C(g(X_\alpha)) \subset C(X_{\alpha+1})$. We claim that $g\delta \in C(X_\alpha)$. For otherwise there exists a C^∞ plurisubharmonic function f on X_α whose differential at $g\delta$ is nonzero. We can find a smooth increasing convex function σ on $\mathbf{R}$ such that $\sigma \circ \varphi > f$ outside some compact subset of X_α and $\sigma \circ \varphi < f$ at x. Let h be the function on X which agrees with $\sigma \circ \varphi$ on $X - X_\alpha$ and agrees with the maximum of $\sigma \circ \varphi$ and f on X_α. By smoothing out h by using local automorphisms, we obtain a C^∞ plurisubharmonic function on $X_{\alpha+1}$ whose differential at δ is nonzero. This contradicts $g\delta \in C(X_{\alpha+1})$. For $g \in G$ sufficiently close to the identity of G, $g(x)$ covers an open neighborhood of x in $\gamma(D)$. Hence E is open in D. It follows from the connectedness of D that $E = D$ and every tangent vector of $\gamma(D)$ belongs to $C(X_\alpha)$. In particular, the differential of φ is zero at every tangent vector of $\gamma(D)$ and $\varphi \circ \gamma$ is constant on D. Hence $\gamma(D)$ is relatively compact in X_α and $D = \mathbf{C}$, contradicting the nonexistence of any interior integral curve in X.

By using the above result, Hirschowitz [47] proved the following.

(4.11) THEOREM. *Suppose X is a compact homogeneous complex manifold and U is a noncompact domain spread over X which admits a continuous plurisubharmonic exhaustion function. If X is rational* (*i.e. birationally equivalent to the projective space*), *then U is holomorphically convex. If X is rational and irreducible* (*as a homogeneous space*), *then U is Stein. In particular, every noncompact subdomain of a Grassmannian which satisfies the Kontinuitätssatz is Stein.*

The case of the projective space was earlier obtained by Takeuchi [84] and Kieselman [49].

Another method of constructing a distance function whose $-\log$ is plurisubharmonic for locally Stein sets is to consider Kähler manifolds with suitable curvature conditions. It was first used by Takeuchi [85]. The following result in this direction was independently obtained by Ellencwajg [19] and Suzuki [83].

(4.12) THEOREM. *Suppose M is a Kähler manifold whose holomorphic bisectional curvature is positive. Then for any open subset Ω of M with the Kontinuitätssatz property*, $-\log$ *of the distance d* (*calculated from the Kähler metric*) *to the boundary of Ω is a continuous strictly plurisubharmonic function on $U \cap \Omega$ for some open neighborhood U of $\partial\Omega$ in M.*

The idea of the proof is as follows. Since locally we can approximate $\partial\Omega$ from within Ω by C^∞ strictly pseudoconvex boundaries and since we can approximate the potential function defining the Kähler metric by real-analytic functions, we can assume without loss of generality that $\partial\Omega$ is C^∞ strictly pseudoconvex and the Kähler metric is real-analytic. Take a point x sufficiently close to $\partial\Omega$ and let γ be the geodesic joining x to the point y on $\partial\Omega$ which is closest to x. Since γ is real-analytic, we can assume that we have a local coordinate chart $z_1, \ldots, z_n$ around y with y as the origin such that

(i) $x = (1, 0, \ldots, 0)$,

(ii) γ lies along the Re z_1-axis,

(iii) $\{z_1 = 0\}$ is tangential to $\partial\Omega$ at y and lies outside Ω, and

(iv) the given Kähler metric $\sum g_{jk} dz_j \otimes d\bar{z}_k$ satisfies $dg_{jk} = 0$ and $\sum g_{jk} dz_j \otimes d\bar{z}_k = \sum dz_j \otimes d\bar{z}_j$ at 0.

For $(z_1, \ldots, z_n) \in \Omega$ let $\delta(z_1, \ldots, z_n)$ be the distance of the curve

$$t \to (tz_1, z_2, \ldots, z_n), \qquad 0 \leq t \leq 1,$$

calculated with respect to the given Kähler metric, that is,

$$\delta(z_1, \ldots, z_n) = |z_1| \int_0^1 \sqrt{g_{1\bar{1}}(tz_1, z_2, \ldots, z_n)}\, dt.$$

Since δ agrees with d at x and $\delta \geq d$ everywhere, to prove the plurisubharmonicity of $-\log d$ at x, it suffices to prove the plurisubharmonicity of $-\log \delta$. Since

$$\partial\bar{\partial}(-\log\delta) = \frac{1}{\delta^2}\, \partial\delta \wedge \bar{\partial}\delta - \frac{1}{\delta}\, \partial\bar{\partial}\delta,$$

it suffices to look at $\partial\bar{\partial}\delta$.

$$\begin{aligned}\partial\bar{\partial}\delta(z_1, \ldots, z_n) = &-\frac{|z_1|}{4} \int_0^1 g_{1\bar{1}}(tz_1, z_2, \ldots, z_n)^{-3/2} \partial\left(g_{1\bar{1}}(tz_1, z_2, \ldots, z_n)\right) \\ &\qquad \wedge \bar{\partial}\left(g_{1\bar{1}}(tz_1, z_2, \ldots, z_n)\right) dt \\ &+ \frac{|z_1|}{2} \int_0^1 g_{1\bar{1}}(tz_1, z_2, \ldots, z_n)^{-1/2}\, \partial\bar{\partial}\left(g_{1\bar{1}}(tz_1, z_2, \ldots, z_n)\right) dt.\end{aligned}$$

Since $\sum_{j,k} \partial^2 g_{1\bar{1}}(0)\lambda_j\bar{\lambda}_k / \partial z_j \partial\bar{z}_k$ (for $\sum|\lambda_j|^2 = 1$) is the holomorphic bisectional curvature at 0 in the direction of $(1, 0, \ldots, 0)$ and $(\lambda_1, \ldots, \lambda_n)$ and since $dg_{jk} = 0$ at 0, it follows that $\partial\bar{\partial}\delta/\delta$ is strictly positive when x is sufficiently close to y. Moreover, when the holomorphic bisectional curvature is not positive, this proof shows that the smallest eigenvalue of the complex Hermitian of $-\log\delta$ is $\geq$ some positive number depending only on n times the lower bound of the holomorphic bisectional curvature. Hence we have the following corollaries [19].

(4.13) COROLLARY. *If X is a complex manifold admitting a continuous strictly plurisubharmonic function φ then every locally Stein relatively compact open subset Ω is Stein.*

(4.14) COROLLARY. *Suppose π: $X \to Y$ is a holomorphic submersion of complex manifolds such that every $y \in Y$ admits an open neighborhood U with $\pi^{-1}(U)$ Stein. Assume that Y is Stein. Then every locally Stein relatively compact open subset Ω of X is Stein.*

The proof of (4.13) is as follows. By Richberg's result [72], continuous strictly plurisubharmonic functions can be approximated on compact subsets by C^∞ strictly plurisubharmonic functions. Hence we can assume that φ is C^∞. We use the Kähler metric whose Kähler form is $\partial\bar{\partial}\varphi$. Then $-\log d + A\varphi$ is a strictly plurisubharmonic exhaustion function on Ω for A sufficiently large.

Corollary (4.14) is proved by constructing a strictly plurisubharmonic function on a neighborhood of $\bar{\Omega}$ in the following way. Find a locally finite open cover $\{U_i\}$ of Y with $\pi^{-1}(U_i)$ Stein. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$, let ψ_i be a C^∞ strictly plurisubharmonic function on $\pi^{-1}(U_i)$ and let φ be a C^∞ strictly plurisubharmonic function on Y. Then for A sufficiently large, $A(\varphi \circ \pi) + \Sigma(\rho_i \circ \pi)\psi_i$ is strictly plurisubharmonic on a neighborhood of Ω.

By using Ellencwajg's result instead of Hirschowitz's result, Brun **[10]** showed that in Theorem (4.8), F can be assumed to be a compact Riemann surface instead of a compact homogeneous manifold.

5. Increasing sequence of Stein open subsets.

(5.1) Because of the Oka theorem, a domain in $\mathbf{C}^n$ (or a domain spread over $\mathbf{C}^n$) is Stein if it is the union of an increasing sequence of Stein open subsets. This was first proved by Behnke-Stein **[4]** prior to Oka's theorem.

It is natural to ask when a manifold which is the union of an increasing sequence of Stein open subsets is Stein.

By the result of Docquier-Grauert (4.3), such a manifold is Stein if it is an open subset of a Stein manifold. By Hirschowitz's results (4.7) and (4.10) such a manifold is Stein if it is a relatively compact open subset of an infinitesimally homogeneous manifold. By the result of Ellencwajg and Suzuki (4.12), such a manifold is Stein if it is a relatively compact open subset of another manifold which either admits a continuous strictly plurisubharmonic function or carries a Kähler metric of positive holomorphic bisectional curvature.

As a partial answer to this question, in **[18]** Docquier-Grauert proved the following.

