THE PURE PHASES (HARMONIC FUNCTIONS) OF GENERALIZED PROCESSES[1] OR: MATHEMATICAL PHYSICS OF PHASE TRANSITIONS AND SYMMETRY BREAKING

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Contents

I. Introduction. This paper is written for mathematicians and mathematical physicists with some knowledge of stochastic processes and of the basic notions of statistical mechanics, but I have tried to explain what I believe are all major concepts, notions and definitions required for the understanding of the main results, i.e. I have tried to write these notes for the nonexpert at the risk of boring the expert and, perhaps, being a little imprecise here and there. (The expert may find some new results in §§IV and V.) *All* major recent or new results I am describing in this paper were obtained in collaboration with B. Simon, T. Spencer, E. H. Lieb and R. Israel. The reader is advised to consult references [1]–[4] for statements of the original results and complete proofs. Reviews of some of the material contained in these references and applications to relativistic quantum field theory may be found in [5], [6]. The reader may consult [7], [8], [1] for the original results on phase transitions in

relativistic quantum field theory and their proofs. The general point of view adopted in this paper is developed in [9]–[11] and references given there (see also [12]–[14]). Some of the weighting of different concepts and a few results have grown out of a course I have taught at Princeton University in the fall semester 1976.

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I.1. *Description of the problem.* In these notes I try to outline a new mathematically rigorous theory of phase transitions and symmetry breaking which is rather general. It applies to Gibbs random fields and noncommutative generalizations of these, namely some class of quantum lattice systems and Fermion (Grassmann) lattice systems; the general concepts and methods involved may however equally well be applied to other physical theories, in particular relativistic quantum field theory. Many of the results I am going to describe were actually first obtained in the context of relativistic quantum field theory or at least motivated by it.

This illustrates once again that mathematics can sometimes profit a lot from theoretical physics. The few proofs contained in these notes also show that, to use some words of Mark Kac, "in the right hands, Schwarz's inequality and integration by parts are still among the most powerful tools of analysis".[3]

Mathematically speaking, we shall be concerned, in this talk, with certain aspects of the theory of stochastic processes and their noncommutative versions; aspects that are somewhat related to probabilistic potential theory. In particular I want to discuss an analogy between *phase transitions* and the existence of *nonconstant harmonic functions* of a *generalized process.* The simplest example of a generalized process is a (multi-time) Markov process (or a multi-dimensional Markov chain), but the concept of generalized processes such as has emerged from the work of the past few years [9], [10], [15], [16], [12], [13], [11], [17] is more general and includes "Gibbs lattice fields" which are some sort of noncommutative random fields.

The general problem I shall discuss may be posed as follows: Suppose we are given the local *characteristics* of a generalized process, in the commutative case e.g. a system of conditional probabilities or some equilibrium equations of the *Dobrushin-Lanford-Ruelle* (DLR) type [9]–[11], in the noncommutative case e.g. a *"Gibbs condition"* [13] or a *"Gibbs variational equality"* (all cases), can we prove general theorems giving a complete description of all harmonic functions of the generalized process (i.e. all Gibbs lattice fields with given local characteristics) or, in a physicist's language, the *pure phases of the process?*

Our results are two-fold:

1. *Uniqueness theorems:* Dobrushin's theorem [15], [11], [18] and its noncommutative versions [19], [20].

2. A general method for proving the existence of *"nonconstant"* harmonic

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[3] M. Kac, *On applying mathematics: Reflections and examples,* Quart. Appl. Math. **30** (1972).

functions, or, in other words, of *several distinct pure phases* with identical local characteristics.

For aesthetical and educational reasons I shall emphasize the discussion of *symmetries* and *symmetry breaking*, by which I mean that the local characteristics of a generalized process (and in particular its "Gibbs potential", resp. its Hamiltonian) may be invariant under a symmetry group which does *not* leave invariant some or all its nonconstant harmonic functions, i.e. which permutes the pure phases of the process among themselves.

I briefly want to motivate this emphasis on symmetries and symmetry breaking.

I.2. *The role of symmetry in mathematics and physics.* For a beautiful discussion of the significance and the history of symmetry as a concept in mathematics, the natural sciences and the arts I refer the reader to Hermann Weyl's book entitled *Symmetry* [21]. (A new, somewhat more modern treatise on this subject would be desirable.) There are two aspects of "symmetry" of direct relevance to this paper:

1. A purely geometric aspect.
2. A dynamical aspect.

The symmetries of geometric objects lead to the *mathematical concept* of symmetry and are one of several major roots for the development of group theory. (One might recall, here, Felix Klein's program of characterizing geometries by their invariance groups.)

Geometric symmetries played and still play an important role in chemistry, crystallography, biology and other natural sciences. Historically they also played a role in dynamics, especially celestial mechanics. The Greeks believed that the motions of the planets and the moon would necessarily have to be circular or a superposition of circular motions, as the circle is a geometric object of maximal symmetry.

The Platonic solids, namely the tetrahedron, the cube, the octahedron, the pentagon dodecahedron and the icosahedron found their way into celestial mechanics: Kepler tried to reduce the distances in the planetary system to the shapes of these solids which he alternatingly inscribed and circumscribed to spheres. The six spheres correspond to the six planets Saturn, Jupiter, Mars, Earth, Venus and Mercurius, known at that time, separated in this order by cube, tetrahedron, dodecahedron, octahedron and icosahedron.

Kepler's attempt to understand laws of nature in terms of static, geometric symmetries is typical for natural philosophy in pre-Galilean and pre-Newtonian times.

One of the most significant steps in the history of human thinking may well have occurred when the static, geometric concept of symmetry and its rather successless applications to dynamics were abandoned in favour of a *dynamical concept of symmetry*.

With Newton physicists started to conceive the idea that it is the laws of physics describing the motion of particles, e.g. the planets, which are invariant under certain symmetry groups rather than the orbits of the particles themselves. This *dynamical* concept of "symmetry" is at the basis of some

of the major revolutions in the physics of this century and is the one of main importance for the following.

I.3. *Spontaneously broken symmetries.* The idea that the symmetry group leaving invariant the laws that describe a physical system may be *broken in the space of states of the system* is in some sense the main theme of these notes. This idea will take a mathematically precise shape in the following discussion.

The statement that some symmetry of the laws describing a physical system is broken means that the states of the system fall into equivalence classes invariant under the time evolution and under all possible measurements one can do at such systems which are however *not* invariant under a symmetry operation. Rather, symmetry operations permute these equivalence classes among themselves.

A very striking example of a broken symmetry is found in biology: Living organisms contain the dextro-rotatory form of glucose and the laevo-rotatory form of fructose. There seems to be no a priori reason why it should not be the opposite or why living organisms of both kinds should not coexist (though perhaps coexistence might necessarily result in the extinction of one kind).

We all know that this striking asymmetry in the chemical constitution of living organisms has been preserved over centuries and has so far not been destroyed by any changes of the environmental conditions. Thus it really presents an example of a broken symmetry.

A trivial example of a broken symmetry in every day physics is a dumbbell shaped balloon with two distinct, asymmetric equilibrium shapes.

The physical laws describing the balloon do not distinguish between left and right.

Without going into any detail I want to recall the fundamental role played by dynamical symmetries (spatial or internal) and their breaking in *elementary particles physics*. The idea that symmetries of physical laws may be broken spontaneously (or dynamically) in the state space of a system is a fundamental ingredient in all recent theories of elementary particle physics [22]. To conclude this introduction let me mention some examples of symmetry breaking in *solid state physics* that have a certain bearing on the subject of my talk:

The first example is a ferromagnet, i.e. a system of bulk matter (e.g. iron) that has the property that when an external magnetic field in a fixed direction is turned off and the temperature is low enough it remains magnetized in the direction of the turned off field. Quantum mechanically, this phenomenon is not yet well understood.

Another theoretically closely related phenomenon is *Bose-Einstein condensation.* In a quantum gas of particles satisfying Bose-Einstein statistics the ground state of the gas may have, at low temperatures, a macroscopic occupation. This is accompanied by the spontaneous breaking of a *gauge*

group of the first kind isomorphic to $SO(2)$ which leaves the physical laws describing the gas invariant.

We shall also meet examples where a discrete symmetry is spontaneously broken.

One of the most striking and fundamental phenomena is however, no doubt, the existence of crystals in nature, that is to say of states of matter which break the translational invariance of all physical laws.

In the past two years mathematically rigorous theoretical understanding of the phenomenon of phase transitions and symmetry breaking in the framework of admittedly somewhat too simple models has made great progress. What I intend to do is to describe some of the mathematical and analytical aspects of this progress. I hope this introduction has convinced the reader that the problems I am going to discuss are important and that it has indicated what kind of mathematics is involved (generalized processes, Gibbs random fields, probabilistic potential theory).

II. Lattice systems and generalized processes.

II.1. *Description of the mathematical structure.* Let $\mathcal{L}$ denote some ν-dimensional lattice. For simplicity I shall in general assume in these notes that $\mathcal{L} = \mathbf{Z}^\nu$; the simple cubic lattice.

Many of the results I am going to indicate in the following depend however only on a certain reflection invariance property of the lattice $\mathcal{L}$, i.e. a *geometric symmetry* property of $\mathcal{L}$. (Some of the results, e.g. the uniqueness theorems, do not depend on any special properties of the lattice, at all.) Since there are only finitely many crystallographic groups in ν dimensions (17 for $\nu = 2$, 230 for $\nu = 3$), it is a matter of consulting a table of these groups in order to give a complete list of all lattices having the required reflection invariance.

At each site $i \in \mathcal{L}$ we are given an algebra $\mathfrak{A}_i$ of operators. We must distinguish two cases (if we included Fermions it would be three):

(C) *classical case*

$$\mathfrak{A}_i \cong C(\Omega_i),$$

where Ω_i is a copy of some fixed, compact Hausdorff space Ω_0. In these notes $\Omega_0 \subseteq \mathbf{\dot{R}}^{N4}$, $N = 1, 2, 3, \ldots$, but in interesting cases (lattice gauge theories) Ω_0 may be a nonabelian compact group. We equip $\mathfrak{A}_i$ with the sup norm and complex conjugation as an involution*.

(QM) *quantum mechanical case*

$$\mathfrak{A}_i \cong B(\mathcal{H}_i);$$

here $\mathcal{H}_i$ is an isomorphic copy of some fixed, finite dimensional Hilbert space $\mathcal{H}_0$. The norm and the * operation on $\mathfrak{A}_i$ are defined in the usual way.

For $X \in \mathcal{P}_f(\mathcal{L})$ (the algebra of bounded subsets of the lattice $\mathcal{L}$), we define

$$\mathfrak{A}_X = \bigotimes_{i \in X} \mathfrak{A}_i.$$

If $X \subset X'$ we consider $\mathfrak{A}_X$ to be the subalgebra of $\mathfrak{A}_{X'}$ defined by

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[4] Here $\mathbf{\dot{R}}^N$ is the one-point compactification of $\mathbf{R}^N$.

$$\left(\bigotimes_{i \in X} \mathfrak{A}_i\right) \otimes \left(\bigotimes_{i \in X' \backslash X} \mathbf{1}_i\right)$$

where $\mathbf{1}_i$ is the identity element in $\mathfrak{A}_i$. Technically speaking, $\{\mathfrak{A}_X: X \in \mathcal{P}_f(\mathcal{L})\}$ is a family of C^* algebras (in case (QM) von Neumann algebras), and they are called "*local algebras*".

