BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 82, Number 1, January 1976

DIRECT SUM PROPERTIES OF QUASI-INJECTIVE MODULES

BY K. R. GOODEARL

Communicated by Barbara Osofsky, October 15, 1975

ABSTRACT. A functorial method is described by which certain problems can be transferred from quasi-injective modules to nonsingular injective modules. Applications include the uniqueness of nth roots: If A and B are quasi-injective modules such that $A^n \cong B^n$, then $A \cong B$.

All rings in this paper are associative with unit, all modules are unital right modules, and endomorphism rings act on the left. The letter R denotes a ring. We use $J(-)$ to denote the Jacobson radical.

Recall that a module A is *quasi-injective* provided any homomorphism of a submodule of A into A extends to an endomorphism of A. For example, all injective modules and all semisimple (completely reducible) modules are quasi-injective.

THEOREM 1. *Let A be a quasi-injective right R-module, and set $Q = \mathrm{End}_R(A)$. Then $Q/J(Q)$ is a regular, right self-injective ring, and idempotents can be lifted modulo $J(Q)$.*

PROOF. Regularity and idempotent-lifting were proved by Faith and Utumi [2, Theorems 3.1, 4.1]. Self-injectivity was proved by Osofsky [6, Theorem 12] and Renault [7, Corollaire 3.5]. □

PROPOSITION 2. *Let A be a quasi-injective right R-module, and set $Q = \mathrm{End}_R(A)$. Let $\mathfrak{A}$ denote the category of all direct summands of finite direct sums of copies of A, and let $\mathcal{P}$ denote the category of all finitely generated projective right $(Q/J(Q))$-modules. Then there exists an additive (covariant) functor $F: \mathfrak{A} \to \mathcal{P}$ with the following properties.*

(a) *For all $B, C \in \mathfrak{A}$, the induced map $\mathrm{Hom}_{\mathfrak{A}}(B, C) \to \mathrm{Hom}_{\mathcal{P}}(F(B), F(C))$ is surjective.*

(b) *Given any $P \in \mathcal{P}$, there exists $B \in \mathfrak{A}$ such that $F(B) \cong P$.*

(c) *A map $f \in \mathfrak{A}$ is an isomorphism if and only if $F(f)$ is an isomorphism in $\mathcal{P}$.*

PROOF. If $\mathcal{P}_0$ denotes the category of all finitely generated projective right Q-modules, then $\mathrm{Hom}_R(A, -)$ defines a category equivalence $G: \mathfrak{A} \to \mathcal{P}_0$. Second, $(-) \otimes_Q (Q/J(Q))$ gives us an additive functor $H: \mathcal{P}_0 \to \mathcal{P}$, and we set $F = HG$.

AMS (MOS) subject classifications (1970). Primary 16A52.

Properties (a) and (c) hold without any hypotheses on A, while (b) follows from the regularity of $Q/J(Q)$ and the fact that idempotents lift modulo $J(Q)$. □

Over a regular, right self-injective ring, all finitely generated projective right modules are injective and nonsingular. Thus the functor F in Proposition 2 enables us to transfer problems from the quasi-injective module A to the nonsingular injective module $F(A)$.

THEOREM 3. *Let A, B be quasi-injective right R-modules, and let n be a positive integer.*

(a) *If A^n is isomorphic to a direct summand of B^n, then A is isomorphic to a direct summand of B.*

(b) *If $A^n \cong B^n$, then $A \cong B$.*

PROOF. Setting $Q = \mathrm{End}_R(B)$, we use Proposition 2 to transfer the problem to nonsingular injective right $(Q/J(Q))$-modules, where the required properties follow from [5, Proposition 9.1]. □

DEFINITION. A module A is *directly finite* provided A is not isomorphic to any proper direct summand of itself.

THEOREM 4 [1, Proposition 5]. *Let A be a directly finite quasi-injective right R-module. If B and C are any right R-modules such that $A \oplus B \cong A \oplus C$, then $B \cong C$.*

PROOF. If P is any directly finite nonsingular injective module, then [8, Corollary 8] (or [5, Theorem 3.8]) shows that isomorphic direct summands of P have isomorphic complements. Using Proposition 2, the module A has the same property. In addition, [3, Theorem 3] shows that A has the exchange property, hence cancellation follows from [4, Theorem 2]. □

COROLLARY 5. *If $A_1, \ldots, A_n$ are directly finite quasi-injective right R-modules, then $A_1 \oplus \cdots \oplus A_n$ is directly finite (but not necessarily quasi-injective).*

PROOF. Obviously cancellation carries over from the A_i to their direct sum. On the other hand, $\mathbf{Z}/2\mathbf{Z}$ and $\mathbf{Q}$ are directly finite quasi-injective $\mathbf{Z}$-modules whose direct sum is not quasi-injective. □

THEOREM 6. *If A is a quasi-injective right R-module, then there exists a decomposition $A = B \oplus C$ such that B is directly finite and $C \cong C^2$.*

PROOF. The corresponding decomposition for nonsingular injective modules is given by [5, Proposition 8.4 and Theorem 7.2]. □

COROLLARY 7. *Let A be a quasi-injective right R-module. Then A is directly finite if and only if A has no nonzero direct summands C for which $C \cong C^2$.* □

REFERENCES

1. G. F. Birkenmeier, *On the cancellation of quasi-injective modules,* Communications in Algebra (to appear)

2. C. Faith and Y. Utumi, *Quasi-injective modules and their endomorphism rings,* Arch. Math. **15** (1964), 166–174. MR **29** #3503.

3. L. Fuchs, *On quasi-injective modules,* Ann. Scuola Norm. Sup. Pisa (3) **23** (1969), 541–546. MR **41** #3518.

4. ———, *The cancellation property for modules,* Lectures on Rings and Modules (Tulane Univ. Ring and Operator Theory Year 1970–1971, Vol. I), Lecture Notes in Math., vol. 246, Springer-Verlag, Berlin, 1972, pp. 191–212. MR **49** #2846.

5. K. R. Goodearl and A. K. Boyle, *Dimension theory for nonsingular injective modules* (to appear).

6. B. L. Osofsky, *Endomorphism rings of quasi-injective modules,* Canad. J. Math. **20** (1968), 895–903. MR **38** #184.

7. G. Renault, *Anneau associé à un module injectif,* Bull. Sci. Math. (2) **92** (1968), 53–58. MR **38** #1129.

8. Y. Suzuki, *On automorphisms of an injective module,* Proc. Japan Acad. **44** (1968), 120–124. MR **37** #2806.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UTAH 84112