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GROUP ACTIONS AND CURVATURE

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1. **Introduction.** The purpose of this note is to outline a proof of the following result: Any isometric action of a compact Lie group G on a 1-connected, compact Riemannian manifold M whose curvature tensor R is sufficiently close to the curvature tensor R_0 of the standard sphere S^n of the same dimension is equivalent to an isometric action of G on S^n.

We measure the proximity of R and R_0 in terms of the eigenvalues of the curvature transformation $R: V\wedge V\to V\wedge V$, where $V=T(M)$. A Riemannian manifold M is called *strongly δ-pinched* if the eigenvalues λ of the curvature transformation at all points of M satisfy the condition $\delta<\lambda\leqq 1$.

2. **Statement of results.** The main result is as follows:

THEOREM. *There exists a $\delta_0<1$, such that for any 1-connected, compact, strongly δ-pinched n-dimensional Riemannian manifold M, and any compact Lie group G the following holds:*
If $\delta>\delta_0$ and $\mu: G\times M\to M$ is an isometric action of G on M, then
(1) *there exists a diffeomorphism $F: M\to S^n$;*
(2) *there exists a homomorphism $\omega: G\to O(n+1)$ such that*
(3) *$\omega(g)=F\circ\mu(g,\cdot)\circ F^{-1}$ for all $g\in G$.*

The following two corollaries are immediate consequences.

COROLLARY 1. *Any compact, strongly δ-pinched Riemannian manifold M with $\delta>\delta_0$ is diffeomorphic to a space of constant curvature* 1.

Together with Wolf's [4] classification of manifolds with constant curvature 1, this corollary gives a classification up to diffeomorphism of compact, strongly δ-pinched Riemannian manifolds with $\delta>\delta_0$. In addition, the isometry group of such a manifold is isomorphic to a subgroup of the isometry group of the corresponding manifold with constant curvature.

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COROLLARY 2. *Let M be a 1-connected, compact, strongly δ-pinched Riemannian manifold of dimension $2n+1$ with $\delta>\delta_0$. If S^1 operates freely on M by isometries then the quotient M/S^1 is diffeomorphic to the complex projective space CP^n.*

3. **Outline of proof.** We prove the theorem in the following steps:

(I) Construction of a preliminary diffeomorphism $f: M\to S^n$.

(II) Construction of an "almost homomorphism" $\omega_0: G\to O(n+1)$ with the property $\omega_0(g)$ is C^1-close to $f\circ\mu(g,\cdot)\circ f^{-1}$ for all $g\in G$.

(III) Construction of a homomorphism $\omega: G\to O(n+1)$ close to ω_0.

From (II) and (III) it follows that $\omega(g)\in O(n+1)\subset \mathrm{Diff}(S^n)$ and $f\circ\mu(g,\cdot)\circ f^{-1}\in\mathrm{Diff}(S^n)$ are C^1-close for all $g\in G$. The corresponding actions on S^n are therefore conjugate (see Grove and Karcher [**1**] or Palais [**2**]), thus there exists $S\in\mathrm{Diff}(S^n)$ such that $\omega(g)=S\circ(f\circ\mu(g,\cdot)\circ f^{-1})\circ S^{-1}$, i.e., $S\circ f: M\to S^n$ is the desired diffeomorphism F.

The ideas in (I) and (II) are based on Ruh [**3**]. The main tool in (III) is the notion of *center of mass* for *almost constant maps* introduced in Grove and Karcher [**1**]. Since (III) might be of independent interest we state it in full generality.

Let G and H be compact Lie groups with bi-invariant metrics normalized so that $\mathrm{Vol}(G)=1$, $\|[X,Y]\|\leqq\|X\|\cdot\|Y\|$ for all $X, Y\in T_eH$ and such that the injectivity radius of $\exp: T_eH\to H$ is $\geqq\pi$ (this choice is always possible; $\langle X, Y\rangle=\mathrm{trace}\, X\circ Y^*$ in the case $H=O(n+1)$).

PROPOSITION. *If G and H are as above and $\omega_0: G\to H$ is a (continuous) map which is an almost homomorphism in the sense that*

$$d_H(\omega_0(g\cdot g')\cdot\omega_0(g')^{-1}, \omega_0(g))\leqq q\leqq\pi/6\quad\forall(g,g')\in G\times G$$

then there exists a (continuous) homomorphism $\omega: G\to H$ close to ω_0. In fact $d_H(\omega(g),\omega_0(g))\leqq 1.5q\ \forall g\in G$.

Now we give a sketch of the steps in the proof of the main result.