(5.2) THEOREM. *The union of an increasing* 1*-parameter family of Stein manifolds is Stein.*

The idea of their proof is as follows. Suppose $X = \bigcup_{t\in[0,1)} X_t$. It suffices to show that for any $0 < t_1 < t_2 < 1$, (X_{t_2}, X_{t_1}) is a Runge pair. Take $t_2 < t_3 < 1$. Embed X_{t_3} as a submanifold of some $\mathbf{C}^N$. We can find a Stein open neighborhood U of X_{t_3} in $\mathbf{C}^N$ so that there is a holomorphic retraction π: $U \to X_{t_3}$. Let K be an arbitrary compact subset of X_{t_1}. Find a strictly plurisubharmonic exhaustion function φ on U such that $K \subset \{\varphi < 0\}$. There exists $\varepsilon > 0$ with the following property: for $t_1 \leqslant s < t \leqslant t_2$, with $t - s < \varepsilon$, one can find a positive number γ such that

$$K \subset \{\varphi < 0\} \cap \{d_t > \gamma\} \subset \pi^{-1}(X_s)$$

where $d_t(x)$ for $x \in \pi^{-1}(X_t)$ is the Euclidean distance in $\mathbf{C}^N$ from x to the boundary of $\pi^{-1}(X_t)$. Since $\pi^{-1}(X_t)$ and $\{\varphi < 0\} \cap \{d_t > \gamma\}$ form a Runge pair, it follows that for every holomorphic function f on X_s, the function $f \circ \pi$ can be approximated on K by holomorphic functions on $\pi^{-1}(X_t)$. By choosing $t_1 = s_0 < s_2 < \cdots < s_k = t_2$ with $s_{j+1} - s_j < \varepsilon$, we conclude that every holomorphic function on X_{t_1} can be approximated uniformly on K by holomorphic functions on X_{t_2}.

In order for the above proof to work, it suffices to require as the definition of a 1-parameter family the following two conditions for every $0 \leqslant t_0 < 1$.

(i) $\bigcup_{0<t<t_0} X_t$ is a union of components of X_{t_0},

(ii) X_{t_0} is a union of components of the interior of $\bigcap_{t_0<t<1} X_t$.

Moreover, it suffices to assume that X_t is Stein for t in a dense subset of $[0, 1)$.

(5.3) Very recently Fornaess [23], [24], [25] (and Fornaess-Stout [27]) gave examples of a non-Stein union of an increasing sequence of Stein open subsets. We sketch below one of his examples. Choose natural numbers m_n for $n > 1$ such that

$$\varphi(z) = \sum_{n=1}^{\infty} \frac{1}{m_n} \log\left|\frac{z - 1/n}{2}\right|$$

is a subharmonic function. Let

$$\Omega = \{(z, w) \in \mathbf{C}^2 | 0 < |w| < 2, |z| < 2, \varphi(z) - \log|w| < 0\}.$$

Blow up $\mathbf{C}^2 - \{0\}$ at the set $\{(1/n,0)\}_{n=1}^{\infty}$ to get π: $M \to \mathbf{C}^2 - \{0\}$. Let X be the interior of the closure of $\pi^{-1}(\Omega)$. For $k \geqslant 1$, let $X_k = X - \pi^{-1}(\{(1/n,0)\}_{n \geqslant k})$. Then each X_k is Stein. For, the pullback of $|w|^2 + \Sigma_{n<k}|(z - 1/n)/w|^2$ (respectively $\varphi(z) - \log|w|$) by π can be extended to plurisubharmonic functions σ (respectively τ) on X_k, and

$$-\frac{1}{\tau} + \frac{1}{2 - |z|} + \frac{1}{2 - |w|} + \sigma$$

is a strictly plurisubharmonic exhaustion function on X_k. However, X is not Stein. For X contains the closed disc $\overline{\pi^{-1}(\{0 < |w| \leqslant 1, z = 1/n\})}$ for every n and this sequence of closed discs contradicts the Kontinuitätssatz property.

(5.4) Markoe [58] announced that the union X of an increasing sequence of Stein spaces X_ν is Stein if and only if $H^1(X, \mathcal{O}_X) = 0$, which is also equivalent to the condition that for any compact subset K of X_ν there exists $\mu > \nu$ such that holomorphic functions on X_ν can be approximated on K by holomorphic functions on X_μ.

The answer to the following question is still unknown.

(5.5) *Question.* Suppose $X_\nu \subset X_{\nu+1}$ are open subsets of a Stein space X. Is $\bigcup_{\nu=1}^{\infty} X_\nu$ Stein?

6. The Serre problem. Serre [73] raised in 1953 the question whether a holomorphic fiber bundle with a Stein base and a Stein fiber is Stein. In the same paper he proved the following case by using the Theorems A and B of Oka-Cartan.

(6.1) THEOREM. *A principal holomorphic fiber bundle whose base is a Stein manifold and whose structure group is a closed complex subgroup of a complex general linear group is Stein.*

Very recently Skoda [79], [80] gave a counterexample in which the base is an open subset in $\mathbf{C}$ and the fiber is $\mathbf{C}^2$. In the intervening years many affirmative answers to special cases of Serre's problem were given. We will discuss these special cases and then sketch Skoda's example.

Stein [81] proved the following.

(6.2) THEOREM. *The topological covering X of a Stein space Y is Stein.*

This is the special case of the Serre problem where the fiber is 0-dimensional. The proof is as follows. We can assume without loss of generality that Y is embedded as a complex subspace of some $\mathbf{C}^n$. One can find a Stein open neighborhood U of Y in $\mathbf{C}^n$ and a *continuous* retraction $\sigma: U \to Y$. We pull back X by σ to form a topological covering W of U. By using $-\log$ of the Euclidean distance and applying Oka's theorem, we conclude that W is Stein. Being a complex subspace of W, X is Stein.

Recently LeBarz [54] generalized Stein's result to the following.

(6.3) THEOREM. *Let $\pi: X \to Y$ be a holomorphic map of complex spaces. Assume that every point of Y admits an open neighborhood U such that the restriction of π to every component of $\pi^{-1}(U)$ is proper and has only finite fibers. If Y is Stein, then X is Stein.*

The proof is as follows. We can assume without loss of generality that Y is $\mathbf{C}^n$ and X is connected. Cover $\mathbf{C}^n$ by a locally finite cover $\{U_j\}$ of bounded Stein open subsets such that the restriction of π to every component $W_{j,m}$ of $\pi^{-1}(U_j)$ is proper and has only finite fibers. Let $\{\rho_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Fix some W_{j_0,m_0}. Let $k_{j,m}$ be the length of the shortest chain $W_{j_0,m_0}, W_{j_1,m_1}, \ldots, W_{j_l,m_l}$ such that W_{j_ν,m_ν} intersects $W_{j_{\nu+1},m_{\nu+1}}$ and $(j_l, m_l) = (j, m)$. Define $f = \Sigma_{j,m}(\rho_j \circ \pi)k_{j,m}$. Since $f = k_{j',m'} + \Sigma_{j,m}(\rho_j \circ \pi)(k_{j,m} - k_{j',m'})$ and $f = \pi \circ g$ locally, by calculating $\partial^2 g/\partial z_j \partial \bar{z}_k$ we conclude that there exists a C^∞ increasing convex function σ on $\mathbf{R}$ such that $f + \sigma \circ |z|^2 \circ \pi$ is a strictly plurisubharmonic exhaustion function on X.

Matsushima and Morimoto [60] solved the Serre problem for the case of the structure group being a connected complex Lie group.

(6.4) THEOREM. *A holomorphic fiber bundle E whose base B and fiber F are Stein manifolds and whose structure group G is a connected complex Lie group is Stein.*

The idea of their proof is as follows. A complex Lie group H is said to have property (P) if no compact subgroup of H contains a positive-dimensional complex subgroup of H. By using Lie group theory, they first showed that any complex Lie group H with property (P) contains a positive-dimensional, invariant, connected, closed, complex subgroup A of H such that H/A has property (P) and A is isomorphic to a closed complex subgroup of a complex general linear group. By considering $P \to P/A$ and using the preceding results of Serre and Stein, we conclude by induction on the dimension of H that every principal holomorphic bundle P whose base is Stein and whose structure group H is a complex Lie group with property (P) is Stein. Now, we can assume without loss of generality that G operates effectively on F. G must have property (P). Otherwise any orbit C of any relatively compact positive-dimensional complex subgroup of G is the relatively compact holomorphic image of C in F and by Liouville's theorem any holomorphic function on F must be constant on C. Let K be a maximal connected compact subgroup of G. Since G has property (P), G has a connected complex subgroup H whose Lie algebra is the complexification of the Lie algebra of K. Since the structure group G of E is continuously reducible to K, by Grauert's result on the equivalence of holomorphic and continuous bundles [31] it follows that the structure group G can be holomorphically reducible to H. Thus we have a principal holomorphic bundle P of base B and structure group H so that E is an associated bundle of P. Since H has property (P), P is Stein. Consider the principal bundle $P \times F \to E$ of structure group H. We can find a finite number of holomorphic functions $f_1, \ldots, f_N$ so that they embed $P \times F$ as a closed submanifold of $\mathbf{C}^N$. We average f_i over the action of K on $P \times F$ to obtain $\tilde{f}_i$. Since $\tilde{f}_i$ is constant on the subgroup K in each fiber of $P \times F \to E$ and since H is the complexification of K, it follows that $\tilde{f}_i$ is constant on each fiber of $P \times F \to E$ and gives rise to a holomorphic function g_i on E. The functions $g_1, \ldots, g_N$ embed E as a closed submanifold of $\mathbf{C}^N$. We would like to remark that in this proof the connectedness of G is very important. It is used in reducing the structure group G to H.