If a is a translation in the lattice $\mathcal{L}$ and X is a finite subset of $\mathcal{L}$ then $X + a$ denotes the translate of X by a.

The natural identification of $\mathfrak{A}_X$ with $\mathfrak{A}_{X+a}$ is denoted τ_a.

The $*$ algebra $\mathring{\mathfrak{A}} \equiv \bigcup_{X \subset \mathcal{L}} \mathfrak{A}_X$ is called the *algebra of all local observables*; $\mathring{\mathfrak{A}}$ is normed in the obvious way with the norm denoted $\|\cdot\|$. The group $\{\tau_a: a \in \mathcal{L}\}$ acts as a $*$ automorphism group on $\mathring{\mathfrak{A}}$.

The completion of $\mathring{\mathfrak{A}}$ in the norm $\|\cdot\|$ is denoted $\mathfrak{A}$ and is called the algebra of all quasi-local observables. This algebra is a C^* algebra. In the classical case, $\mathfrak{A}$ is isomorphic to $C(\Omega)$, with

$$\Omega = \underset{i \in \mathcal{L}}{\times} \Omega_i$$

(Stone-Weierstrass theorem).

A state ρ on $\mathfrak{A}$ is a positive linear functional on $\mathfrak{A}$ normalized such that

$$\rho(\mathbf{1}) = 1,$$

with $\mathbf{1} = \bigotimes_{i \in \mathcal{L}} \mathbf{1}_i$ (the identity in $\mathfrak{A}$). The space of states on $\mathfrak{A}$ is denoted $\mathfrak{A}^*$. The structure of $\mathfrak{A}^*$ is analyzed in [14].

In the classical case, where $\mathfrak{A} = C(\Omega)$, $\mathfrak{A}^*$ is simply the class of all regular Borel probability measures on Ω.

For all $A \in \mathfrak{A}_X$, define

(C) $$\mathrm{tr}(A) = \int_{\Omega_X} \prod_{i \in X} d\mu(\omega_i) A(\omega_X),$$

where $\Omega_X = \times_{i \in X} \Omega_i$, $\omega_X = \{\omega_i: i \in X\}$, and $d\mu$ is some probability measure on Ω_0;

(QM) $$\mathrm{tr}(A) = (1/d)\mathrm{Tr}_{\mathcal{K}_X}(A),$$

where $\mathcal{K}_X = \bigotimes_{i \in X} \mathcal{K}_i$, d is the dimension of $\mathcal{K}_X$, i.e. $d = (\dim \mathcal{K}_0)^{|X|}$, with $|X|$ the number of sites in X, and $\mathrm{Tr}_{\mathcal{K}_X}$ is the usual trace on $B(\mathcal{K}_X) = \bigotimes_{i \in X} B(\mathcal{K}_i)$.

In the definition of tr, X is an arbitrary finite subset of $\mathcal{L}$. Hence tr extends by continuity to a *state* on $\mathfrak{A}$ (note that tr is linear, $\mathrm{tr}(A^*A) \geqslant 0$, for all $A \in \mathring{\mathfrak{A}}$, $\mathrm{tr}(\mathbf{1}) = 1$).

II.2. *Interactions* ("Gibbs potentials"). An interaction Φ is a function on the class $\mathcal{P}_f(\mathcal{L})$ of all finite subsets of $\mathcal{L}$ with values in $\mathfrak{A}$ such that, for $X \in \mathcal{P}_f(\mathcal{L})$,

(Φ1) $\Phi: X \mapsto \Phi(X) \in \mathfrak{A}_X$.

(Φ2) $\Phi(X + a) = \tau_a(\Phi(X))$ (translation invariance).

(Φ3) $\Phi(X)^* = \Phi(X)$,   for all X.

The interactions Φ form a real Banach space $\tilde{\mathfrak{B}}$ with norm $\|\Phi\|_{\sim} = \Sigma_{X \ni 0} \|\Phi(X)\|$ (0 is the origin in $\mathcal{L}$).

An interaction Φ is of *finite range* if $\Phi(X) = 0$ when diam $X > r$, for some finite $r > 0$.

The iteractions of finite range are dense in $\tilde{\mathfrak{B}}$.

Symmetries. Let G be some compact, topological group acting as a group of continuous * automorphisms on $\mathfrak{A}_0$. Clearly the action of G as a * automorphism group of $\mathfrak{A}_0$ has a natural extension to a representation $\{\tau_g: g \in G\}$ of G by continuous * automorphisms of the algebra $\mathfrak{A}$.

We say that the interaction Φ is *G-invariant* (or: G is a *symmetry* of Φ) if

$$\tau_g(\Phi(X)) = \Phi(X), \quad \text{for all } X \in \mathcal{P}_f(\mathcal{L}).$$

II.3. *Finite systems.* We are now prepared to define the systems considered in the following. They represent a class of *dynamical systems* characterized by

–a C^* algebra of observables,

—a one parameter * automorphism group of "time-translations" on this algebra,

–the "states of interest" on the algebra of observables (in these notes we concentrate on the analysis of equilibrium states, defined below, see also [9], [13], [14]).

First we study *finite (dynamical) systems*: For simplicity we assume that

$$\mathcal{L} = \mathbf{Z}^\nu_{1/2} \equiv \mathbf{Z}^\nu + (1/2, \ldots, 1/2).$$

Let Λ be some finite rectangle in $\mathbf{Z}^\nu_{1/2}$. We identify opposite faces of Λ, i.e. we wrap Λ on a torus (recovering in this way a group of translations). We then regard $\mathfrak{A}_\Lambda$ as the *algebra of observables of a finite dynamical system* in the region Λ. Given an interaction Φ we construct the time-translation automorphisms of the system by means of a *Hamiltonian*

$$H^\Phi_\Lambda = \sum_{X \subset \Lambda} \Phi(X).$$

Property ($\Phi 3$) guarantees that, for all finite regions Λ, H^Φ_Λ is selfadjoint. One may therefore define the time-evolution of an observable $A \in \mathfrak{A}_\Lambda$ by

$$A \mapsto \alpha^\Lambda_t(A) = e^{itH^\Phi_\Lambda} A e^{-itH^\Phi_\Lambda}.$$

In case (C) α^Λ_t is trivial, but in case (QM) it is in general *not*.

The *equilibrium state* ("state of interest" in these notes) of a finite, dynamical system specified by the region Λ and the interaction Φ is then defined by

$$\rho^{\beta,\Phi}_\Lambda(A) \equiv \text{tr}(e^{-\beta H^\Phi_\Lambda})^{-1} \text{tr}(e^{-\beta H^\Phi_\Lambda} A), \quad (A \in \mathfrak{A}_\Lambda).$$

It is *unique* and it is *invariant* under all * automorphism groups of $\mathfrak{A}_\Lambda$ commuting with H^Φ_Λ and leaving tr invariant. This means that "finite systems do *not* have phase transitions or symmetry breaking".

Here some *examples of interactions* that are interesting for physics:

(C1) $\Omega_0 = \{-1, 1\}$; S_0 is the function on Ω_0 defined by $S_0(\pm 1) = \pm 1$; $S_i = \tau_i(S_0)$; S_i is called the *(Ising) spin* at site i. $\Phi(\{i\}) = -hS_i$, h real; $\Phi(\{i,j\}) = -J_{i-j}S_iS_j$, with $J_{i-j} \geqslant 0$ and $J_{i-j} > 0$, for $|i-j| = 1$;

$$\Phi(X) = 0, \quad \text{for } |X| \geqslant 3;$$

$$\mu(\{\pm 1\}) = \tfrac{1}{2}.$$

This is the so-called *ferromagnetic Ising model*. We note that, for $h = 0$, H_Λ^Φ has a *discrete symmetry*

$$H_\Lambda^\Phi(S_\Lambda) = H_\Lambda^\Phi(-S_\Lambda),$$

i.e. the dynamics is invariant under flipping all spins in Λ. This symmetry is shared by $d\mu$ and tr.

(C2) Here $\Omega_0 = S^{N-1}$, the unit sphere in $\mathbf{R}^N$; S_0^i is the function assigning to a unit vector in Ω_0 its ith component, and $\mathbf{S}_0 = (S_0^1, \ldots, S_0^N)$, $\mathbf{S}_i = \tau_i(\mathbf{S}_0)$. $\Phi(\{i\}) = -\mathbf{h} \cdot \mathbf{S}_i$, $\mathbf{h} \in \mathbf{R}^N$; $\Phi(\{i,j\}) = -J_{i-j}\mathbf{S}_i \cdot \mathbf{S}_j$, with e.g. $J_{i-j} \geqslant 0$, and $J_{i-j} > 0$, for $|i - j| = 1$; $\Phi(X) = 0$, for $|X| \geqslant 3$;

$$d\mu(\mathbf{S}_0) = \delta(|\mathbf{S}_0| - 1)d^N S_0.$$

This is the classical, ferromagnetic N-vector model (when $N = 3$ it is frequently called the *classical Heisenberg model*). For $\mathbf{h} = 0$, its symmetry group is obviously $O(N)$.

REMARK. If in the definition of the interaction Φ in models (C1) and (C2) we set $J_{i-j} = 0$, for $|i - j| \geqslant 2$, we obtain examples of *multidimensional Markov chains*. (The equilibrium expectations of these models have the *local Markov* property, [16], [17], [23].)

(QM1) $\mathcal{K}_0 = \mathbf{C}^{2S+1}$; $S = 1/2, 1, 3/2, \ldots$;

$\{S_0^i: i = x, y, z\}$ a $2S + 1$ dimensional, irreducible representation of the Lie algebra of $SU(2)$; $\mathbf{S}_0 = (S_0^x, S_0^y, S_0^z)$; $\mathbf{S}_i = \tau_i(\mathbf{S}_0)$; Φ as in (C2).

This is the spin-S Heisenberg ferromagnet. For $\mathbf{h} = 0$ it has $O(3)$ as its symmetry group. This is a difficult model which is not completely understood yet.

(QM2) The same as (QM1), but in the definition of Φ we require $J_{i-j} < 0$, for $|i - j| = 1$, $J_{i-j} = 0$, otherwise.

This is the spin-S Heisenberg antiferromagnet. Again the symmetry group is $O(3)$ (for $\mathbf{h} = 0$). For results see [2].

The main part of these notes is devoted to the discussion of new results concerning the *equilibrium statistical mechanics of a dynamical system* specified by the observable algebra $\mathfrak{A}_\Lambda$ and the dynamics H_Λ^Φ. We are mainly interested in the systems obtained by taking the thermodynamic limit $\Lambda \to \mathbf{Z}_{1/2}^\nu$. This limit must be taken before one can start to discuss phase transitions and symmetry breaking. (Finite systems never exhibit symmetry breaking!)

I shall now introduce some basic objects of thermodynamics and statistical mechanics, in particular the so called *thermodynamic functions*. They are needed to define the "states of interest" for the infinite systems ($\Lambda = \mathbf{Z}_{1/2}^\nu$).

