BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, November 1974

ON THE DETERMINATION OF A HILL'S EQUATION FROM ITS SPECTRUM

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Communicated by Cathleen Morawetz, March 26, 1974

A Hill's equation is an equation of the form:

$$y'' + [\lambda - q(z)]y = 0, \qquad q(z + \pi) = q(z), \tag{1}$$

where $q(z)$ is assumed to be integrable over $[0, \pi]$. Without loss of generality, it is customary to assume that $\int_0^\pi q(z)\,dz = 0$. The discriminant of (1) is defined by

$$\Delta(\lambda) = y_1(\pi) + y_2'(\pi),$$

where y_1 and y_2 are solutions of (1) satisfying $y_1(0) = y_2'(0) = 1$ and $y_1'(0) = y_2(0) = 0$.

The set of values of λ for which $|\Delta| > 2$ consists of a finite or an infinite number of finite disjoint intervals and one infinite interval. These intervals are called instability intervals, since (1) has no solution which is bounded for all real z in these intervals. When $|\Delta| < 2$, all solutions of (1) are bounded for all real z and the corresponding intervals are called stability intervals. Pertinent information about stability and instability intervals of (1) can be found in Magnus and Winkler [**1**].

The following result has been proved:

THEOREM. *If $q(z)$ is real and integrable, and if precisely n finite instability intervals fail to vanish, then $q(z)$ must satisfy a differential equation of the form*

$$q^{(2n)} + H(q, q', \cdots, q^{(2n-2)}) = 0, \quad a.e. \tag{2}$$

where H is a polynomial of maximal degree $n+2$.

Borg [**2**], Hochstadt [**3**] and Ungar [**4**] proved this theorem for the case $n=0$, i.e. when all finite instability intervals vanish, and found that

$$q(z) = 0, \quad \text{a.e.} \tag{3}$$

For the case $n=1$, Hochstadt [**3**] showed that $q(z)$ is the elliptic function which satisfies

$$q'' = 3q^2 + Aq + B, \quad \text{a.e.} \tag{4}$$

AMS (MOS) subject classifications (1970). Primary 34B30; Secondary 34E05.

[1] I thank my advisor, Harry Hochstadt, for his generous advice and guidance.

where A and B are constants. (3) and (4) are equivalent to (2) for the cases $n=0$ and 1, respectively. In particular, for the case $n=2$, the explicit expression of (2) is

$$(5)\quad q^{(4)} = 10qq'' + Aq'' + 5(q')^2 - 10q^3 + Bq^2 + Cq + D, \quad \text{a.e.}$$

where A, B, C and D are constants.

Erdélyi [5] investigated a Hill's equation where $q(z)$ is a Lamé function and discovered situations where all but a finite number of finite instability intervals vanish. (4) provides a converse to some of his results.

Lax [6], through the study of partial differential operators, derived sufficient conditions for the vanishing of all but n finite instability intervals. These conditions coincide with (3), (4) and (5) for the cases $n=0$, 1 and 2, respectively. Whether there exist equivalent necessary conditions for higher values of n is still an open question.

The proof of the Theorem is accomplished by investigating the related problem

$$(6)\qquad u'' + [\lambda - q(z+\tau)]u = 0; \quad \tau \text{ real, arbitrary}$$

and by assuming the result [3] that $q(z)$ is infinitely differentiable a.e. when n finite instability intervals fail to vanish.

In [3], Hochstadt showed that (6), when subject to $u(0)=u(\pi)$, has eigenvalues $\mu_n(\tau)$, where $\mu_i(\tau)$ lies in the ith finite instability interval of (1). Furthermore, when precisely n finite instability intervals fail to vanish

$$u_2(\pi)\prod_{i=1}^{n}[\lambda - \mu_i(0)] = y_2(\pi)\prod_{i=1}^{n}[\lambda - \mu_i(\tau)],$$

where $u_2(t)$ denotes the solution of (6) which satisfies $u_2(0)=0$ and $u_2'(0)=1$. Suitable asymptotic expressions of $u_2(\pi)$ and $y_2(\pi)$ have been developed and our result follows upon their substitution into this equation. The details will appear in a later paper.

References

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