(6.5) Because of the condition that the structure group is complex and connected, the result of Matsushima-Morimoto cannot be applied even to the case where the fiber is a bounded Stein domain in $\mathbf{C}^n$. For the automorphism group of a bounded domain is a real Lie group which cannot be made into a complex Lie group unless it is regarded as a discrete group. With the case of a bounded Stein domain in $\mathbf{C}^n$ in mind, Fischer [20], [21], [22] introduced the concept of a Banach-Stein space and gave an affirmative answer to the Serre problem when the fiber is Banach-Stein. He called a Stein space F *Banach-Stein* if there exists a Banach space H of holomorphic functions which are invariant under the automorphism group Aut F of F and which satisfy the following conditions:

(i) H separates points of F,

(ii) for every sequence $\{x_\nu\}$ of F which has no accumulation point in F, there is an element of H which is unbounded on $\{x_\nu\}$,

(iii) for any map from a complex space $Y \to$ Aut F such that the associated map $Y \times F \to F$ is holomorphic, the induced map $Y \to$ Aut H is holomorphic.

THEOREM (FISCHER). *A holomorphic fiber bundle π: $X \to B$ with Stein base B and Banach-Stein fiber F is Stein.*

The idea of the proof is as follows. From the bundle X we can in a natural manner construct a holomorphic fiber bundle $\mathcal{H} \to B$ with base B and fiber H. Theorems A and B of Oka-Cartan can be generalized to the case of holomorphic Banach bundles (see [11]). The set $\Gamma(B, \mathcal{H})$ of all holomorphic cross sections of $\mathcal{H}$ over B can be identified in a natural way with a subset of the set of all holomorphic functions on X. We can thus conclude that holomorphic functions on X separate points. To prove the holomorphic convexity of X, we take a sequence of points $\{x_\nu\}$ in X having no accumulation point in X and want to produce a holomorphic function on X which is unbounded on $\{x_\nu\}$. We can assume without loss of generality that $\pi(x_\nu)$ approaches a limit point b in B. Let E_n be the set of all $f \in \Gamma(B, \mathcal{H})$ which, when regarded as functions on X, are bounded on $\{x_\nu\}$ by n. By the Baire category theorem, it suffices to show that the complement of E_n is dense in $\Gamma(B, \mathcal{H})$. Take $g \in E_n$, $\varepsilon > 0$, and a seminorm $\|\cdot\|$ in $\Gamma(B, \mathcal{H})$. We want to find $h \in \Gamma(B, \mathcal{H}) - E_n$ such that $\|h - g\| < \varepsilon$. For some open neighborhood U of b in B, we have a trivialization Φ: $\pi^{-1}(U) \to U \times F$. For $f \in H$ denote by f' the holomorphic function on $\pi^{-1}(U)$ obtained by pulling back f through

$\pi^{-1}(U) \to U \times F \to F$ and denote by $f'' \in \Gamma(U, \mathcal{H})$ the element corresponding to f'. By Theorem B applied to $\mathcal{H}$ and the open mapping theorem, there exists a constant C independent of ν with the property that for every $f \in H$ there exists $\tilde{f} \in \Gamma(B, \mathcal{H})$ such that $\tilde{f}$ agrees with f'' at $\pi(x_\nu)$ and $\|\tilde{f}\| \leqslant C\|f\|_H$. Write $\Phi(x_\nu) = (\pi(x_\nu), y_\nu)$. Choose $f \in H$ so that f is unbounded on $\{y_\nu\}$. Choose ν such that $|f(y_\nu)| > 2nC\varepsilon^{-1}\|f\|_H$. Then $h = (C\|f\|_H)^{-1}\varepsilon\tilde{f} + g$ satisfies the requirement.

It is not easy to verify whether a given Stein space is Banach-Stein.

(6.6) THEOREM (FISCHER [22]). *A bounded domain Ω in $\mathbf{C}^n$ with strictly pseudoconvex boundary is Banach-Stein.*

For one can find a smooth function φ on $\bar{\Omega}$ which is plurisubharmonic on Ω and whose zero-set is precisely $\partial\Omega$. By Hopf's lemma, for $g \in \operatorname{Aut}\Omega$, the lower derivate of $\varphi \circ g$ in the outward normal direction of $\partial\Omega$ is positive at every point of $\partial\Omega$. One can actually refine the argument to conclude that $\varphi(g(x))/d(x) \geqslant c$ for $d(x) < \varepsilon$, where d is the distance to the boundary of $\partial\Omega$ and c, ε are positive constants. Hence $d(x) \leqslant (1/c)\varphi(g(x)) \leqslant Cd(g(x))$ for some positive constant C when $d(x) < \varepsilon$. This shows that the Banach space H_k of all holomorphic functions f on Ω with $|f(x)| \leqslant M_f d(x)^{-k}$ for some M_f is invariant under Aut Ω. For a suitable k, H_k separates points and contains elements unbounded on sequences of points of Ω approaching $\partial\Omega$. The above proof is not Fischer's original proof. This proof is in Pflug [70] and is known also to R. M. Range.

Another example of a Banach-Stein space is any plane domain.

(6.7) THEOREM (SIU [76]). *Every domain Ω in $\mathbf{C}$ is Banach-Stein.*

When Ω is $\mathbf{C}$, one simply uses as the invariant Banach space H the set of all polynomials of degree $\leqslant 1$. When Ω is not $\mathbf{C}$, let $d(z)$ be the distance from $z \in \Omega$ to $\mathbf{C} - \Omega$ and let $a(z)$ be the infimum of $|g'(0)|^{-2}$, where g is a univalent holomorphic function from the open unit 1-disc into Ω such that $g(0) = z$. By the $\frac{1}{4}$-theorem of Koebe-Bieberbach, $a(z) \geqslant (4d(z))^{-2}$. Fix $z_0 \in \Omega$ and for $z \in \Omega$ let $h(z)$ be the distance from z_0 to z measured by the invariant metric $a(z)dz \otimes d\bar{z}$ of Ω. By considering the functions z and $(z-b)^{-1}$ for $b \in \mathbf{C} - \Omega$ and the Banach space H of all holomorphic functions f on Ω with $|f(z)| \leqslant M_f \exp(4h(z))$, one easily concludes that Ω is Banach-Stein. Hence every holomorphic fiber bundle over a Stein space whose fiber is an open subset of $\mathbf{C}$ is Stein [76]. This result was also obtained by Hirschowitz [46] and Sibony [75] by other means.

(6.8) THEOREM (HIRSCHOWITZ [44]). *A bounded domain Ω in $\mathbf{C}^n$ is Banach-Stein if it is strongly complete with respect to the Carathéodory metric d in the sense that for $x_0 \in \Omega$ and $c \in \mathbf{R}$, the set of all $x \in \Omega$ with $d(x, x_0) \leqslant c$ is compact.*

The invariant Banach space used is the set of all holomorphic functions f on Ω with $|f(x)| \leqslant M_f \exp(d(x, x_0))$ for some $x_0 \in \Omega$ and $M_f \in \mathbf{R}$.

Stehlé [82] introduced another method of proving the Steinness of a holomorphic fiber bundle. The method is to piece together plurisubharmonic functions to get a global plurisubharmonic exhaustion function.

(6.9) THEOREM (STEHLÉ [82]). *A holomorphic fiber bundle $\pi: X \to B$ with Stein base B is Stein if its fiber F is hyperconvex in the sense that there exists a strictly plurisubharmonic function φ on F which defines a proper map from F to $[c, 0)$ for some $c < 0$.*

We sketch below the main idea of his proof. In general, the product of two plurisubharmonic functions is not plurisubharmonic. In order to guarantee the plurisubharmonicity of a product, one has to introduce a stronger notion of plurisubharmonicity. A positive function f is called m-plurisubharmonic if $(1 - m)^{-1} f^{1-m}$ for $0 \leq m \neq 1$ (or $\log f$ for $m = 1$) is plurisubharmonic. The usual plurisubharmonic functions are precisely the 0-plurisubharmonic functions. For $m > m'$, every m-plurisubharmonic function is m'-plurisubharmonic. The maximum of m-plurisubharmonic functions is also m-plurisubharmonic. The function $(-\varphi)^{-1}$ is a 2-plurisubharmonic exhaustion function on F. For $1/(p - 1) + 1/(q - 1) = 1/(r - 1)$, the product of a p-plurisubharmonic function f and a q-plurisubharmonic function g is r-plurisubharmonic. We prove this for the case $p > 1$, $q < 1$, and $pq > 1$, which is the only case we need. By the convexity of the exponential function and because $p - 1 > 1 - q$ and $r < 1$, one has

$$\frac{1-r}{1-q} e^{s} - \frac{1-r}{p-1} e^{t} \leq \exp\left(\frac{1-r}{1-q} s - \frac{1-r}{p-1} t\right)$$

and, setting

$$s = (1 - q)\log \frac{g(x)}{g(a)} \quad \text{and} \quad t = (1 - p)\log \frac{f(x)}{f(a)},$$

one obtains

$$\frac{1}{1-p}\left(\frac{f(x)}{f(a)}\right)^{1-p} + \frac{1}{1-q}\left(\frac{g(x)}{g(a)}\right)^{1-q} \leq \frac{1}{1-r}\left(\frac{f(x)g(x)}{f(a)g(a)}\right)^{1-r}$$

with equality at $x = a$. The sub-mean-value property of the right-hand side for circles centered at a follows from the sub-mean-value property of the left-hand side. Now we describe the key step in the procedure of piecing together plurisubharmonic functions. Let $m > 1$. Suppose U, V are two relatively compact open subsets of B and we have a plurisubharmonic function g on $\pi^{-1}(U)$ whose restriction to every fiber is m-plurisubharmonic and which is relatively exhausting on $\pi^{-1}(U)$ in the sense that for every compact subset K of U and every $c \in \mathbf{R}$, $\{g \leq c\} \cap \pi^{-1}(K)$ is compact. We want to construct a relatively exhausting plurisubharmonic function g' on $\pi^{-1}(U \cup V)$ whose restriction to every fiber is m-plurisubharmonic. We have to assume that

(i) X is trivial over an open neighborhood W of V^-,

(ii) g is continuous up to $\pi^{-1}(U^-)$,

(iii) $(U - V)^-$ and $(V - U)^-$ are disjoint,

(iv) there exists a plurisubharmonic function h on $U \cup V$ such that $h > 1$ on $V - U$ and $h < 0$ on $U - V$.