II.4. *Thermodynamic functions.* We define the canonical partition function for a system in the region Λ with interaction Φ by

$$Z_\Lambda(\beta, \Phi) = \text{tr}(e^{-\beta H_\Lambda^\Phi}),$$

and the *free energy* $f_\Lambda(\beta, \Phi)$ per unit volume by

$$\beta f_\Lambda(\beta, \Phi) = -(1/|\Lambda|) \ln Z_\Lambda(\beta, \Phi).$$

Here β is the *inverse temperature*.

Let ρ be a translation invariant state of the infinite system, i.e.

$$\rho(\tau_i(A)) = \rho(A),$$

for all $i \in \mathbf{Z}'$, all $A \in \mathfrak{A}$. We set

$$\rho_\Lambda = \rho/\mathfrak{A}_\Lambda \quad \text{(the restriction of } \rho \text{ to } \mathfrak{A}_\Lambda\text{)}.$$

We define the *internal energy per site* by

$$u_\Lambda(\rho, \Phi) = (1/|\Lambda|)\rho(H_\Lambda^\Phi),$$

and the *specific entropy* by

$$s_\Lambda(\rho) = -(1/|\Lambda|)\mathrm{tr}(\rho_\Lambda \ln \rho_\Lambda).$$

(In case (QM) we consider arbitrary *translation-invariant* states on $\mathfrak{A}$, in case (C) only those states ρ with the property that ρ_Λ is absolutely continuous with respect to $\Pi_{i \in \Lambda}\, d\mu(\omega_i)$. The class of all these states is denoted E^I.)

The following results (see [9] and references given there) summarize some rigorous thermodynamics for lattice systems with interactions $\Phi \in \tilde{\mathfrak{B}}$.

THEOREM 1. *For all real β, $\Phi \in \tilde{\mathfrak{B}}$, $\rho \in E^I$, the following limits exist (and are "independent of boundary conditions" and of the sequence $\{\Lambda\} \to \mathbf{Z}'$; $\Lambda \to \mathbf{Z}'$ "in the sense of van Hove").*

$$f(\beta, \Phi) = \lim_{\Lambda \to \mathbf{Z}'} f_\Lambda(\beta, \Phi),$$

$$u(\rho, \Phi) = \lim_{\Lambda \to \mathbf{Z}'} u_\Lambda(\rho, \Phi) = \sum_{X \ni 0} |X|^{-1} \rho(\Phi(X)),$$

$$s(\rho) = \lim_{\Lambda \to \mathbf{Z}'} s_\Lambda(\rho).$$

(2) *The function $s(\rho)$ is affine and upper semicontinuous.*
(3) *Gibbs (variational) ienqualtiy;*

$$s(\rho) \leqslant \beta u(\rho, \Phi) - \beta f(\beta, \Phi).$$

THEOREM 2. *For all real β and $\Phi \in \tilde{\mathfrak{B}}$ there exists at least one translation invariant state ρ on $\mathfrak{A}$ such that*

$$s(\rho) = \beta u(\rho, \Phi) - \beta f(\beta, \Phi)$$

(Gibbs variational equality [9]).

Any cluster point of the sequence of states in E^I

$$\left\{ Z_\Lambda(\beta, \Phi)^{-1}\mathrm{tr}_\Lambda(e^{-\beta H_\Lambda^\Phi} -) \otimes \mathrm{tr}_{\Lambda^c}(-) \right\}_{\{\Lambda\} \nearrow \mathbf{Z}'_{1/2}}$$

satisfies the Gibbs variational equality, i.e. any *limiting state* of the *equilibrium states* of the *finite* systems, as $\Lambda \nearrow \mathbf{Z}'_{1/2}$ satisfies this equality.

REMARK 3. *Existence* of at least one such limiting state follows from a standard compactness argument. Moreover, under suitable assumptions on the interaction Φ, one can prove that

$$\alpha_t(A) = \underset{\Lambda \nearrow \mathbf{Z}'_{1/2}}{n\text{-lim}}\ \alpha_t^\Lambda(A) \text{ exists}, \quad \text{for all } A \in \mathfrak{A};$$

see [9]. A different construction of the time-translations for infinite systems is discussed in [24].

We are now prepared to define the *"states of interest"* for the *infinite* systems.

II.5. *Equilibrium states, uniqueness theorems.* Any state $\rho \in E^I$ satisfying the *Gibbs variational equality* for an interaction $\Phi \in \widetilde{\mathfrak{B}}$ and inverse temperature β is called an *equilibrium state* of the infinite system (specified by Φ) at inverse temperature β. These are the "states of interest" in these notes. For people familiar with thermodynamics this definition looks most reasonable. I should however emphasize that the justification of this definition and its consequences is the subject of deep and difficult work that is still partly in "statu nascendi" [9], [10], [13], [14]. In particular it is possible to prove the equivalence of this definition with a characterization of equilibrium states in terms of local *characteristics* (systems of conditional probabilities in case (C)). This establishes a connection to the theory of generalized processes.

Moreover there is a deep connection of the *theory of equilibrium states* and *Tomita-Takesaki theory* [25]. Since $s(\rho)$ and $u(\rho, \Phi)$ are affine in ρ, we immediately conclude that the set $\Delta^{\beta,\Phi}$ of all equilibrium states with given Φ and β is *convex*.

As a matter of fact one has

THEOREM 4. $\Delta^{\beta,\Phi}$ *is a Choquet simplex, i.e. each equilibrium state* $\rho \in \Delta^{\beta,\Phi}$ *is the resultant of (a unique probability measure supported on) extremal elements in* $\Delta^{\beta,\Phi}$ *(see e.g.* [9], [31]; *a simple proof in the classical case* (C) *is outlined in* [19]*)*.

REMARKS. The extremal elements of $\Delta^{\beta,\Phi}$ are called the *pure phases* of the infinite system with interaction Φ, at inverse temperature β. If $\Delta^{\beta,\Phi}$ happens to be the family of all stationary states of a multidimensional Markov chain (see the Remark in §II.3, Example (C2))–or some more general stochastic process–then the probability measures supported on the extreme points of $\Delta^{\beta,\Phi}$ are in 1-1 correspondence with the *harmonic functions* of the Markov chain.

The next result asserts that, for small β, $\Delta^{\beta,\Phi}$ contains typically only *one* state.

THEOREM 5. *Let* Φ *be of finite range. Then, for sufficiently small* β *("high temperature"),* $\Delta^{\beta,\Phi}$ *contains precisely one state* $\rho^{\beta,\Phi}$. *If A and B are arbitrary operators in some local algebra then*

$$\left| \rho^{\beta,\Phi}\left(A\tau_j(B)\right) - \rho^{\beta,\Phi}(A)\rho^{\beta,\Phi}(B) \right|$$

decays exponentially in $|j|$ *("exponential clustering")*.

REMARKS. 1. A more general result has been proven in the classical case (C) by Dobrushin [15]: For any interaction $\Phi \in \widetilde{\mathfrak{B}}$ $(\|\Phi\|_{\sim} < \infty)$ $\Delta^{\beta,\Phi}$ contains precisely *one* state, for β small enough (but generally *no* exponential clustering); see also [11], [18].

2. In the quantum case (QM) Theorem 5 is due to the author [19]. In *one* dimension $(\nu = 1)$ *Araki* has proven a more general result of this type, for *all* β [20]. Preliminary results of this genre were obtained in [26], [27].

3. Let G be a connected Lie group acting as a nontrivial, local, continuous * automorphism group on $\mathfrak{A}$. Let Φ be an interaction of finite range that is *G-invariant. For* $\nu = 1$ *and* 2 Dobrushin and Shlosman [28] have proven that

in the classical case (C) all states in $\Delta^{\beta,\Phi}$ are *G-invariant*. Hence there is *no* spontaneous symmetry breaking. We shall show that if Φ has *very long* range then this conclusion is in general *false*.

4. The proofs of Theorem 5 and the results mentioned in Remarks 1–3 involve a great deal of concrete, hard analysis, in particular expansion methods, fixed point theorems, trace-inequalities, etc.

III. The general notion of phase transition. We consider an infinite lattice system characterized by an interaction Φ and the family $\Delta^{\beta,\Phi}$ of all its translation-invariant equilibrium states at inverse temperature β.

DEFINITION OF PHASE TRANSITIONS. We say that a system with interaction Φ has a *phase transition* if the number of extremal equilibrium states in $\Delta^{\beta,\Phi}$ is *not* constant as a function of β.

From Theorem 5 we already know that, under suitable assumptions on Φ and for $|\beta|$ small enough, $\Delta^{\beta,\Phi}$ contains precisely one state. In this situation we speak of a phase transition if, for sufficiently large $|\beta|$, $\Delta^{\beta,\Phi}$ contains more than one extremal state.

Thus we will have proven the existence of a phase transition if we can find some β and a state $\rho^{\beta,\Phi} \in \Delta^{\beta,\Phi}$ which is *not* an extremal state. We must therefore formulate a criterion which permits us to decide whether some state $\rho^{\beta,\Phi}$ is extremal or not. Since in the following $\rho^{\beta,\Phi}$ is some fixed element of $\Delta^{\beta,\Phi}$, we do not need the labels β and Φ and write $\langle A \rangle$ for $\rho^{\beta,\Phi}(A)$, $A \in \mathfrak{A}$.

We now state this criterion, then indicate why it is correct and finally discuss the connections between phase transitions and symmetry breaking.

Let $A \in \mathfrak{A}$ be some quasi-local observable and $A_i = \tau_i(A)$ the translate of A by the vector i. We define

$$c \equiv c(A) := \lim_{\Lambda \nearrow Z_{1/2}^i} \frac{1}{|\Lambda|^2} \sum_{i \in \Lambda} \sum_{j \in \Lambda} \langle A_i^* A_j \rangle.$$

We define a truncated expectation A^*A by

$$\langle A^*A \rangle^T = \langle A^*A \rangle - c.$$

The state $\langle - \rangle$ is *not* an extremal invariant equilibrium state (extreme point of $\Delta^{\beta,\Phi}$) if and only if, for some $A \in \mathfrak{A}$,

(PT) $$\langle A^*A \rangle^T < \langle A^*A \rangle - |\langle A \rangle|^2.$$

Clearly this inequality is equivalent to

$$c > |\langle A \rangle|^2,$$

or–if we define

$$\Delta A_\Lambda^\# = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \{ A_i^\# - \langle A_i^\# \rangle \} = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \{ A_i^\# - \langle A^\# \rangle \}$$

(MF) $$\lim_{\Lambda \nearrow Z_{1/2}^i} \langle \Delta A_\Lambda^\# \Delta A_\Lambda \rangle = c - |\langle A \rangle|^2 > 0.$$

A theoretical physicist interprets this inequality as showing the presence of *macroscopic fluctuations* (or long range order) in the state $\langle - \rangle$, see e.g. [2].