Step (I). As in Ruh [**3**] we construct a flat connection ∇' on the bundle $E=T(M)\oplus 1(M)$, where $1(M)$ is the trivial line bundle $M\times\boldsymbol{R}$. First we define a connection ∇'' with small curvature by

$$\begin{aligned}\nabla''_X Y &= \nabla_X Y-\langle X, Y\rangle e, \qquad \forall X, Y\in C^\infty(TM),\\ \nabla''_X e &= X,\end{aligned}$$

where ∇ is the Riemannian connection on $T(M)$ and e is the section $e_m=(O_m, 1)$. We use ∇'' to construct a cross section u' of the principal bundle P of orthonormal $(n+1)$-frames associated to E, and ∇' is the corresponding flat connection of E. The difference $\nabla'-\nabla'': V\to\mathfrak{o}(n+1)$ is small for $1-\delta$ small. We define the preliminary diffeomorphism $f: M\to S^n$

by $f(m)=\langle e, u'\rangle_m$, where $\langle e, u'\rangle_m$ denotes the coordinates of e_m in the basis u'_m. Since $df(X)=\langle\nabla'_X e, u'\rangle$, $\nabla''_X e=X$ and $\|\nabla'-\nabla''\|$ is small, f is a diffeomorphism.

Step (II). We extend the action $\mu:G\times M\to M$ to an action of G on E as follows: $g\cdot(X_m+te_m)=\mu(g,\cdot)_*X_m+te_{\mu(g,m)}$. With the trivialization $E=M\times\boldsymbol{R}^{n+1}$ determined by u' we obtain a map $\Omega:M\times G\to O(n+1)$; fix an arbitrary $m_0\in M$ and define $\omega_0=\Omega(m_0,\cdot):G\to O(n+1)$. Now we estimate the deviation of ω_0 from a homomorphism. The main observation in this estimate is that ∇'' is invariant under the action of G and the difference $\|\nabla'-\nabla''\|$ is small. Now, $\omega_0(g)$ and $f\circ\mu(g,\cdot)\circ f^{-1}\in\mathrm{Diff}(S^n)$ are C^1-close for all $g\in G$ because the maps $\Omega(m,\cdot):G\to O(n+1)$ are almost independent of $m\in M$.

Step (III). Let M and N be compact Riemannian manifolds. There exists a $\rho'>0$ such that for any continuous map $f:M\to N$ whose image is contained in a ball $B_{\rho'}$ of radius ρ' (f is called *almost constant*) there is a unique point $\mathscr{C}(f)\in B_{\rho'}$ (the *center of* f) with the property

$$\int_M \exp^{-1}_{\mathscr{C}(f)}(f(m))\,dm=0.$$

If $A:N\to N$ is an isometry

$$(*)\qquad \mathscr{C}(A\circ f)=A(\mathscr{C}(f))$$

and if $k:M\to M$ is a volume preserving diffeomorphism

$$(**)\qquad \mathscr{C}(f\circ k)=\mathscr{C}(f)\text{ holds,}$$

see Grove and Karcher [**1**].

Now let G, H and $\omega_0:G\to H$ be as in the proposition. From ω_0 we construct inductively a sequence $\{\omega_k\}$ of almost homomorphisms as follows:

$$\omega_{k+1}(g)=\mathscr{C}(g'\to\omega_k(g\cdot g')\omega_k(g')^{-1})\quad\forall g\in G.$$

We prove that ω_{k+1} is an "improvement" of ω_k and that the sequence converges uniformly to a homomorphism $\omega:G\to H$. The equations (*) and (**) applied to inversion, left- and right-translations reduce the proof to estimating the center $\mathscr{C}(\eta_1\cdot\eta_2)$ of the product of almost constant maps with $\mathscr{C}(\eta_1)=\mathscr{C}(\eta_2)=e\in H$. The tools used here are the Campbell-Hausdorff formula together with the comparison theorems of Rauch and Toponogov. To conclude Step (III) we apply the above proposition to the map $\omega_0=\Omega(m_0,\cdot):G\to O(n+1)$. The details, as well as an estimate for the pinching constant δ_0, will be furnished in a subsequent paper.

ADDED IN PROOF. In the meantime the paper has appeared under the same title in Invent. Math. **23** (1974), 31–48. Furthermore by using a Finsler norm on the orthogonal group $O(n)$ instead of the Riemannian metric the theorem has been proved under the weaker assumption of sectional curvature pinching (this will appear in Math. Ann. under the title, *Jacobi fields and Finsler metrics on compact Lie groups with an application to differentiable pinching problems*).

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