Every point of $\pi^{-1}(W)$ can be denoted by (b, y) with $b \in W$ and $y \in F$.

Define f on $\pi^{-1}(W)$ by

$$f(b, y) = \sup\{ g(b', y) | b' \in U^- \cap V^- \}.$$

Then f is a relatively exhausting m-plurisubharmonic function on $\pi^{-1}(W)$. Choose a positive integer ν such that $m(1 - 1/\nu) > 1$. Then h^ν is $(1 - 1/\nu)$-plurisubharmonic at points where h is positive and $(h \circ \pi)^\nu f$ is m'-plurisubharmonic on $\pi^{-1}(\{h > 0\})$, where

$$m' = \frac{m(1 - 1/\nu) - 1}{m - 1/\nu - 1} > 0.$$

Define g' on $\pi^{-1}(U \cup V)$ as the maximum of g and $(h \circ \pi)^\nu f$ in the obvious sense. Then g' satisfies the requirement. By using this procedure, one can construct a strictly plurisubharmonic exhaustion function on X.

The following example of a hyperconvex space is due to Diederich-Fornaess [16].

(6.10) THEOREM. *A bounded Stein domain in* $\mathbf{C}^n$ *with* C^2 *boundary is hyperconvex.*

They proved this by showing that $-d^\eta \exp(-\eta L|z|^2)$ for sufficiently large positive L and sufficiently small positive η (in relation to L) is strictly plurisubharmonic at points of Ω which are sufficiently close to $\partial\Omega$, where d is the distance function to $\mathbf{C}^n - \Omega$. The idea of their proof is as follows. First, using the plurisubharmonicity of $-\log d$, they showed that the Levi form $\mathcal{L}_d(p; t)$ of d is positive semidefinite on the subspace of all tangent vector t satisfying $(\partial d)(t) = 0$ at points p sufficiently close to $\partial\Omega$. Hence for some negative constant C, $\mathcal{L}_d(p; t)$ is bounded from below by the product of C and the length of $((\partial d)(t))t$ for all tangent vector t at points p sufficiently close to $\partial\Omega$. Then a direct computation of the Levi form of $-d^\eta \exp(-\eta L|z|^2)$ gives the desired plurisubharmonicity. It follows from this and Stehlé's result that every holomorphic fiber bundle whose base is Stein and whose fiber is a bounded Stein domain in $\mathbf{C}^n$ with C^2 boundary is Stein.

Stehlé's technique of piecing together plurisubharmonic functions yields also the following result [82].

(6.11) THEOREM. *A holomorphic fiber bundle with Stein base is Stein if there exists a strictly plurisubharmonic exhaustion function* φ *on the fiber so that for every automorphism* g *of the fiber* $\varphi \circ g - \varphi$ *is bounded.*

Earlier, Königberger [53], using the simpler technique of piecing together plurisubharmonic functions by a partition of unity obtained the same result with the stronger assumption that the fiber is a bounded domain in $\mathbf{C}^n$ and $\varphi \circ g - \varphi$ has bounded first-order derivatives. However, it is difficult to determine whether a given domain satisfies these conditions.

In [77] Siu obtained the following result:

(6.12) THEOREM. *A holomorphic fiber bundle* $\pi\colon X \to B$ *with Stein base* B *is Stein if the fiber* F *is a bounded Stein domain in* $\mathbf{C}^n$ *with zero first Betti number.*

This result is very close to the case of the fiber being a general bounded

Stein domain, because the additional condition is only topological. It is quite likely that this approach can be modified to get rid of the additional topological condition. We sketch the idea of the proof below. Without loss of generality we can assume that B is a domain in some $\mathbf{C}^k$ with coordinates $w_1, \ldots, w_k$. By using the holomorphic Banach bundle over B obtained by replacing each fiber of X by the Banach space of all bounded holomorphic functions on that fiber, we conclude that every bounded holomorphic function on a fiber of X can be extended to a holomorphic function on X. Hence holomorphic functions on X separate points and give local coordinates. To prove holomorphic convexity, we need only prove that for every relatively compact Stein open subset B' of B and every sequence $\{x_\nu\}$ in $\pi^{-1}(B')$ without accumulation points, we can find a plurisubharmonic function on X which is unbounded on $\{x_\nu\}$. Without loss of generality we can assume that $\pi(x_\nu)$ approaches a limit point b_0 in B'. For some open neighborhood U of b in B, we have a trivialization Φ: $\pi^{-1}(U) \to U \times F$. Write $\Phi(x_\nu) = (\pi(x_\nu), y_\nu)$. We claim that it suffices to produce a plurisubharmonic function f on X which is unbounded on $\Phi^{-1}(b_0, y_\nu)$. We can find a positive number r such that every ball $B(b, r)$ with radius r and center $b \in B'$ is contained in B and X is trivial over $B(b, r)$ with a trivialization map Φ_b: $\pi^{-1}(B(b, r)) \to B(b, r) \times F$. For $x \in \pi^{-1}(B')$, let $b = \pi(x)$ and $\Phi_b(x) = (b, y)$ and define $\varphi(x)$ to be the supremum of $f(\Phi_b^{-1}(b', y))$ for all $b' \in B(b, r/2)$. The function φ is a well-defined plurisubharmonic function on $\pi^{-1}(B')$, because the transition functions of the fiber bundle X are necessarily locally constant. Clearly φ is unbounded on $\{x_\nu\}$. To construct f, we first extend the n coordinate functions on the fiber $\pi^{-1}(b_0) \subset \mathbf{C}^n$ to holomorphic functions $g_1, \ldots, g_n$. Let θ: $X \to \mathbf{C}^{k+n}$ be defined by $w_1, \ldots, w_k, g_1, \ldots, g_n$ and let Z be the subvariety of X where θ is not locally biholomorphic. For $x \in X - Z$ let $d(x)$ be the largest positive number such that θ maps an open neighborhood of x biholomorphically onto the ball of radius $d(x)$ centered at $\theta(x)$. Consider the projection σ: $X \times_B \tilde{B} \to X$, where $\tilde{B}$ is the universal covering of B. Since $X \times_B \tilde{B}$ is simply $\tilde{B} \times F$, it follows that the covering $X \times_B \tilde{B} - \sigma^{-1}(Z)$ of $X - Z$ is Stein and $-\log d$ is plurisubharmonic on $X - Z$. Clearly $-\log d$ is unbounded on $\{x_\nu\}$. The trouble is that $-\log d$ becomes ∞ on Z. But the Jacobian determinant $J(\theta)$ of θ vanishes at Z and locally $-\log d + 3 \log|J(\theta)|$ is plurisubharmonic even at points of Z. Unfortunately $J(\theta)$ is only a holomorphic $(n + k)$-form and not a function. We will make a holomorphic function out of $J(\theta)$ by using a holomorphic $(n + k)$-vector field v on X. Let $\{g_{ij}\}$ be the transition functions for X, that is, we have a covering $\mathfrak{U} = \{U_i\}_{i=0}^\infty$ of convex subsets of B and trivializations φ_i: $\pi^{-1}(U_i) \to U_i \times F$ such that $(\varphi_i\varphi_j^{-1})(b, y) = (b, g_{ij}(y))$. We assume that $b_0 \in U_0$ and $b_0 \notin U_i$ for $i > 0$. We can regard g_{ij} as a holomorphic function on $\pi^{-1}(U_j)$ by defining it to be the composite of g_{ij} and $\pi^{-1}(U_j) \to U_i \times F \to F$. A holomorphic $(n + k)$-vector field v is the same as $\{v_i\}$ with $v_i \in \Gamma(\pi^{-1}(U_i), \Theta_X)$ and $v_i = g_{ij}^{-1}v_j$. We will obtain $\{v_i\}$ by taking log in $v_i = g_{ij}^{-1}v_j$ and solving the Cousin problem for some holomorphic Banach bundle we now define. Fix $y_0 \in F$ and let $\log J(g)$ be the branch of the log of the Jacobian determinant $J(g)$ defined by $-\pi < \operatorname{Im} \log J(g)(y_0) < \pi$. Define