If $\langle - \rangle$ were extremal invariant then, clearly,

$$\lim_{\Lambda \to \mathbf{Z}_{1/2}^{\nu}} \langle \Delta A_{\Lambda}^{*} \Delta A_{\Lambda} \rangle = 0, \quad \text{for all } A \in \mathfrak{A}.$$

Therefore the state $\langle - \rangle$ is *not* extremal invariant if and only if there exists some quasi-local observable A satisfying inequality (PT) or, equivalently, inequality (MF).

We now develop some geometric notions that may be used to make this criterion plausible (and, as a matter of fact, to prove it; see [9], [10], [1], [2], [5]). Given any translation-invariant state $\langle - \rangle$ on the algebra $\mathfrak{A}$, there exists a Hilbert space $\mathcal{H}$, a cyclic vector $\Omega \in \mathcal{H}$, a representation π of $\mathfrak{A}$ on $\mathcal{H}$ and a continuous, unitary representation $\{U_a \colon a \in \mathbf{Z}^{\nu}\}$ of the translation group such that

$$\langle A \rangle = (\Omega, \pi(A)\Omega),$$

$$\langle A\tau_a(B) \rangle = (\Omega, \pi(A) U_a \pi(B)\Omega),$$

$$U_a \Omega = \Omega, \quad \text{for all } a \in \mathbf{Z}^{\nu}.$$

This is simply the Gel'fand-Naimark-Segal construction; see e.g. [31].

Let $\mathcal{H}^I$ be the closed subspace of $\mathcal{H}$ consisting of *all translation-invariant* vectors in $\mathcal{H}$, i.e. for $\Psi \in \mathcal{H}^I$, $U_a \Psi = \Psi$. Clearly dim $\mathcal{H}^I \geqslant 1$, as $\Omega \in \mathcal{H}^I$.

THEOREM 6. $\langle - \rangle \equiv \rho^{\beta,\Phi}$ *is extremal* (more precisely: extremal invariant) *if and only if* dim $\mathcal{H}^I = 1$, *i.e.* $\mathcal{H}^I = \{\Omega\}$.

The *proof* of this result involves using the definition of equilibrium states in terms of local characteristics (the *DLR-equations* in the classical case, the *KMS-* or Araki's *Gibbs condition* in the quantum case). Thus, in order to prove that $\langle - \rangle$ is not extremal, it suffices to show that dim $\mathcal{H}^I \geqslant 2$.

Let P^I denote the orthogonal projection onto the orthogonal complement of $\mathcal{H}^I$.

For A and B in $\mathfrak{A}$ we define a "truncated expectation" of A and B by

$$\langle BA \rangle^T \equiv \left(\pi(B)^* \Omega, P^I \pi(A) \Omega \right).$$

(For $B = A^*$ this definition coincides with the one given previously. This follows from the spectral theorem.)

If dim $\mathcal{H}^I = 1$ then P^I is simply the orthogonal projection onto $\{\Omega\}^{\perp}$ which we denote by $1 - P^{\Omega}$. Hence

$$\langle AB \rangle^T = \langle AB \rangle - \langle A \rangle \langle B \rangle.$$

Since $\Omega \in \mathcal{H}^I$, $P^I \leqslant 1 - P^{\Omega}$, so that

$$\langle A^* A \rangle^T \leqslant \langle A^* A \rangle - |\langle A \rangle|^2.$$

Thus, in order to show that $\langle - \rangle$ is not extremal, it suffices to show that

$$\langle A^* A \rangle^T < \langle A^* A \rangle - |\langle A \rangle|^2,$$

for some suitable $A \in \mathfrak{A}$, i.e. inequality (PT)!

Let $A_i \equiv \tau_i(A)$. The Fourier transform of $\langle A_0^* A_i \rangle^{(T)}$ is denoted $d\omega^{(T)}(k)$. It is a positive measure on the *first Brillouin zone* $B = \{k \colon k^i \in [-\pi, \pi], i = 1, \ldots, \nu\}$. Clearly

$$\langle A_0^* A_i \rangle = \langle A_0^* A_i \rangle^T + \left(\pi(A_0)\Omega, (1 - P^I)\pi(A_i)\Omega \right).$$

The second term is *independent* of i, since $(1 - P^I)$ projects onto the space of *translation-invariant* vectors. Therefore

$$d\omega(k) = c\delta(k)d''k + d\omega^T(k),$$

for some $c \geqslant 0$; if $c > |\langle A \rangle|^2$ then (PT) and (MF) hold.

Our *strategy for the proof of the existence of a phase transition* can now be formulated as follows:

(0) Choose a suitable local observable A.

(I) Derive an upper bound for $\langle A^*A \rangle^T$:

$$\langle A^*A \rangle^T = \int_B d\omega^T(k) \leqslant c_1.$$

(II) Derive a lower bound on $\langle A^*A \rangle - |\langle A \rangle|^2$:

$$\langle A^*A \rangle - |\langle A \rangle|^2 \geqslant c_2.$$

If $c_1 < c_2$ then

$$\langle A^*A \rangle - |\langle A \rangle|^2 > \langle A^*A \rangle^T,$$

therefore dim $\mathcal{H}^I \geqslant 2$, i.e. $\langle - \rangle \equiv \rho^{\beta,\Phi}$ is not extremal (that means that $\langle - \rangle$ has macroscopic fluctuations in the sense of inequality (MF)). Next we want to show why a *phase transition* may be accompanied by the *spontaneous breaking of a symmetry* of the system.

Let G be some compact topological group acting as a nontrivial, local * automorphism group $\{\tau_g : g \in G\}$ on the algebra $\mathfrak{A}$ of all quasi-local observables, and $\tau_g(\mathfrak{A}_X) = \mathfrak{A}_X$, for all $X \in \mathcal{P}_f(\mathcal{L})$.

Suppose now that the interaction Φ of the system is G-invariant, i.e.

$$\tau_g(\Phi(X)) = \Phi(X), \quad \text{for all } X \in \mathcal{P}_f(\mathcal{L}).$$

This is a precise expression for "G is a symmetry of the system" (dynamical concept of symmetry). Next, assume that the equilibrium state $\langle - \rangle$ is G-invariant, i.e.

$$\langle \tau_g(C) \rangle = \langle C \rangle, \quad \text{for all } g \in G, C \in \mathfrak{A}.$$

There exists always *at least one G-invariant equilibrium state* if the interaction Φ is G-invariant; see Theorem 2. Let dg be the normalized Haar measure on G and let $\tilde{A}$ be some observable of the system with the property that

$$A \equiv \tilde{A} - \int_G \tau_g(\tilde{A}) \, dg \neq 0.$$

Suppose now that A satisfies Estimates (I) and (II) with $c_1 < c_2$, i.e.

$$\langle A^*A \rangle > \langle A^*A \rangle^T.$$

(Estimate (II) is simplified because $\langle A \rangle = 0$.) In this case $\langle - \rangle$ is not extremal, so that by Theorem 4,

$$\langle A \rangle = \int_{\mathcal{E}(\Delta^{\beta,\Phi})} d\rho(\chi)\langle A \rangle_\chi \quad \text{(for all } A \in \mathfrak{A}),$$

where $\mathcal{E}(\Delta^{\beta,\Phi})$ denotes the set of extremal states in $\Delta^{\beta,\Phi}$, and $d\rho$ is a proba-

bility measure on $\mathcal{E}(\Delta^{\beta,\Phi})$ with at least two *different* extremal states in its support. For $d\rho$-almost all χ, $\langle - \rangle_\chi$ is extremal, i.e.

$$\langle A^* A \rangle_\chi^T = \langle A^* A \rangle_\chi - |\langle A \rangle_\chi|^2,$$

and therefore

$$\int_{\mathcal{E}(\Delta^{\beta,\Phi})} d\rho(\chi) |\langle A \rangle_\chi|^2 = \int_{\mathcal{E}(\Delta^{\beta,\Phi})} d\rho(\chi) \left[\langle A^* A \rangle_\chi - \langle A^* A \rangle_\chi^T \right]$$

$$= \langle A^* A \rangle - \langle A^* A \rangle^T > 0,$$

i.e. $\langle A \rangle_\chi \neq 0$, for a set Γ of χ's of positive $d\rho$-measure.

From $\langle A \rangle_\chi \neq 0$ and $\int_G dg \langle \tau_g(A) \rangle_\chi = 0$, for all $\chi \in \Gamma$, one immediately concludes that the states $\{ \langle - \rangle_\chi : \chi \in \Gamma \}$ are not G-invariant. All the states

$$\{ \langle \tau_g(-) \rangle_\chi : g \in G, \chi \in \Gamma \}$$

are *equilibrium states* of the system (they satisfy the Gibbs variational equality!).

Thus the symmetry group G of the interaction Φ (the "dynamics of the system") is *broken* by the states $\{ \langle \tau_g(-) \rangle_\chi : g \in G, \chi \in \Gamma \}$; G permutes different pure phases of the system among themselves.

An interesting special case in this situation is the following: There exists some distinct, extremal equilibrium state $\langle - \rangle_+$ in the set of all equilibrium states $\Delta^{\beta,\Phi}$ of the system such that every state $\langle - \rangle$ in $\Delta^{\beta,\Phi}$ is of the form

$$\langle - \rangle = \int_G d\mu(g) \langle \tau_g(-) \rangle_+,$$

for some probability measure $d\mu$ on G.

In this case information on the structure of $\Delta^{\beta,\Phi}$ is obviously rather complete.

As proven by Slawny [29] and Lebowitz [30] this special situation is met in the ferromagnetic Ising models (of the form of Example (C1), §II.3) at all but possibly countably many temperatures.

It follows from our discussion that the concept of phase transition is in principle more basic than the one of symmetry breaking (see [1], [6], [28] for a precise discussion of this point). In many important physical theories phase transitions and symmetry breaking come however in pairs.

The remaining part of these notes is devoted to an outline of a general theory for the derivation of Estimate (I) and the application of our strategy (Estimates (I) and (II)) to the *proof of existence* of *phase transitions* in *specific models*.

The starting point for our proofs of Estimate (I) is the following chain of simple observations:

To get an upper bound on $\langle A^* A \rangle^T = \int_B d\omega^T(k)$ it suffices to prove a pointwise upper bound on $d\omega^T(k)$. Let $\hat{J}(k)$ be some continuous function on the first Brillouin zone B with $\hat{J}(k = 0) = 0$. Then

$$(1) \qquad \hat{J}(k)^2 \, d\omega(k) = \hat{J}(k)^2 \, d\omega^T(k),$$

because $\hat{J}(k)^2 \delta(k) = 0$, as a measure on B. Taking the Fourier transform of equation (1) we conclude that

(2) $$\langle (J * A^*)_0 (J * A)_i \rangle = \langle (J * A^*)_0 (J * A)_i \rangle^T.$$

Let $C(h) \equiv \Sigma C_i h(i) = \Sigma \tau_i(C) h(i)$, where C is a local observable and $h(i)$ is some function on $\mathbf{Z}^{\nu}_{1/2}$. Then we obtain from (2)

$$\langle (J * A^*)(\bar{h})(J * A)(h) \rangle = \langle (J * A^*)(\bar{h})(J * A)(h) \rangle^T.$$

Hence if we can find some J such that its Fourier transform $\hat{J}$ is nonpositive, $-\hat{J}(k)^{-1}$ is $d^{\nu}k$-integrable, and for all summable functions h

$$\langle (J * A^*)(\bar{h})(J * A)(h) \rangle \leqslant -\text{const} \sum_{i,j} \overline{h(i)} \, J_{i-j} h(j)$$

then

$$\hat{J}(k)^2 \, d\omega^T(k) \leqslant -\text{const} \, \hat{J}(k) \, d^{\nu}k, \quad \text{or}$$

$$d\omega^T(k) \leqslant -\text{const} \, \hat{J}(k)^{-1} \, d^{\nu}k$$

which is the required upper bound.