$$s(y) = \sup_{g \in \operatorname{Aut} F} \frac{|\log J(g)(y)| + 1}{\log|J(y)(y_0)| + 1}$$

for $y \in F$. It follows easily from the Theorem of Borel-Carathéodory that s is a finite-valued function on F. Moreover, $s \circ g \leqslant C_g s$ on F for every $g \in \operatorname{Aut} F$. Let E be the Banach space of all holomorphic functions h on F satisfying $|h| \leqslant C_h s$. Then E is invariant under Aut F and $\log J(g)$ belongs to E. Let $\mathfrak{S}$ be the holomorphic fiber bundle over B obtained by replacing each fiber of X by the Banach space of holomorphic functions on that fiber which are elements of E. Now $\{\log g_{ij}\}$ almost defines an element in $Z^1(\mathfrak{U}, \mathfrak{S})$. Suppose it does. Then by applying Theorem B to $\mathfrak{S}$, we can find $\{t_i\} \in C^0(\mathfrak{U}, \mathfrak{S})$ whose coboundary is $\{\log g_{ij}\}$ and the value of t_0 at b_0 is the zero function on $\pi^{-1}(b_0)$. Let $v_i = \exp t_i$. Then $\{v_i\}$ defines a holomorphic $(n+k)$-vector field v on X and $-\log d + 3 \log|\langle J(\theta), v\rangle|$ is a plurisubharmonic function on X which is unbounded on $\{\Phi^{-1}(b_0, y_\nu)\}$. When $\{\log g_{ij}\}$ does not define an element in $Z^1(\mathfrak{U}, \mathfrak{S})$, since its coboundary always defines an element of $C^2(\mathfrak{U}, \mathbf{Z})$, we can modify $\{\log g_{ij}\}$ by adding to it an element of $C^1(\mathfrak{U}, \mathcal{O}_B)$ and the above argument goes through after this modification.

(6.13) In the proof sketched above, the vanishing of the first Betti number is needed to define $\log J(g)$. One can replace the condition of the vanishing of the first Betti number by the condition that the structure group X is the identity component of Aut F. We define $\log J(g)$ simply for the purpose of producing the holomorphic $(n+k)$-vector field v. We can avoid this if there is an invariant Banach space of holomorphic n-vector fields on F including $\partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n$. This is the case, for example, when the Jacobian determinants of elements of Aut F are bounded on F. Earlier, Pflug [69] showed that a holomorphic fiber bundle with Stein base is Stein if the fiber is a bounded Stein domain in $\mathbf{C}^n$ whose every automorphism has bounded Jacobian determinant and which satisfies certain boundary conditions. It is very difficult to determine whether every automorphism of a given bounded domain has bounded Jacobian determinant.

One can modify the above argument so that the condition of the vanishing of the first Betti number can be relaxed to the following: F^- is contained in a bounded Stein domain $\tilde{F}$ such that $H_1(F, \mathbf{R}) \to H_1(\tilde{F}, \mathbf{R})$ is injective. We would like to mention also that the argument still works if, instead of assuming that F is a bounded Stein domain, we assume that F is a relatively compact Stein open subset of a Stein manifold whose canonical line bundle is trivial.

One is tempted to try to conclude the Steinness of X from the Steinness of $X \times_B \tilde{B}$ by proving that a complex manifold whose universal covering is Stein is holomorphically convex. Unfortunately, this is not true. Morimoto [62] gave an example of a lattice Γ in $\mathbf{C}^2$ which is of rank 3 over $\mathbf{Z}$ such that $\mathbf{C}^2/\Gamma$ is noncompact and admits no nonconstant holomorphic functions.

Morimoto's example has another significance. Hirschowitz [47] used it in the following way to give a topological covering of a compact manifold which satisfies the Kontinuitätssatz but is not holomorphically convex. Take a lattice $\tilde{\Gamma}$ of $\mathbf{C}^2 \oplus \mathbf{C}$ such that $\Gamma = \tilde{\Gamma} \cap \mathbf{C}^2$ and $X = (\mathbf{C}^2 \oplus \mathbf{C})/\tilde{\Gamma}$ is compact.

Then $(\mathbf{C}^2/\Gamma) \times \mathbf{C}$ is a topological covering of X. It satisfies the Kontinuitätssatz and is not holomorphically convex. The important point about this example is that the Kontinuitätssatz is satisfied, because it is easy to construct a topological covering of a compact manifold that is not holomorphically convex. For example, $\mathbf{C}^2 - 0$ is the universal covering of the Hopf manifold which is the quotient of $\mathbf{C}^2 - 0$ by the cyclic group action whose generator sends (z_1, z_2) to $(2z_1, 2z_2)$. In conjunction with this, we would like to mention the result of Carleson-Harvey [12] that a domain spread over a Stein manifold is Stein if it is biholomorphic to a topological covering of a compact Moišezon manifold (see (7.3) for definition). It is also very easy to show that a domain spread over a Stein manifold is Stein if it is biholomorphic to a topological covering of a compact Kähler manifold.

(6.14) We now discuss Skoda's counterexample [79], [80] to the Serre problem. The key point of Skoda's counterexample is the following result of Lelong [55]. Let Ω be a domain in $\mathbf{C}^d$, let ω_1, ω_2 be compact subsets of Ω with nonempty interiors, let V be a plurisubharmonic function on $\Omega \times \mathbf{C}^n$, and $M_V(r, \omega) = \sup\{V(x, z) | x \in \omega, |z| < r\}$. Then there exist positive numbers σ and C such that $M_V(r, \omega_1) \leqslant M_V(r^\sigma, \omega_2) + C$. To get Lelong's result, it suffices to consider the special case where ω_j is the ball $B^d(R_j)$ of radius R_j centered at 0, with $R_1 > R_2$, because the general case follows from considering a sequence of consecutively intersecting balls. The special case follows from the fact that $M_V(r, B^d(R))$ is a convex function of the two variables $(\log r, \log R)$. The construction of Skoda's example is as follows. Take domains Ω_j $(0 \leqslant j \leqslant N)$ of C such that $\Omega_0 \cap \Omega_j$ has two connected components Ω_j', Ω_j'' for $1 \leqslant j \leqslant N$ and Ω_j is disjoint from Ω_k for $1 \leqslant j < k \leqslant N$. The base of the bundle is $B = \bigcup_{j=0}^N \Omega_j$. Take $g_j \in \operatorname{Aut} \mathbf{C}^n$ $(1 \leqslant j \leqslant N)$. Define the fiber bundle X by collating $\Omega_j \times \mathbf{C}^n$ together so that the transition function at Ω_j' is g_j and the transition function at Ω_j'' is the identity of $\operatorname{Aut} \mathbf{C}^n$. We will show that for suitable choice of g_j, X admits no nonconstant plurisubharmonic function. A plurisubharmonic function on X is the same as $\{V_j\}$ where V_j is a plurisubharmonic function on $\Omega_j \times \mathbf{C}^N$ such that $V_0(x, z) = V_j(x, g_j(z))$ for $x \in \Omega_j'$, and $V_0(x, z) = V_j(x, z)$ for $x \in \Omega_j''$. It follows from Lelong's result that for any given compact subset ω_0 of Ω_0 with nonempty interior, one can find positive number σ, C such that

$$M_{V_0 \circ g_j}(r, \omega_0) \leqslant M_{V_0}(r^\sigma, \omega_0) + C$$

for all $1 \leqslant j \leqslant N$ and all $r > 0$. That is,

$$\sup\left\{V_0(x, z) | x \in \omega, z \in \textstyle\bigcup_j g_j(B^n(r))\right\} \leqslant M_{V_0}(r^\sigma, \omega_0) + C$$

where $B^n(r)$ is the ball in $\mathbf{C}^n$ with radius r and center 0. Let $\tilde{r}$ be the largest positive number such that $B^n(\tilde{r})$ is contained in the holomorphically convex hull of $\bigcup_j g_j(B^n(r))$ in $\mathbf{C}^n$. Then, since $M_{V_0}(r, \omega_0)$ is a strictly increasing function of r when V_0 is nonconstant, it follows that $\tilde{r} \leqslant r^{2\sigma}$ for r sufficiently large. One obtains a contradiction when the automorphisms g_j distort enough the shape of $B^n(r)$. For example, when $n = 2$, $N = 8$, and $g_j(z_1, z_2) = (z_1, z_2 \exp(\alpha_j z_1))$, $g_{j+4}(z_1, z_2) = (z_1 \exp(\alpha_j z_2), z_2)$, $1 \leqslant j \leqslant 4$, where the α_j's are the four 4th roots of unity.

In view of Skoda's example, what remains from the Serre problem is the following conjecture.

(6.15) CONJECTURE. A holomorphic fiber bundle with Stein base is Stein if its fiber is a relatively compact Stein open subset of a Stein manifold.