These are the estimates derived–under suitable *assumptions on the interaction* Φ–in the subsequent sections.

IV. Reflection positivity. In this section we consider a certain *cone of interactions* which satisfy a positivity property called *reflection positivity* [4]. This property can only be formulated for lattices which have a certain reflection invariance (alluded to in §II.1). For simplicity we only consider simple, cubic lattices; but see [4] for more general results in the classical case (C).

In the language of a mathematical physicist reflection positivity expresses the existence of a *selfadjoint transfer matrix*.

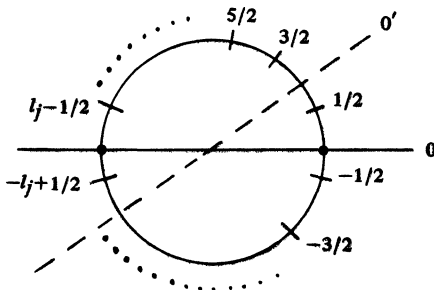IV.1. *Reflection positivity in finite systems.* Unless otherwise stated all the following results are proven in [4].

We let $\Lambda \subset \mathbf{Z}^{\nu}_{1/2} \equiv \mathbf{Z}^{\nu} + (1/2, \ldots, 1/2)$ be the rectangle

$$\underset{i=1}{\overset{\nu}{\times}} [-l_i + 1/2, l_i - 1/2], \quad l_i = 1, 2, 3, \ldots,$$

and we identify $l_i + 1/2$ with $-l_i + 1/2$, i.e. we wrap Λ on a torus. We then define

$$\Lambda^j_{\pm} = \underset{i \neq j}{\times} [-l_i + 1/2, l_i - 1/2] \times [\pm 1/2, \pm l_i \mp 1/2].$$

The following figure represents a cross section of Λ (viewed as a torus):

For $i = (i_1, \ldots, i_\nu) \in \mathbf{Z}_{1/2}^\nu$ we define

$$\theta^j i = (i_1, \ldots, -i_j, \ldots, i_\nu).$$

Thus if X is some subset of $\mathbf{Z}_{1/2}^\nu$, $\theta^j X$ will denote the reflection of X at the plane $i_j = 0$.

We now define Θ^j to be the $*$ automorphism of $\mathfrak{A}_\Lambda$ which when restricted to $\mathfrak{A}_i$ is the identification map: $\mathfrak{A}_i \to \mathfrak{A}_{\theta^j i}$, and $\Theta^j(\mathfrak{A}_X) = \otimes_{i \in X} \Theta^j(\mathfrak{A}_i)$. Obviously

$$\Theta^j \mathfrak{A}_{\Lambda_\pm^j} = \mathfrak{A}_{\Lambda_\mp^j}.$$

In the following we generally suppress the superscript j. Any statement that does not contain explicit reference to a distinct j is true for all $j = 1, \ldots, \nu$.

LEMMA 7. *For all* $A \in \mathfrak{A}_{\Lambda_+}$, $\mathrm{tr}(A\Theta(A^*)) \geqslant 0$.

PROOF.

$$\mathrm{tr}(A\Theta(A^*)) = \mathrm{tr}_{\Lambda_+}(A)\mathrm{tr}_{\Lambda_-}(\Theta(A^*))$$

$$= \mathrm{tr}_{\Lambda_+}(A)\mathrm{tr}_{\Lambda_+}(A^*) = \mathrm{tr}_{\Lambda_+}(A)\,\overline{\mathrm{tr}_{\Lambda_+}(A)} \geqslant 0. \quad \text{Q.E.D.}$$

Note that because of translation invariance of Λ and tr the choice of an origin on the torus Λ is *arbitrary*, as indicated in the figure!

DEFINITION. An interaction Φ satisfies *reflection positivity* iff, for all finite X, $\Theta(\Phi(X)) = \Phi(\theta X)$, and

($\Phi 4$) *classical case* (C): For all finite rectangles Λ,

$$\sum_{\substack{X \cap \Lambda_+ \neq \varnothing \\ X \cap \Lambda_- \neq \varnothing}} \Phi(X) = -\sum B_i \Theta(B_i) \quad \left(\text{or} = -\int B_x \Theta(B_x)\,dx\right),$$

where all the operators B_i (resp. B_x) are *selfadjoint* elements of $\mathfrak{A}_{\Lambda_+}$.

($\Phi 4$) *quantum mechanical case* (QM): For all $X \in \mathscr{P}_f(\mathbf{Z}_{1/2}^\nu)$, $\Phi(X)$ is *real* with respect to some complex conjugation of $\mathfrak{A}$ (i.e. $\Phi(X)$ is a matrix with real entries, in a suitable representation). Moreover, for all finite rectangles Λ,

$$\sum_{\substack{X \cap \Lambda_+ \neq \varnothing \\ X \cap \Lambda_- \neq \varnothing}} \Phi(X) = -\sum B_i \Theta(B_i) \quad \left(\text{or} = -\int B_x \Theta(B_x)\,dx\right),$$

where all operators B_i (resp. B_x) are *real* elements of $\mathfrak{A}_{\Lambda_+}$ and $B_i^* = \pm B_i$ ($B_x^* = \pm B_x$); see [2], [4].

REMARKS. 1. A similar (somewhat more complicated) condition defines reflection positivity in Fermion lattice systems; see [19], [4].

2. From now on we shall *only* consider the classical case (C). All results extend however to the quantum mechanical case (QM) if the interaction Φ satisfies ($\Phi 4$) and if we only consider *real* observables $A \in \mathfrak{A}$, replacing $\Theta(A^*)$ systematically by $\Theta(A)$. These conditions seem to *exclude* the treatment of the quantum mechanical ferromagnet (model (QM1)), permit however the analysis of the antiferromagnet (model (QM2)) and the so-called

quantum mechanical x-y model within the gerneral theory described in this and the next section; see [2].

LEMMA 8. *If Φ satisfies reflection positivity ($\Phi 4$) then*

(H) $$H_\Lambda^\Phi = C + \Theta(C) - \sum B_i \Theta(B_i)$$

with $C = C^* \in \mathfrak{A}_{\Lambda_+}$ and $B_i \in \mathfrak{A}_{\Lambda_+}$, for all i.

PROOF.

$$H_\Lambda^\Phi = \sum_{X \in \Lambda} \Phi(X)$$

$$= \sum_{X \in \Lambda_+} \Phi(X) + \sum_{X \in \Lambda_-} \Phi(X) + \sum_{\substack{X \cap \Lambda_+ \neq \varnothing \\ X \cap \Lambda_- \neq \varnothing}} \Phi(X).$$

We set $C = \sum_{X \in \Lambda_+} \Phi(X)$. Then

$$\Theta(C) = \sum_{X \in \Lambda_+} \Theta(\Phi(X)) = \sum_{X \in \Lambda_+} \Phi(\theta X)$$

$$= \sum_{X \in \Lambda_-} \Phi(X).$$

This and reflection positivity ($\Phi 4$) yield equation (H). The remaining part of Lemma 8 follows from the selfadjointness of $\Phi(X)$, for all X, and from $\Phi(X) \in \mathfrak{A}_X$.   Q.E.D.

THEOREM 9. *If Φ satisfies ($\Phi 4$) then*

$$\left\langle A\Theta(A^*) \right\rangle_\Lambda \equiv Z_\Lambda(\beta, \Phi)^{-1} \operatorname{tr}\left(e^{-\beta H_\Lambda^\Phi} A\Theta(A^*) \right) \geqslant 0,$$

for all $A \in \mathfrak{A}_{\Lambda_+}$, arbitrary Λ and all $\beta \geqslant 0$.

Before we prove Theorem 9 we pause for a

REMARK. In the *classical* case (C) we can derive Theorem 9 from a more general notion of reflection positivity:

($\Phi 4'$) An interaction Φ is *reflection positive* (in the generalized sense) if $\Theta(\Phi(X)) = \Phi(\theta X)$ and, for all finite rectangles Λ,

$$- \sum_{\substack{X \cap \Lambda_+ \neq \varnothing \\ X \cap \Lambda_- \neq \varnothing}} \operatorname{tr}\left(A\Phi(X)\Theta(A^*) \right) \geqslant 0,$$

for all $A \in \mathfrak{A}_{\Lambda_+}$ with $\operatorname{tr}(A) = 0$.

This version of reflection positivity is *weaker* than condition ($\Phi 4$). One can prove that ($\Phi 4'$) is *equivalent* to the inequality

$$\left\langle A\Theta(A^*) \right\rangle_\Lambda \geqslant 0, \quad \text{for all } A \in \mathfrak{A}_{\Lambda_+},$$

arbitrary Λ and *all $\beta \geqslant 0$.*

Finally there is a third (even more general) notion of reflection positivity that is applicable and useful in the *classical* case: Without loss of generality we may assume that the interaction Φ is normalized such that

(T) $\text{tr}(\Phi(X)) = 0$,   for all $X \in \mathcal{P}_f(\mathbf{Z}^\nu_{1/2})$.

(If (T) does not hold we may introduce a *physically equivalent* interaction $\tilde{\Phi}$ defined by

$$\tilde{\Phi}(X) = \Phi(X) - \text{tr}(\Phi(X)).$$

Clearly $\tilde{\Phi}$ determines the same time translation automorphisms as Φ, and Φ and $\tilde{\Phi}$ have the same equilibrium states: $\Delta^{\beta,\Phi} = \Delta^{\beta,\tilde{\Phi}}$.)

($\Phi4''$) An interaction Φ satisfying (T) is *reflection positive* if $\Theta(\Phi(X)) = \Phi(\theta X)$ and

$$- \sum_{\substack{X \cap (\mathbf{Z}^\nu_{1/2})_+ \neq \varnothing \\ X \cap (\mathbf{Z}^\nu_{1/2})_- \neq \varnothing}} \text{tr}(A\Phi(X)\Theta(A^*)) \geq 0,$$

for all $A \in \bigcup_{\Lambda \text{ finite}} \mathfrak{A}_{\Lambda_+}$.

For a large class of classical interactions Φ one can use correlation inequalities to prove that–under the hypothesis that Φ satisfy ($\Phi4''$)–Theorem 9 holds in the *thermodynamic limit* $\Lambda = \mathbf{Z}^\nu_{1/2}$; (in this case Theorem 9 may *fail* for finite Λ). Our general theory of phase transitions applies under these circumstances (§§IV.3 and IV.4; see [4] for proofs).

PROOF OF THEOREM 9. We only consider the classical case which is somewhat simpler than the quantum case, since all operators commute. Because $Z_\Lambda(\beta, \Phi) > 0$ we must only show that

$$\text{tr}(e^{-\beta H^\Phi_\Lambda} A\Theta(A^*)) \geq 0.$$

Using Lemma 8 we obtain

$$\text{tr}(e^{-\beta H^\Phi_\Lambda} A\Theta(A^*)) = \text{tr}(e^{-\beta C}e^{-\beta\Theta(C)}e^{\beta\Sigma B_i\Theta(B_i)}A\Theta(A^*))$$

$$= \sum_{\substack{m_i = 0,\ldots,\infty, \\ \text{all } i}} \text{tr}\left[e^{-\beta C} A \prod_i \frac{B_i^{m_i}}{\sqrt{m_i!}} \Theta\left(\left(e^{-\beta C} A \prod_i \frac{B_i^{m_i}}{\sqrt{m_i!}} \right)^* \right) \right].$$

Each term in the sum on the r.h.s. is positive, by Lemma 7.   Q.E.D.

COROLLARY 10 [7], [34], [3] ("CHESSBOARD ESTIMATE"). *Assume that Φ satisfies reflection positivity ($\Phi4$). Let $A_i \in \mathfrak{A}_i$ be selfadjoint, for all $i \in \Lambda$. Then*

$$\left| \left\langle \prod_{i \in \Lambda} A_i \right\rangle_\Lambda \right| \leq \prod_{i \in \Lambda} \left\langle \prod_{j \in \Lambda} \tau_{j-i}(A_i) \right\rangle_\Lambda^{1/|\Lambda|}.$$

The *proof* of Corollary 10 is based on the following inequality (one of the fundamental inequalities of L_p- and $\mathcal{T}_p$-theory).

THEOREM [3] (GENERALIZED HÖLDER INEQUALITY). *Let V be some complex vector space (e.g. an algebra) and $*$ a conjugation on V. Let ω be a multilinear functional on $V^{\times 2l}$ with the following three properties*

(A) $\omega(B_1,\ldots,B_{2l}) = \omega(B_{2l}, B_1,\ldots,B_{2l-1})$

(cyclic invariance)

(B) *for arbitrary* $B_1^i, \ldots, B_l^i$ *in* V, $i = 1, 2, 3, \ldots,$

$$M_{ij} \equiv \omega\left((B_l^i)^*, \ldots, (B_1^i)^*, B_1^j, \ldots, B_l^j\right)$$

defines a positive semidefinite matrix; in particular

(C) $\quad \omega(A_l^*, \ldots, A_1^*, B_1, \ldots, B_l) = \overline{\omega(B_l^*, \ldots, B_1^*, A_1, \ldots, A_l)}$.

Then

$$\left|\omega(B_1, \ldots, B_{2l})\right| \leqslant \prod_{\alpha=1}^{2l} \omega(B_\alpha, B_\alpha^*, \ldots, B_\alpha, B_\alpha^*)^{1/2l},$$

and

$$\|B\|_{2l} \equiv \omega(B, B^*, \ldots, B, B^*)^{1/2l}$$

is a seminorm on V.

We give the proof for the case $l = 2$:
By properties (B) and (C) of ω we have the Schwarz inequality

(S) $\quad \omega(B_1, B_2, B_3, B_4) \leqslant \omega(B_1, B_2, B_2^*, B_1^*)^{1/2} \, \omega(B_4^*, B_3^*, B_3, B_4)^{1/2}.$

Using (S) and the cyclic invariance of ω, i.e. (A), we obtain

$$\omega(A, B, C, D) \leqslant \omega(A, B, B^*, A^*)^{1/2} \, \omega(D^*, C^*, C, D)^{1/2}$$

$$= \omega(A^*, A, B, B^*)^{1/2} \, \omega(D, D^*, C^*, C)^{1/2}$$

$$\leqslant \omega(A^*, A, A^*, A)^{1/4} \, \omega(B, B^*, B, B^*)^{1/4}$$

$$\times \omega(D, D^*, D, D^*)^{1/4} \, \omega(C^*, C, C^*, C)^{1/4}$$

$$= \omega(A, A^*, A, A^*)^{1/4} \, \omega(B, B^*, B, B^*)^{1/4}$$

$$\times \omega(C, C^*, C, C^*)^{1/4} \, \omega(D, D^*, D, D^*)^{1/4}$$

which completes the proof for $l = 2$.

The proof for general l is similar, but involves an additional maximization argument; see [3].

APPLICATION 1. For $\alpha = 1, \ldots, 2l_j$ and i_j the jth component of $i \in \mathbf{Z}_{1/2}^\nu$, define

$$B_\alpha = \prod_{i_j = -l_j - 1/2 + \alpha} A_i, \quad \text{with } A_i \in \mathfrak{A}_i, \text{ and}$$

$$\omega_j(B_1, \ldots, B_{2l_j}) = \left\langle \prod_{i \in \Lambda} A_i \right\rangle_\Lambda,$$

where $\langle - \rangle_\Lambda$ is the equilibrium state of a finite system with an interaction Φ satisfying reflection positivity. Then, by Theorem 9 and the translation invariance of $\langle - \rangle_\Lambda$, ω_j has properties (A)–(C). If we now apply the generalized Hölder inequality to ω_j and let j vary from 1 to ν we immediately obtain Corollary 10.

APPLICATION 2 (*Hölder inequality for traces*). Let $C_1, \ldots, C_n$ be arbitrary matrices (or functions) and tr the usual trace (integral), as above. Let

$m_1, \ldots, m_n$ be rational numbers $\geqslant 1$ with

$$\sum_{i=1}^{n} m_i^{-1} = 1.$$

Choose a positive, even integer $2l$ such that $m_i^{-1}2l$ is a positive integer, for all i. We define (for arbitrary matrices (functions) $B_1 \ldots B_{2l}$)

$$\omega(B_1, \ldots, B_{2l}) = \mathrm{tr}\left(\prod_{i=1}^{2l} B_i\right)$$

which has obviously properties (A)–(C).

Let $C_i = U_i|C_i|$ be the polar decomposition of C_i and define

$$\hat{D}_i = U_i|C_i|^{m_i/2l}, \qquad D_i = |C_i|^{m_i/2l};$$

then

$$C_i = \hat{D}_i D_i^{(2l/m_i)-1}, \quad \text{for all } i = 1, \ldots, n.$$

Hence

$$\mathrm{tr}(C_1 \ldots C_n) = \mathrm{tr}\left(\prod_{i=1}^{n} \hat{D}_i D_i^{(2l/m_i)-1}\right)$$

$$= \omega(\hat{D}_1, D_1, \ldots, D_1, \ldots, \hat{D}_n, D_n, \ldots, D_n)$$

$$\leqslant \prod_{i=1}^{n} \left[\omega(\hat{D}_i, \hat{D}_i^*, \ldots)^{1/2l}\right]^{2l/m_i}$$

$$= \prod_{i=1}^{n} \mathrm{tr}(|C_i|^{m_i})^{1/m_i}$$

$$\equiv \prod_{i=1}^{n} \|C_i\|_{m_i},$$

and we have used that

$$\omega(D_i, D_i^*, \ldots) = \omega(\hat{D}_i, \hat{D}_i^*, \ldots) = \mathrm{tr}(|C_i|^{m_i}).$$

The inequality just proven and a simple continuity argument yield the general Hölder inequality for traces (resp. arbitrary central states). Other applications of the generalized Hölder inequality include proofs of most of the important inequalities for traces and KMS states.

IV.2. *Examples of interactions satisfying reflection positivity.* Again, we restrict ourselves to the classical case.

THEOREM 11. *Let Φ be the interaction defined by*

$$\Phi(\{n\}) = h\tau_n(A), \qquad \Phi(\{n, m\}) = -\sum_\alpha J^{(\alpha)}_{n-m}\tau_n(B_\alpha)\tau_m(B_\alpha),$$

where n, m are lattice sites, and A, B_α selfadjoint operators in $\mathfrak{A}_0$. Furthermore

$$\Phi(X) = 0, \quad \text{for } |X| \geqslant 3.$$

Assume, in addition, that for arbitrary $\{c_n\} \subset \mathbf{C}$,

(R) $\qquad \sum_{\substack{n_j > 0 \\ m_j > 0}} J_{n-\Theta m}^{(\alpha)} c_n \bar{c}_m \geq 0, \quad \text{for all } j \text{ and all } \alpha.$

Then Φ satisfies reflection positivity ($\Phi 4$).

REMARK. In the classical case this theorem was first proven by the author [32]; see also [33]. The general case appears in [4].

PROOF. We only consider the case $\nu = 1$ (the general case is hardly more difficult). Then condition (R) is

$$\sum_{\substack{i > 0 \\ j > 0}} J_{i+j}^{(\alpha)} c_i \bar{c}_j \geq 0.$$

By a version of Bochner's theorem and the fact that $J_j^{(\alpha)} \to 0$, as $|j| \to \infty$, this implies that for $|n| \geq 1$

(R') $\qquad J_n^{(\alpha)} = \int_{-1}^{+1} \lambda^{|n|-1} d\rho^{(\alpha)}(\lambda),$

for some positive measure $d\rho^{(\alpha)}$ on $[-1, 1]$. We now consider a fixed α and set

$$J_n^{(\alpha)} \equiv J_n, \quad d\rho^{(\alpha)} \equiv d\rho \quad \text{and} \quad \tau_n(B_\alpha) \equiv S_n.$$

Then, using (R'),

$$\sum_{\substack{X \cap \Lambda_+ \neq \varnothing \\ X \cap \Lambda_- \neq \varnothing}} \Phi(X) = \sum_{\substack{i > 0 \\ j > 0}} J_{i+j} S_i S_{\theta j} + \sum_{\substack{i > 0 \\ j > 0}} J_{2l-(i+j)} S_i S_{\theta j}$$

$$= \int_{-1}^{+1} \left(\sum_{n=1/2}^{l-1/2} \lambda^{n-1/2} S_n \right) \left(\sum_{m=1/2}^{l-1/2} \lambda^{m-1/2} S_{\theta m} \right) d\rho(\lambda)$$

$$+ \int_{-1}^{+1} \left(\sum_{n=1/2}^{l-1/2} \lambda^{l-1/2-n} S_n \right) \left(\sum_{m=1/2}^{l-1/2} \lambda^{l-1/2-m} S_{\theta m} \right) d\rho(\lambda)$$

which, in view of $\Theta(S_m) = S_{\theta m}$, is precisely of the form ($\Phi 4$).

REMARK. Conditions analogous to (R) and (R') can also be derived for many body interactions ($\Phi(X) \neq 0$, for some X with $|X| \geq 3$); see [4].

An explicit example of a J_{i-j} in ν dimensions satisfying condition (R) is

$$J_{i-j} = \begin{cases} \text{const } |i - j|^{-(\nu-2+\eta)}, & i \neq j, \\ \text{const}', & i = j, \end{cases}$$

where η is an arbitrary, positive number. For J to define an admissible interaction (i.e. $\Phi \in \tilde{\mathfrak{B}}$) we must require $\eta > 2$; see [4].

For a field theorist the verification of condition (R) for this choice of J_{i-j} is a rather easy exercise (catch word: *conformal invariant* two point functions).