(6.16) Before Skoda's example was discovered, there was a general conjecture that unified the various forms of the Levi problem. The conjecture was the following: Suppose π: $X \to Y$ is a holomorphic map of complex spaces with Y Stein and every point of Y admits an open neighborhood U with $\pi^{-1}(U)$ Stein. Then X is Stein. The special case where X is an open subset of Y and π is the inclusion is the problem of proving that every locally Stein open subset of a Stein space is Stein. The special case where π: $X \to Y$ is a holomorphic fiber bundle is the Serre problem. Besides Skoda's example, the example of Fornaess given in (5.3) is also a counterexample to the general conjecture. For if we let f: $\Omega \to \mathbf{C}$ be defined by $f(z, w) = w$ and if we choose φ so that for some $c \in R$ and some mutually disjoint closed neighborhoods K_n of $1/n$ in $\{0 < |z| < 2\}$, $\varphi > c$ on ∂K_n, then the inverse images of $\mathbf{C} - 0$ and $\{|w| < e^c\}$ under $(f \circ \pi)|X$ are Stein.

It is also possible to modify X in the example of (5.3) so that there is a holomorphic map σ: $X \to \mathbf{C}^2$ with finite fibers and every point of $\mathbf{C}^2$ admits an open neighborhood U in $\mathbf{C}^2$ with $\sigma^{-1}(U)$ Stein (see [25]). This exhibits the pathological properties of a branched Riemann domain (see [34], [35] for other examples of pathological branched Riemann domains).

7. Weakly pseudoconvex boundaries.

(7.1) After Grauert constructed a non-Stein domain in a torus whose every boundary point is weakly pseudoconvex, Narasimhan [65] raised the question whether a domain with smooth pseudoconvex boundary which is strictly pseudoconvex at at least one boundary point is holomorphically convex. In giving a negative answer to this question, Grauert [33] constructed the following example.

Let R be a compact, projective algebraic manifold with $H^1(R, \mathcal{O}_R) \neq 0$. Let G be a negative holomorphic line bundle over R in the sense of [32]. Let $Y \subset G$ be the zero cross section of G. By adding an infinite point to every fiber of G, we obtain a holomorphic fiber bundle α: $X \to R$ with $\mathbf{P}_1$ as fiber. Let β: $X \to X'$ be the map which blows down Y to a point. X' is a projective algebraic space. Let L be a negative holomorphic line bundle over X'. We can give L a Hermitian metric whose curvature form is strictly negative. Let P be a holomorphic line bundle over R such that P is topologically trivial but no tensor power of P is holomorphically trivial (for example, when R is a torus, P is a point in the Picard variety whose integral multiples are all nonzero). Then the transition functions of P can be chosen to be constant functions of absolute value 1. Hence we can furnish P with a Hermitian metric whose curvature form is identically zero. Let γ: $F \to X$ be the holomorphic line bundle $\beta^*(L) \otimes \alpha^*(P)$, carrying a Hermitian metric induced by those of L and P. Let Ω be the set of vectors of F with norm < 1. Then the boundary of Ω is pseudoconvex everywhere and is strictly pseudoconvex at points of $\partial\Omega - \gamma^{-1}(Y)$. We claim that $\Omega \cap \gamma^{-1}(Y)$ admits no nonconstant holomorphic function and hence cannot be holomorphically convex. For α

induces a biholomorphic map from $\Omega \cap \gamma^{-1}(Y)$ onto the open subset D of P consisting of all vectors of norm < 1 in P. If there is a nonconstant holomorphic function on $\Omega \cap \gamma^{-1}(Y)$, then there is a nonconstant holomorphic function f on D. We now expand f in power series along the fibers of P. The kth coefficient f_k in this power series is a holomorphic cross section of P^{-k} over R. Since the Chern class of P^{-k} is zero and P^{-k} is not holomorphically trivial for $k \neq 0$, it follows that f_k is identically zero for $k \neq 0$, contradicting that f is nonconstant.

On the positive side, recently Michel [61] gave an affirmative answer to Narasimhan's question in the case of a compact homogeneous manifold.

(7.2) THEOREM. *Let Ω be a locally Stein open subset of a compact homogeneous manifold X. If the boundary of Ω at at least one point b is smooth and strictly pseudoconvex, then Ω is Stein.*

The main idea of the proof is as follows. Suppose Ω is not Stein. Then by Hirschowitz's result (4.10), there is an interior integral curve Γ in Ω. We join a point $a \in \Gamma$ to b by a smooth curve γ: $[0, 1] \to X$ so that $\gamma(t) \in \Omega$ for $0 \leqslant t < 1$. We can find a smooth curve σ: $[0, 1] \to G$ such that $\sigma(0)$ is the identity element of G and $\gamma(t)$ is the image $\sigma(t)a$ of a under $\sigma(t)$. We claim that $\sigma(t)\Gamma \subset \Omega$ for $0 \leqslant t < 1$. For this claim, it suffices to show that if t approaches some $t_* \in [0, 1)$ from the left such that each $\sigma(t_\nu)\Gamma \subset \Omega$, then $\sigma(t_*)\Gamma \subset \Omega$. By Hirschowitz's result we can construct some distance function d to $X - \Omega$ such that $-\log d$ is plurisubharmonic on Ω. Let c be the infimum of all $\gamma(t_\nu)$. Since $\sigma(t_\nu)\Gamma$ is a relatively compact holomorphic image of $\mathbf{C}$, the plurisubharmonic function $-\log d$ is constant on each $\sigma(t_\nu)\Gamma$ and $\sigma(t_\nu)\Gamma \subset \{d \geqslant c\}$. Hence $\sigma(t_*)\Gamma \subset \Omega$. It follows from $\bigcup_{0 \leqslant t < 1}\sigma(t)\Gamma \subset \Omega$ that $\sigma(1)\Gamma$ is a nonconstant holomorphic curve which contains b and is contained in Ω^-, contradicting the strict plurisubharmonicity of $\partial\Omega$ at b.

Narasimhan's question is related to the following conjecture of Grauert-Riemenschneider [36], certain special cases of which are known (see [78]).

(7.3) CONJECTURE. Suppose L is a holomorphic line bundle over a compact complex manifold X of dimension n and suppose L carries a Hermitian metric whose curvature form is positive semidefinite everywhere and positive definite on a nonempty open subset G of X. Then X is Moišezon in the sense that the meromorphic function field of X has transcendence degree n over $\mathbf{C}$.

Consider the open subset D of the dual L^* of L which consists of all vectors of length < 1. Then ∂D is weakly pseudoconvex everywhere and strictly psuedoconvex at every point of $\pi^{-1}(G) \cap \partial\Omega$, where π is the projection $L^* \to X$. If we have an affirmative answer to Narasimhan's question, then D is holomorphically convex and, by expanding holomorphic functions on D in power series along the fibers of L^*, we obtain coefficients which are holomorphic cross sections of L^k over X whose quotients give rise to enough meromorphic functions on X to conclude that X is Moišezon. Actually it suffices to know that for every sequence x_ν in D approaching some point of $\pi^{-1}(G) \cap \partial\Omega$ there is a holomorphic function on D which is unbounded on $\{x_\nu\}$. The reason is as follows. Take an open neighborhood U in L^* of some point of $\pi^{-1}(G) \cap \partial\Omega$ and a holomorphic function f on U such that $W := \{|f| < 1\} \cap D$ is relatively compact in U and $\partial\Omega$ is strictly pseudoconvex at every point of $\{|f| \leqslant 1\} \cap \partial\Omega$. Let F be the Fréchet algebra of holomorphic functions on W generated by $1, f$, and the restrictions of all the holomorphic functions on D. We can find $g_1, \ldots, g_N \in F$ which define a proper map Φ: $U \to \mathbf{C}^N$. Since every fiber of Φ is of dimension 0, the Jacobian rank of Φ is $n + 1$ almost everywhere. Hence, at almost every point $x \in W$ we can find n holomorphic functions on D whose gradients are linearly independent at x. Since we have much freedom in the choice of f, by expanding the holomorphic functions on D in power series along the fibers of L^* and taking the quotients of the coefficients, we obtain enough meromorphic functions on X to conclude that X is Moišezon.

(7.4) *Question.* For which complex manifolds X is it true that if D is a relatively compact domain in X whose boundary is everywhere pseudoconvex and is strictly pseudoconvex at some point x_* and if x_ν is a sequence in D approaching x_*, then there is a holomorphic function on D which is unbounded on $\{x_\nu\}$?