IV.3. *Extension to the thermodynamic limit.* Let $\langle - \rangle$ denote some equilibrium state which is an arbitrary cluster point of the family $\{\langle - \rangle_\Lambda \otimes \text{tr}_{\Lambda^c}(-)\}_{\Lambda \subset \mathbf{Z}_{1/2}^\nu}$ of equilibrium states of finite systems with an interaction Φ satisfying ($\Phi 1$)–($\Phi 3$) and $\sum_{X \ni (1/2, \ldots, 1/2)} \|\Phi(X)\| < \infty$.

A standard compactness argument shows that such a family of states has always at least one cluster point. Any cluster point is automatically translation invariant. We let $\mathfrak{A}_\pm$ denote the normclosure of $\bigcup_{\Lambda \subset \mathbf{Z}^\nu_{1/2}} \mathfrak{A}_{\Lambda_\pm}$.

THEOREM 12. *Assume, in addition, that Φ satisfies reflection positivity* ($\Phi 4$). *Then*

(1) $$\langle A\Theta(A^*) \rangle \geqslant 0, \quad \text{for all } A \in \mathfrak{A}_+.$$

(2) *For any finite subset $B \subset \mathbf{Z}^\nu_{1/2}$ and operators $A_i = A_i^* \in \mathfrak{A}_i$, $i \in B$,*

$$\left| \left\langle \prod_{i\in B} A_i \right\rangle \right| \leqslant \prod_{i\in B} \lim_{\Lambda\to\mathbf{Z}^\nu_{1/2}} \left\langle \prod_{i\in\Lambda} \tau_{j-i}(A_i) \right\rangle_\Lambda^{1/|\Lambda|}$$

$$= \prod_{i\in B} \lim_{\Lambda\to\mathbf{Z}^\nu_{1/2}} \left\langle \prod_{i\in\Lambda} \tau_{j-i}(A_i) \right\rangle^{1/|\Lambda|}.$$

REMARKS. Part (1) is an immediate consequence of Theorem 9. Part (2) follows from Corollary 10 and a well-known theorem which asserts that the free energy per site $f(\beta, \Phi)$ (see Theorem 1, (1)) of infinite lattice systems with interactions Φ of the type considered here is independent of "boundary conditions." See [3] for a detailed proof.

IV.4. *Infrared bounds.* In this section we prove a basic estimate on the Fourier transform $d\omega^T(k)$ of the truncated expectation $\langle A_0^* A_i \rangle^T$ considered in §III. This estimate is a precise version of Estimate (I) in the strategy for proving the existence of phase transitions described there.

Let Φ be an interaction satisfying reflection positivity ($\Phi 4$) and

$$\Phi(\{i,j\}) = -J_{i-j}\, \mathbf{S}_i \cdot \mathbf{S}_j,$$

where $\mathbf{S}_i = \tau_i(\mathbf{B})$, $\mathbf{B} = (B_1, \ldots, B_N)$ is an N-tuple of selfadjoint operators in $\mathfrak{A}_0$, and J satisfies condition (R).

THEOREM 13 (INFRARED BOUND). *Let $\hat{J}$ denote the Fourier transform of J and $d\omega^{(T)}(k)$ the one of $\langle \mathbf{S}_0 \cdot \mathbf{S}_i \rangle^{(T)}$. Then, under the assumptions on Φ stated above and in the classical case* (C),

$$d\omega^T(k) \leqslant \frac{N}{2\beta\left(\hat{J}(0) - \hat{J}(k)\right)}\, d^\nu k,$$

where β^{-1} is the temperature.

In the quantum mechanical case (QM) the analogous estimate is somewhat more complicated, but see [2], [4].

PROOF. We prove Theorem 13 only for a special class of classical models (the general case is treated in [4]). They are defined as follows: $\Omega_0 = \mathbf{R}^N$, $\mathbf{S}_0 = (S_0^1, \ldots, S_0^N)$, with $S_0^i(\mathbf{x}) = x^i$ (the ith component of $\mathbf{x}$), for all $\mathbf{x} \in \mathbf{R}^N$; $\mathbf{S}_j = \tau_j(\mathbf{S}_0)$, for $j \in \mathbf{Z}^\nu_{1/2}$.

The a priori measure $d\mu$ used in the definition of the expectation tr $(\text{tr}_X(\cdot) \equiv \int_{\Omega_X} \prod_{i\in X} d\mu(\mathbf{S}_i))$ is assumed to be quasi-invariant under the translations of $\mathbf{R}^N$ (this is no loss of generality, since the general case will follow from the one considered here by a limiting argument). The interaction Φ is given by $\Phi(\{n\}) = -\mathbf{h} \cdot \mathbf{S}_n$, $\mathbf{h} \in \mathbf{R}^N$ (independent of n); $\Phi(\{n, m\}) = -J_{n-m}\mathbf{S}_n \cdot \mathbf{S}_m$, $\Phi(X) = 0$, for $|X| \geqslant 3$.

Without loss of generality we may normalize J_{n-m} such that

(N) $$\sum_m J_{n-m} = \hat{J}(0) = 0.$$

(This can be achieved by a suitable choice of J_0 and hence amounts to a trivial redefinition of $d\mu$.)

For each $\mathbf{g} \in \mathbf{R}^N$, we define

$$F_{\mathbf{g}}(\mathbf{S}) = \frac{e^{\mathbf{h}\cdot(\mathbf{S}+\mathbf{g})}d\mu(\mathbf{S}+\mathbf{g})}{e^{\mathbf{h}\cdot\mathbf{S}}d\mu(\mathbf{S})} = e^{\mathbf{h}\cdot\mathbf{g}}\frac{d\mu(\mathbf{S}+\mathbf{g})}{d\mu(\mathbf{S})}.$$

Consider now the equilibrium expectation

$$\left\langle e^{-\beta\sum_{n,m}J_{n-m}(2\mathbf{S}_n\cdot\mathbf{g}_m - \mathbf{g}_n\cdot\mathbf{g}_m)}\right\rangle.$$

We note that

$$-\sum_{n,m}\left[J_{n-m}(2\mathbf{S}_n\cdot\mathbf{g}_m - \mathbf{g}_n\cdot\mathbf{g}_m) - \Phi(\{n,m\})\right]$$

$$= \sum_{n,m}J_{n-m}(\mathbf{S}_n - \mathbf{g}_n)\cdot(\mathbf{S}_m - \mathbf{g}_m).$$

This suggests a change of variables $\mathbf{S}'_n = \mathbf{S}_n - \mathbf{g}_n$. Then

$$e^{\mathbf{h}\cdot\mathbf{S}_n}\,d\mu(\mathbf{S}_n) = F_{\mathbf{g}_n}(\mathbf{S}'_n)e^{\mathbf{h}\cdot\mathbf{S}'_n}\,d\mu(\mathbf{S}'_n).$$

Let $\langle - \rangle_\Lambda$ denote, as usual, the equilibrium expectation of the system with interaction Φ in the region $\Lambda \subseteq \mathbf{Z}^\nu_{1/2}$, at inverse temperature β. Then the substitution $\mathbf{S}_n \to \mathbf{S}'_n + \mathbf{g}_n$ yields

$$\left\langle e^{-\beta\sum_{n,m}J_{n-m}(2\mathbf{S}_n\cdot\mathbf{g}_m - \mathbf{g}_n\cdot\mathbf{g}_m)}\right\rangle_\Lambda$$

$$= \left\langle \prod_{i\in\Lambda} F_{\mathbf{g}_i}(\mathbf{S}_i)\right\rangle_\Lambda, \quad \text{for all } \Lambda \subseteq \mathbf{Z}^\nu_{1/2}.$$

We now apply the chessboard estimate (Corollary 10, Theorem 12) to the r.h.s. of this identity. This yields

$$\left\langle \prod_{i\in\Lambda} F_{\mathbf{g}_i}(\mathbf{S}_i)\right\rangle_\Lambda \leqslant \prod_{i\in\Lambda}\left\langle \prod_{j\in\Lambda} F_{\mathbf{g}_i}(\mathbf{S}_j)\right\rangle_\Lambda^{1/|\Lambda|},$$

for all bounded Λ. Now we undo the substitution $\mathbf{S}_n \to \mathbf{S}'_n + \mathbf{g}_n$ on the r.h.s. of this inequality and obtain

$$\left\langle \prod_{j\in\Lambda} F_{\mathbf{g}_i}(\mathbf{S}_j)\right\rangle_\Lambda = \left\langle e^{-\beta\sum_{n,m}J_{n-m}(2\mathbf{S}_n\cdot\mathbf{g}_i - \mathbf{g}_i\cdot\mathbf{g}_i)}\right\rangle_\Lambda.$$

Using the normalization (N) of J_{n-m} we find

$$\sum_m J_{n-m}(2\mathbf{S}_n\cdot\mathbf{g}_i - \mathbf{g}_i\cdot\mathbf{g}_i) = 0,$$

whence the r.h.s. of the equation above equals 1. This still holds in the limit $\Lambda = \mathbf{Z}^\nu_{1/2}$ (use Theorem 12!). If we put everything together we conclude

$$\left\langle e^{-\beta\sum_{n,m}J_{n-m}(2\mathbf{S}_n\cdot\mathbf{g}_m - \mathbf{g}_n\cdot\mathbf{g}_m)}\right\rangle \leqslant 1.$$

Next, scale g_i to εg_i and expand the l.h.s. of this inequality in powers of ε. This yields

$$1 - 2\varepsilon\beta \sum_{n,m} J_{n-m}\langle S_n\rangle \cdot g_m$$

$$+ \frac{1}{2}\, 4\varepsilon^2\beta^2 \sum_{\substack{n,m \\ n',m'}} J_{n-m}J_{n'-m'}\langle (S_n \cdot g_m)(S_{n'} \cdot g_{m'})\rangle$$

$$+ \varepsilon^2\beta \sum_{n,m} J_{n-m}\, g_n \cdot g_m + 0(\varepsilon^3) \leqslant 1.$$

Now note that $\sum_n J_{n-m}\langle S_n\rangle \cdot g_m = 0$. This follows from (N) and the fact that $\langle S_n\rangle \cdot g_m$ is *independent* of n, by translation invariance of $\langle - \rangle$. If we divide the inequality above by ε^2 and let ε tend to 0 we obtain

(IR′)
$$\sum_{\substack{n,m \\ n',m'}} J_{n-m}J_{n'-m'}\langle (S_n \cdot g_m)(S_{n'} \cdot g_{m'})\rangle$$

$$\leqslant -\frac{1}{2\beta} \sum_{n,m} J_{n-m}\, g_n \cdot g_m.$$

Next choose g_n such that its Fourier transform is peaked near some momentum $k \in B$. Then (Fourier transformation of) (IR′) gives

$$\hat{J}(k)^2 d\omega(k) \leqslant -N(2\beta)^{-1}\hat{J}(k).$$

Since $\hat{J}(k = 0) = 0$, by (N), we obtain

(IR) $d\omega(k) = c\delta(k)d^\nu k + d\omega^T(k) \leqslant \left[c\delta(k) - \dfrac{N}{2\beta\hat{J}(k)} \right] d^\nu k.$ Q.E.D.