(7.5) Recently there has been a lot of activity on weakly pseudoconvex boundaries. We briefly discuss here some of the results on the existence of peak functions and Stein neighborhood bases. A domain Ω with strictly pseudoconvex boundary has the property that every boundary point x admits a local holomorphic coordinate chart on an open neighborhood U such that with respect to this chart Ω is strictly Euclidean convex at x. So there is a holomorphic function f on U such that $f(x) = 1$ and $|f(x)| < 1$ on $\overline{\Omega} \cap U - \{x\}$. Such an f is called a *local peak function* for x. If f is defined on an open neighborhood of $\overline{\Omega}$, it is called a *global peak function.* For a long time it was thought that, if Ω is only weakly pseudoconvex at x, one can find a local coordinate chart at x such that Ω is weakly Euclidean convex at x with respect to the local chart; and, therefore, instead of a local peak function, one can find a nonconstant local weak peak function f for x; that is, f is holomorphic on U such that $f(x) = 1$ and $|f(x)| \leqslant 1$ on $\overline{\Omega} \cap U$. In 1973 Kohn and Nirenberg **[52]** found the following example. The domain Ω in $\mathbf{C}^2$ defined by

$$\rho(z, w) = \operatorname{Re} w + |zw|^2 + |z|^8 + \tfrac{15}{7}|z|^2 \operatorname{Re} z^6 < 0$$

whose boundary is strictly pseudoconvex except at 0 does not admit any nonconstant local weak peak function f for 0, even if one requires only that f is C^∞ on some neighborhood U of 0 and is holomorphic only on $\Omega \cap U$. The reason is as follows. By applying the Hopf lemma to $\operatorname{Re} f$, one concludes that the normal derivative of $\operatorname{Re} f$ in the normal direction of $\partial\Omega$ at 0 is nonzero. Hence, $\partial f/\partial w$ is nonzero at 0 and it follows from the implicit function theorem that we can find a C^∞ function $g(z)$ on a neighborhood of 0 such that $f(z, g(z)) \equiv 1$. Since f is holomorphic on Ω, we conclude from $f(z, g(z)) \equiv 1$ that $\partial^{k+l}g(0)/\partial z^k \partial\bar{z}^l = 0$ for $l > 0$. Thus the power series expansion of g at z is of the form

$$g(z) = az^p + O\left(|z|^{p+1}\right),$$

for some $p \geqslant 1$ and $a \neq 0$. It is easy to verify directly that $\rho(z, g(z))$ is negative somewhere on any open neighborhood of 0, contradicting the disjointness of $\{f = 1\}$ and Ω. Actually in [52] it was proved that there is no holomorphic support function h on any open neighborhood U of 0 in $\mathbf{C}^2$ in the sense that $h(0) = 0$ and $\Omega \cap U$ is disjoint from $\{h = 0\}$.

(7.6) By modifying the example of Kohn-Nirenberg [52], Fornaess [26] obtained the domain Ω in $\mathbf{C}^2$ defined by

$$\operatorname{Re} w + |zw|^2 + |z|^6 + \tfrac{3}{2}|z|^2 \operatorname{Re} z^4 < 0$$

whose boundary is strictly pseudoconvex except at 0 and which does not admit any nonconstant local weak peak function f for 0, even if one requires only that f is $\mathbf{C}^1$ on some neighborhood U of 0 and holomorphic on $U \cap \Omega$. However, the answer to the following is unknown.

(7.7) *Question*. Does there always exist a continuous local weak peak function for a nonstrictly pseudoconvex boundary point?

Most recently Bedford and Fornaess [3] succeeded in constructing continuous global weak peak functions for a domain in $\mathbf{C}^2$ with weakly pseudoconvex smooth real-analytic boundary.

For certain type of boundary points, Hakim-Sibony [42] obtained a necessary condition for the existence of a peak function. Before we state their result, we first introduce the concept of type for weakly pseudoconvex boundary points. This concept was first introduced by Kohn [51] in connection with subelliptic estimates. Let Ω be a domain in $\mathbf{C}^n$ defined by $r < 0$ with r C^∞ and dr nowhere zero on the boundary M of Ω. Suppose that Ω is weakly pseudoconvex at every point of M. Let T be the set of all C^∞ tangent vector fields of M of the form $\sum_{i=1}^n a_i \partial/\partial z_i$. Inductively let $\mathfrak{L}_1 = T + \overline{T}$ (where $\overline{T}$ is the set of all conjugates of elements of T) and let $\mathfrak{L}_\mu$ be spanned by the Lie bracket $[X, Y]$ with $X \in \mathfrak{L}_1$ and $Y \in \mathfrak{L}_{\mu-1}$. A point P of M is said to be *of type m* if $\langle \partial r(P), X(P) \rangle = 0$ for every $X \in \mathfrak{L}_m$ and for some $Y \in \mathfrak{L}_{m+1}$, $\langle \partial r(P), Y(P) \rangle \neq 0$. According to a result of Bloom-Graham [7] this is equivalent to the condition that some complex submanifold of codimension 1 in an open neighborhood of P in $\mathbf{C}^n$ is tangential to M at P to the mth order while there is none to a higher order. Another way to phrase this condition is that for some local coordinate system centered at P, $r = \operatorname{Re} z_n + \varphi$, where φ vanishes at P to order $\geqslant 2$ and the lowest order of a term in the power series expansion of φ at P not involving z_n is $m + 1$, and for no other local coordinate system at P we can replace m by a bigger number. Suppose the origin 0 of $\mathbf{C}^n$ belongs to M and $r = \operatorname{Re} z_n + O(|z|^2)$ near 0.

(7.8) THEOREM (HAKIM-SIBONY [42]). *If* 0 *is of type m for some finite m and there exists a nonconstant local weak peak function for* 0 *which is* C^∞ *on* $\overline{\Omega}$ *and holomorphic on* Ω, *then there exists a holomorphic polynomial* $g(z_1, \ldots, z_{n-1})$ *on* $\mathbf{C}^{n-1}$ *such that the first nonzero homogeneous* (*real-analytic*) *polynomial in the power series expansion of* $r(z_1, \ldots, z_{n-1}, g)$ *is nonnegative everywhere on* $\mathbf{C}^{n-1}$.

This means that the existence of a local weak peak function for a boundary point of finite type imposes a certain special condition on the defining function of the domain. In the examples of Kohn-Nirenberg and Fornaess the exceptional points are of finite type, but the defining functions do not satisfy the special condition. So in a way this result of Hakim-Sibony explains why local weak peak functions do not exist in these examples.

The following result relating the existence of peak functions to that of supporting hypersurfaces was obtained by Hakim-Sibony [42].

(7.9) THEOREM. *If there is a complex manifold of codimension* 1 *in an open neighborhood of* 0 *in* $\mathbf{C}^n$ *which intersects* Ω *only at* 0, *then there exists a continuous* (*strong*) *peak function for* 0.

(7.10) It was known to Behnke and Thullen [5] that there are bounded Stein domains whose topological closures do not have a Stein neighborhood basis. A simple example is the domain $|z| < |w| < 1$ in $\mathbf{C}^2$, because any Stein domain containing it and the origin must contain the bidisc. It has been an open question until a counterexample was recently constructed by Diederich-Fornaess [15] whether the topological closure of a bounded Stein domain in $\mathbf{C}^n$ with smooth boundary admits a Stein neighborhood basis. Their counterexample is the domain Ω in $\mathbf{C}^2$ defined by

$$\left|w + \exp\left(i \log|z|^2\right)\right|^2 + \lambda\left(|z|^{-2} - 1\right) + \lambda\left(|z|^2 - r^2\right) < 1,$$

where r is a sufficiently large number and λ is a smooth nonnegative function on $\mathbf{R}$ such that $\lambda(x) = 0$ for $x < 0$, $\lambda'(x) > 0$ for $x > 0$ and λ is sufficiently convex on $\{x > 0\}$. Every Stein domain containing $\bar{\Omega}$ contains the set $\{e^\pi < |z| < e^{2\pi}, |w| < 2\}$, which is not contained in Ω. This phenomenon occurs because for $a \in [1, e^\pi]$ the following 3 sets

$$\left\{|z| = a, \left|w + \exp(ia^2)\right| \leq 1\right\},$$

$$\left\{|z| = ae^{2\pi}, \left|w + \exp(ia^2)\right| \leq 1\right\},$$

$$\left\{a \leq |z| \leq ae^{2\pi}, w = 0\right\}$$

are contained in $\partial\Omega$. The union of these three sets is the "skeleton" of a Hartogs' figure. Hence any function holomorphic on an open neighborhood of these three sets can be extended to a function holomorphic on a neighborhood of

$$K_a := \left\{a \leq |z| \leq ae^{2\pi}, \left|w + \exp(ia^2)\right| \leq 1\right\}.$$

The set $\{e^\pi < |z| < e^{2\pi}, |w| < 2\}$ is contained in $\bigcup\{K_a | 1 \leq a \leq e^\pi\}$.

Most recently Diederich and Fornaess [17] showed that the above strange phenomenon cannot occur when the boundary of the domain is real-analytic.

(7.11) THEOREM. *The topological closure of a bounded Stein domain in* $\mathbf{C}^n$ *with smooth real-analytic boundary admits a Stein neighborhood basis.*

8. Curvature conditons. The main point of the Levi problem is to prove Steinness under the assumption of local Steinness or boundary pseudoconvexity or the existence of a plurisubharmonic exhaustion function. Recently there have been attempts at proving Steinness from differential-geometric conditons, mainly curvature conditions. Up to this point all results on obtaining Steinness from differential-geometric conditions are proved by using known results of some forms of the Levi problem or the methods employed in the proofs of such known results. So these results on sufficient differential-geometric conditions for Stein manifolds can be regarded as applications or consequences of the Levi problem. The earliest result on getting Steinness from differential-geometric conditions is the following theorem of Grauert [29].

(8.1) THEOREM. *Suppose Ω is a domain of (or spread over) $\mathbf{C}^n$ with smooth real-analytic boundary. If Ω admits a complete Kähler metric, then Ω is Stein.*

This theorem is false if there is no smoothness condition on the boundary of Ω. Grauert constructed the following complete Kähler metric on $\mathbf{C}^n - 0$. Let

$$f(x) = x^2 + \int_0^x \frac{d\lambda}{\lambda} \int_0^\lambda \frac{(\mu - 1)^2 d\mu}{\mu(\log \mu)^2}$$

and $\varphi(z) = f(|z|^2)$. Then $\sum_{i,j}(\partial^2\varphi/\partial z_i \partial \bar{z}_j)\, dz_i \otimes d\bar{z}_j$ is the desired Kähler metric.