V. Applications to classical lattice systems: phase transitions for Gibbs random fields. In this final section we consider classical lattice systems (the so-called classical ferromagnets), i.e. models (C1) and (C2) defined in §II.3. More results on these systems and a detailed study of a class of quantum lattice systems may be found in [2], [4]; for applications to Fermion lattice systems we refer the reader to [4], [19]. Some of our results (*Ising models* with long range interactions *in one dimension*) have earlier been obtained by Dyson [35]. In models (C1) and (C2), $\Omega_0 = S^{N-1}$, the unit sphere in $\mathbf{R}^N$, $N = 1, 2, 3, \ldots$. The measure $d\mu$ on Ω_0 used in the definition of the trace tr is given by

$$d\mu(S) = \delta(|S| - 1)d^N S.$$

We assume that the interaction Φ satisfies *reflection positivity* (Φ4); see §IV.1. Moreover

$$\Phi(\{n\}) = -\mathbf{h} \cdot S_n, \qquad \mathbf{h} \text{ a fixed vector in } \mathbf{R}^N,$$

$$\Phi(\{n, m\}) = -J_{n-m}S_n \cdot S_m,$$

where J satisfies the reflection positivity condition (R) (resp. (R′)) of §IV.2.

For $|X| \geqslant 4$, $\Phi(X)$ is an $O(N)$ invariant polynomial in $S_X = \{S_i\}_{i \in X}$ (compatible with (Φ4)).

Obviously $\langle S_j \cdot S_j\rangle = 1$, for any state $\langle - \rangle$.

If $\mathbf{h} = 0$ these models have $O(N)$ as their *symmetry group*.

Let $\langle - \rangle$ denote some infinite volume (thermodynamic) limit of the equilibrium states $\{\langle - \rangle_\Lambda \equiv \rho_\Lambda^{\beta,\Phi}(-)\}$, where $\{\Lambda\}$ is a sequence of rectangles increasing to $\mathbf{Z}'_{1/2}$ (see Theorem 2).

Under these assumptions we may apply the infrared bound of §4. Thus

(IR) $$d\omega(k) \leqslant \left(c\delta(k) + N\left[2\beta\left(\hat{J}(0) - \hat{J}(k)\right)\right]^{-1}\right)d^\nu k.$$

This inequality gives

THEOREM 14 [1], [4]. *If*

$$I(\nu, J) \equiv \int_B \frac{d^\nu k}{\hat{J}(0) - \hat{J}(k)}$$

is finite then c is positive for $\beta > NI(\nu, J)/2$.

For $\mathbf{h} = 0$ $c > 0$ *implies the existence of macroscopic fluctuations in the state* $\langle - \rangle$ *and hence of a phase transition and* $O(N)$*-symmetry breaking in pure phases.*

PROOF. If we integrate inequality (IR) we obtain

$$\langle \mathbf{S}_i \cdot \mathbf{S}_i \rangle = 1 = \int_B d\omega(k) \leqslant c + \frac{N}{2\beta} I(\nu, J).$$

This proves the first part of the theorem.

If $\mathbf{h} = 0$ then the state $\langle - \rangle$ is $O(N)$-invariant, so that

$$\langle \mathbf{S}_i \rangle = 0 \quad \text{and} \quad c = \langle \mathbf{S}_i \cdot \mathbf{S}_i \rangle - \langle \mathbf{S}_i \cdot \mathbf{S}_i \rangle^T > 0$$

implies the existence of a phase transition. The remaining assertions therefore follow from the results of §III.   Q.E.D.

Finally we derive conditions for the finiteness of $I(\nu, J)$. For simplicity we only consider the case where $J_i \geqslant 0$, for all $i \neq 0$ (but see [4] for more general results).

Let δ_j be the unit vector with components $\delta_{ji}, j = 1, \ldots, \nu$.

PROPOSITION 15 [1], [4]. *Suppose that* $J_{n^j\delta_j} > 0$, *for some* $n^j = 1, 2, 3, \ldots$; $j = 1, \ldots, \nu$; *e.g.*

$$J_{\delta_j} = J > 0, \quad \text{for all } j = 1, \ldots, \nu$$

("ferromagnetic nearest neighbor coupling"; see [1], [4]). Then $I(\nu, J)$ *is finite, for all* $\nu \geqslant 3$ *and hence there is a phase transition.*

PROOF.

$$\hat{J}(0) - \hat{J}(k) = \sum_{i \in \mathbf{Z}^\nu} J_i(1 - \cos(k \cdot i))$$

$$\geqslant \sum_{j=1}^\nu J_{n^j\delta_j}\left(1 - \cos(k^j \cdot n^j)\right)$$

$$\geqslant \frac{2}{\pi^2} \min_j \left(J_{n^j\delta_j}(n^j)^2\right)|k|^2 \geqslant \text{const } |k|^2.$$

Thus $(\hat{J}(0) - \hat{J}(k))^{-1}$ is $d^\nu k$-integrable for $\nu \geqslant 3$.   Q.E.D.

Next we briefly discuss phase transitions in $\nu = 1$ and 2 dimensions. From §IV.2 we know that

$$J_i = \begin{cases} c|i|^{-\alpha}, & i \neq 0, \\ c', & i = 0, \end{cases}$$

satisfies reflection positivity, for all $\alpha > 0$ ($\nu = 1$ or 2).

We assume that $J_i \geqslant 0$, for all $i \neq 0$, and

(1) $$J_i \geqslant c|i|^{-\alpha}, \quad \text{as } |i| \to \infty,$$

for some positive constant c.

Existence of the thermodynamic limit requires

(2) $$\alpha > \nu.$$

The angle between a vector $i \in \mathbf{Z}^\nu$ and a vector $k \in B$ is denoted $\measuredangle(i, k)$.

THEOREM 16 [4]. *$I(\nu, J)$ is finite if $\alpha < 2\nu$.*

For $\nu < \alpha < 2\nu$ and $\mathbf{h} = 0$ there exists a phase transition and $O(N)$-symmetry breaking, for β sufficiently large (low temperatures).

PROOF. By Theorem 14 it suffices to show that $I(\nu, J) < \infty$, for $\alpha < 2\nu$.

We first estimate $\hat{J}(0) - \hat{J}(k)$: Clearly $\hat{J}(0) - \hat{J}(k) > 0$, for $k \neq 0$, under our assumptions on J_i. It therefore suffices to estimate the behaviour near $k \approx 0$. We assume that $|k| \ll 1$. Then

$$\hat{J}(0) - \hat{J}(k) = \sum_{i \in \mathbf{Z}^\nu} J_i(1 - \cos(k \cdot i)) \geqslant \sum_{\substack{\measuredangle(i,k) \leqslant \pi/4 \\ |i| \leqslant \varepsilon|k|^{-1}}} J_i(1 - \cos(k \cdot i))$$

$$\geqslant c_1|k|^2 \sum_{\substack{\measuredangle(i,k) \leqslant \pi/4 \\ |i| \leqslant \varepsilon|k|^{-1}}} J_i|i|^2, \quad \text{for } \varepsilon \text{ small enough,}$$

(3)

$$\geqslant c_2|k|^2 \int_1^{\varepsilon|k|^{-1}} x^{\nu+1-\alpha} \, dx, \quad \text{for } \varepsilon|k|^{-1} \gg 1,$$

$$= c_3|k|^2|k|^{-\nu-2+\alpha}, \quad \text{for } \alpha < \nu + 2,$$

$$= c_3|k|^{\alpha-\nu}.$$

Hence $(\hat{J}(0) - \hat{J}(k))^{-1}$ is $d^\nu k$-integrable if

(4) $$\alpha < 2\nu.   \text{Q.E.D.}$$

REMARKS. 1. Let $\mathbf{S}_n$ be an N-dimensional, classical spin. Let Φ be defined by

$$\Phi(\{n, m\}) = -J_{n-m}\mathbf{S}_n \cdot \mathbf{S}_m,$$

where J satisfies the reflection positivity condition (R) (resp. (R')) of §IV.2;

$$\Phi(X) = 0, \quad \text{for } |X| \neq 2.$$

THEOREM 14'. *Under these assumptions and for $N \geqslant 2$ the condition*

$$I(v, J) = \int_B \frac{d^v k}{\hat{J}(0) - \hat{J}(k)} < \infty$$

is necessary and sufficient for the existence of spontaneous magnetization.

Theorem 14' follows directly from Theorem 14 and the Mermin-Wagner theorem [36].

2. For $N = 1$ and $\Phi(X) = -J_X \Pi_{i \in X} S_i$, $J_X \geq 0$, Slawny [29] has obtained sharp upper bounds on the *number of equilibrium states* in $\Delta^{\beta,\Phi}$, for small β. Using recent results of Lebowitz [30] one can extend these bounds to all but possibly countably many values of the inverse temperature β. As an example we mention that for such models for which $\Phi(\{n, m\}) = -J_{n-m} S_n \cdot S_m$, with $J_n > 0$, for $|n| = 1$, there exist at most *two* translation invariant equilibrium states at all but possibly countably many values of β [30].

In the two dimensional Ising model with *nearest neighbor ferromagnetic* interaction one has:

(1) In an external magnetic field (i.e. $\Phi(\{n\}) = -h \cdot S_n$, with $h \neq 0$) there exists precisely *one* translation invariant equilibrium state. The same is true for $h = 0$ and all $\beta < \beta_c$, where $\beta_c^{-1} > 0$ is the critical temperature.

(2) For $h = 0$ and $\beta > \beta_c$ there exist precisely two translation invariant equilibrium states with opposite spontaneous magnetization.

The *proofs* of these results are surprisingly *simple*. We urge the reader to consult [29], [30].

It has recently been shown in [37] that the plane rotator model in zero external field ($N = 2$, $J_{i-j} \geq 0$, $\mathbf{h} = 0$) has a unique, extremal, translation invariant equilibrium state whenever there is *no* spontaneous magnetization.

(3) For application and adaptations of the general theory of §§IV and V to quantum lattice systems, including non-relativistic fermions, see [2], [4].

We conclude with some *open problems*:

1. Is there a generalization of the theory described in §§IV and V which is applicable to the quantum mechanical Heisenberg ferromagnet? (The present theory only covers the anti-ferromagnet and the x-y model; see [2].)

2. How can one analyze phase transitions in systems on irregular lattices and with impurities (*other* than Ising type models with discrete spins?)

3. How close is the connection between reflection positivity (Theorem 9) and the validity of infrared bounds (Theorem 13, §IV.4)? It is known that there exist classical lattice systems which violate reflection positivity (and translation invariance) for which infrared bounds are true (E. H. Lieb, private communication). The only known such examples are however systems without phase transitions.

4. Is there a generalization of the Slawny-Lebowitz theory [29], [30] concerning the number of translation invariant equilibrium states to general classical or quantum lattice systems?

5. Consider a lattice system at the critical temperature (clustering, but *not* exponential). Is there a connection between reflection positivity and scaling behaviour at large distances? It is easy to see that if there is scaling, the scaling limit of the correlation functions of a system satisfying reflection positivity are the Euclidean Green's functions of a relativistic quantum field theory satisfying the Osterwalder-Schrader axioms. (For results concerning

phase transitions and the critical point in relativistic quantum field theory see [7], [8], [1], [4].)

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