We now sketch the proof of Theorem (8.1). One proves the Steinness of Ω by showing that $-\log$ of the Euclidean distance to the boundary of Ω is plurisubharmonic. By using the real-analyticity of the boundary of Ω, one reduces the proof of the theorem to the following statement. If $R_1(z_1)$, $R_2(z_1)$ are C^2 functions on $\{|z_1| < 1\}$ with $R_1(z_1) < R_2(z_1)$ and if the domain D in $\mathbf{C}^2$ defined by $|z_1| < 1$, $R_1(z_1) < |z_2| < R_2(z_1)$ carries a complete Kähler metric, then $-\log R_2(z_1)$ is subharmonic on $\{|z_1| < 1\}$. We prove this statement by contradiction. Suppose the Laplacian of $-\log R_2(z_1)$ is negative at some z_1^0. We can assume without loss of generality that $z_1^0 = 0$. By averaging the Kähler metric over the group action T_θ: $(z_1, z_2) \to (z_1, z_2 e^{i\theta})$ ($\theta \in \mathbf{R}$), we can assume without loss of generality that the Kähler metric is invariant under T_θ. Consider the tube domain G in $\mathbf{C}^2$ (with coordinates w_1, $w_2 = u + iv$) defined by $|w_1| < 1$ and $\log R_1(w_1) < u < \log R_2(w_2)$. G is a topological covering of D through the map $z_1 = w_1$, $z_2 = e^{w_2}$. Thus we have a Kähler metric $\sum h_{\mu\bar{\nu}} dw_\mu \otimes d\bar{w}_\nu$ on G with $h_{\mu\bar{\nu}}$ independent of v. Choose a C^2 function $R(w_1)$ on $|w_1| < 1$ with $R_1 < R < R_2$. We claim that there is a global potential function φ on G for the Kähler metric such that $\partial\varphi/\partial u > 0$ on $\{|w_1| < \frac{1}{2}, \log R(w_1) < u < \log R_2(w_1)\}$. The Kähler conditon implies that the form $\omega = h_{1,\bar{2}} dw_1 + h_{2,\bar{1}} d\bar{w}_1 + 2h_{2,\bar{2}} du$ is closed. Choose $w^0 \in G$ and define $\psi(w) = \int_{w^0}^{w} \omega + C$ (where C is a constant). The Kähler condition implies that $\partial^2\psi/\partial w_1 \partial \bar{w}_1 - h_{1,\bar{1}}$ is independent of w_2. So we can find f on G such that $\partial^2 f/\partial w_1 \partial \bar{w}_1 = \partial^2\psi/\partial w_1 \partial \bar{w}_1 - h_{1,\bar{1}}$. Since $h_{2,\bar{2}} > 0$, it follows that we can choose C so large that $\varphi = \psi - f$ satisfies the conditions in the claim. For some $r \in (0, \frac{1}{2})$, the Laplacian of $-\log R_2(w_1)$ is negative on $\{|w_1| < r\}$. Let $\rho(w_1)$ be a smooth function on $\{|w_1| < 1\}$ such that $\rho \equiv 1$ on $\{|w_1| < r/2\}$ and $\rho(w_1) < 1$ on $\{r/2 < |w_1| < 1\}$. Let $\sigma = \rho \log R_2$. Then the Laplacian of $\sigma(w_1)$ is positive on $\{|w_1| < s\}$ for some $s > r/2$. Let H be the tube domain in $\mathbf{C}^2$ defined by $|w_1| < s$ and $\log R(w_1) - \sigma(w_1) < u < \log R_2(w_1) - \sigma(w_1)$. Let $\Phi(w_1, w_2) = \varphi(w_1, w_2 - \sigma(w_1))$. One can easily verify that, due to the positivity of $\partial^2\sigma/\partial w_1 \partial \bar{w}_1$ and $\partial\varphi/\partial u$, $\sum(\partial^2\Phi/\partial w_\mu \partial \bar{w}_\nu)\, dw_\mu \otimes d\bar{w}_\nu$ is a Kähler metric on H. Moreover, with respect to this metric the boundary points of H with $|w_1| < s$ and $u = \log R_2(w_1) - \sigma(w_1)$ are at infinite distance from any point of H. Take $r/2 < \lambda < s$ and a positive number δ small enough so that $\log R(w_1) - \sigma(w_1) < -\delta$ for $|w_1| < \lambda$. By applying Stokes' theorem to the exterior derivative of the Kähler form of H and to the set

$$\{|w_1| < \lambda, -\delta \leqslant u \leqslant t, v = 0\}$$

for $-\delta < t < 0$, we conclude that for $-\delta < t < 0$ the area of

$$\Delta_t := \{|w_1| < \lambda, w = t\}$$

is bounded by the sum of the areas of $\Delta_{-\delta}$ and

$$\{|w_1| = \lambda, -\delta \leqslant u \leqslant 0, v = 0\}$$

which is a finite number. This leads to a contradiction when we let t approach 0 from the left, because every point of the open subset $\{|w_1| \leqslant r/2, w_2 = 0\}$ of Δ_0 is a boundary point of H which has infinite distance from any point of H.

Since for general domains in $\mathbf{C}^n$ the existence of a complete Kähler metric is not enough to guarantee Steinness, one is led to consider additional conditions. The most natural conditions to consider are the curvature conditions. Since holomorphic sectional (and also bisectional) curvatures of complex submanifolds are $\leqslant$ those of ambient manifolds and since Stein manifolds are complex submanifolds of complex Euclidean spaces, one would suspect that one should impose the negativity of curvatures as a condition. Griffiths and Shiffman showed that this is the case for domains spread over $\mathbf{C}^n$.

(8.2) THEOREM (GRIFFITHS [41] AND SHIFFMAN [74]). *Suppose M is a complete Hermitian manifold with nonpositive holomorphic sectional curvature and Ω is a domain spread over $\mathbf{C}^n$. Then any holomorphic map from Ω to M can be extended to a holomorphic map from the envelope of holomorphy of Ω to M. As a consequence, a domain spread over $\mathbf{C}^n$ is Stein if and only if it admits a complete Hermitian metric of nonpositive holomorphic sectional curvature.*

Not much is known about characterizing Stein manifolds which are not domains by curvature conditions. One has the following conjecture.

(8.3) CONJECTURE. A simply connected complex manifold is Stein if and only if it carries a complete Kähler metric of nonpositive holomorphic bisectional curvature.

There is a stronger conjecture which replaces "bisectional" in the above conjecture by "sectional" and there is a weaker conjecture which replaces "nonpositive" in the above conjecture by "negative".

On the other hand there is a conjecture for the positive curvature case.

(8.4) CONJECTURE. A noncompact complete Kähler manifold with positive holomorphic bisectional curvature is Stein.

Siu and Yau have shown (by a very easy proof) that on such a manifold there are holomorphic functions and holomorphic vector fields with any prescribed values to any finite order at any finite number of points and that any complete Hermitian manifold whose holomorphic sectional curvature approaches zero uniformly at its boundary has the Kontinuitätssatz property if one considers smooth 1-parameter families of discs instead of sequences of discs. One is led to raise the following question.

(8.5) *Question*. Suppose on a complex manifold M there are holomorphic functions and holomorphic vector fields with any prescribed values to any finite order at any finite number of points. If M has the Kontinuitätssatz property (or only the weaker form of it mentioned above), is M Stein?

There are some results obtained by Greene and Wu on curvature conditions sufficient to yield Steinness [**37**], [**38**], [**39**], [**40**].

(8.6) THEOREM (GREENE-WU). *Let M be a complete Kähler manifold. Then M is a Stein manifold if any one of the following holds.*

(A) *M is simply connected and the sectional curvature* $\leqslant 0$.

(B) *M is noncompact, the sectional curvature* $\geqslant 0$ *and moreover* > 0 *outside a compact set.*

(C) *M is noncompact, the sectional curvature* $\geqslant 0$, *and the holomorphic bisectional curvature* > 0.

(D) *M is noncompact, the Ricci curvature* > 0, *the sectional curvature* $\geqslant 0$ *and the canonical bundle is trivial.*

The main ideas of the proof are as follows: In (A) the square of the distance function to a fixed point is a strictly plurisubharmonic exhaustion function, because of well-known comparison theorems in differential geometry. In (B), (C), and (D) the result of Cheeger-Gromoll [**14**] gives a plurisubharmonic exhaustion function φ on M. Moreover, in (B) it says that M is diffeomorphic to $\mathbf{C}^n$ and φ is strictly plurisubharmonic outside a compact set. Since any positive-dimensional compact subvariety in a Kähler manifold represents a nontrivial nonzero homology class, it follows that the M in (B) is Stein. The M in (C) is Stein, because by Ellencwajg's result [**19**], $\{\varphi < c\}$ is Stein for any $c \in \mathbf{R}$. The M in (D) is Stein, because solving $\partial\bar\partial\psi =$ Ricci curvature form by L^2 estimates of $\bar\partial$ yields a strictly plurisubharmonic function on M.

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