BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 3, May 1974

FOLIATIONS

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Introduction. The study of foliations on manifolds has a long history in mathematics, even though it did not emerge as a distinct field until the appearance in the 1940's of the work of Ehresmann and Reeb. Since that time, the subject has enjoyed a rapid development, and, at the moment, it is the focus of a great deal of research activity.

The purpose of this article is to provide an introduction to the subject and present a picture of the field as it is currently evolving.

The treatment will by no means be exhaustive. My original objective was merely to summarize some recent developments in the specialized study of codimension-one foliations on compact manifolds. However, somewhere in the writing I succumbed to the temptation to continue on to interesting, related topics. The end product is essentially a general survey of new results in the field with, of course, the customary bias for areas of personal interest to the author.

Since such articles are not written for the specialist, I have spent some time in introducing and motivating the subject. However, this article need not be read linearly. §§ 1, 2, 3 and 5 fall into the category of "basic material." §§ 4, 8 and the combination 6–7 are essentially independent of each other.

I would like to thank Bill Thurston and André Haefliger for making several valuable suggestions for improving the manuscript.

An address delivered before the 78th Annual Meeting of the Society in Las Vegas, Nevada, on January 17, 1972 by invitation of the Committee to Select Hour Speakers for the Summer and Annual Meetings, under the title *Foliations of compact manifolds*; received by the editors May 7, 1973.

AMS (MOS) subject classifications (1970). Primary 57D30.

[1] Work partially supported by NSF grant GP29697.

1. **Definitions and general examples.** A manifold is, roughly speaking, a space locally modelled on affine space; and a submanifold is a subset locally modelled on an affine subspace. In this spirit, a foliated manifold is a manifold modelled locally on an affine space decomposed into parallel affine subspaces.

DEFINITION 1. By a *p-dimensional, class C^r* foliation of an m-dimensional manifold M we mean a decomposition of M into a union of disjoint connected subsets $\{\mathscr{L}_\alpha\}_{\alpha\in A}$, called the *leaves* of the foliation, with the following property: Every point in M has a neighborhood U and a system of local, class C^r coordinates $x=(x^1, \cdots, x^m): U\to \boldsymbol{R}^m$ such that for each leaf $\mathscr{L}_\alpha$, the components of $U\cap\mathscr{L}_\alpha$ are described by the equations $x^{p+1}=\text{constant}, \cdots, x^m=\text{constant}$.

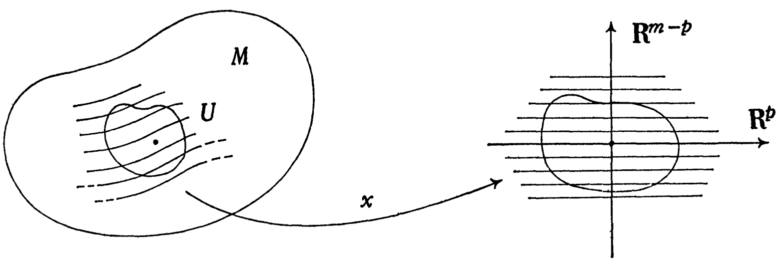

FIGURE 1

We shall denote such a foliation by $\mathscr{F}=\{\mathscr{L}_\alpha\}_{\alpha\in A}$.

It will often be more natural to refer to the *codimension* $q=m-p$ of $\mathscr{F}$ rather than to its dimension p.

Note that every leaf of $\mathscr{F}$ is a p-dimensional, embedded submanifold of M. The embedding, however, may not be proper; in fact, as we shall see, it is possible for a leaf to be dense.

Local coordinates with the property mentioned in Definition 1 are said to be *distinguished* by the foliation. If x and y are two such coordinate systems defined in an open set $U\subset M$, then the functions giving the change of coordinates $y_i=y_i(x^1, \cdots, x^m)$ must satisfy the equations

$$\partial y_i/\partial x_j = 0 \quad \text{for } 1 \leqq j \leqq p < i \leqq m \tag{1.1}$$

in U. Hence, choosing a covering of M by distinguished local coordinates gives rise to a G-structure on M (cf. Chern **[C]**) where $G\subset GL(m, \boldsymbol{R})$ is the group of matrices with zeros in the lower left $(m-p)\times p$ block. That is, G is the subgroup of $GL(m, \boldsymbol{R})$ which preserves the linear subspace $\boldsymbol{R}^p=\{(x^1, \cdots, x^p, 0, \cdots, 0)\}\subset\boldsymbol{R}^m$. One of the reasons that foliations interest people in geometry is that they constitute a class of structures on manifolds which is complicated enough to shed light on the general situation but has certain geometric aspects that make it tractable.

Foliations arise naturally in many ways in mathematics and it should be useful to examine some of the important cases. The first and simplest examples come from nonsingular differentiable mappings.

A. *Submersions.* Let M and Q be manifolds of dimension m and $q \leqq m$ respectively, and let $f: M \to Q$ be a submersion, that is, suppose that $\operatorname{rank}(df) \equiv q$. It follows from the Implicit Function Theorem that f induces a codimension-q foliation on M where the leaves are defined to be the components of $f^{-1}(x)$ for $x \in Q$. Differentiable fiber bundles are examples of this sort.

Note that locally every foliation is defined by submersion.

B. *Bundles with discrete structure group.* Let $M \to^{\pi} P$ be a differentiable fiber bundle with fiber Q. Recall that a bundle is defined by an open covering $\{U_\alpha\}_{\alpha \in A}$ of P, diffeomorphisms $h_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times Q$, and transition functions $g_{\alpha\beta}: U_\alpha \cap U_\beta \to \operatorname{Diff}(Q)$ such that $h_\alpha \circ h_\beta^{-1}(x, y) = (x, g_{\alpha\beta}(x)(y))$. If the transition functions are locally constant, the bundle is said to have *discrete structure group.* Note that under this assumption, the codimension-q ($q = \dim Q$) foliations of $\pi^{-1}(U_\alpha)$ given by the submersions $\pi^{-1}(U_\alpha) \approx U_\alpha \times Q \to Q$ fit together to give a foliation of M.

Every such bundle can be constructed in the following way. Let $\varphi: \pi_1(P) \to \operatorname{Diff}(Q)$ be a homomorphism and denote by $\tilde{P}$ the universal covering space of P. Then $\pi_1(P)$ acts jointly on the product $\tilde{P} \times Q$, and we define $M = \tilde{P} \times Q / \pi_1(P)$. The action preserves the product structure, and so the product foliation of $\tilde{P} \times Q$ (arising from $\tilde{P} \times Q \to Q$) projects to a foliation of M. This is the foliation we described above. Note that each leaf looks like a many-valued cross-section of the bundle $M \to^{\pi} P$. In fact, π restricted to any leaf is a covering map. To see this note that if $\mathscr{L}$ is the leaf corresponding to $\tilde{P} \times \{x\} \subset \tilde{P} \times Q$, then $\mathscr{L} \approx \tilde{P}/\Gamma_x$ where $\Gamma_x = \{g \in \pi_1(P) : \varphi(g)(x) = x\}$.

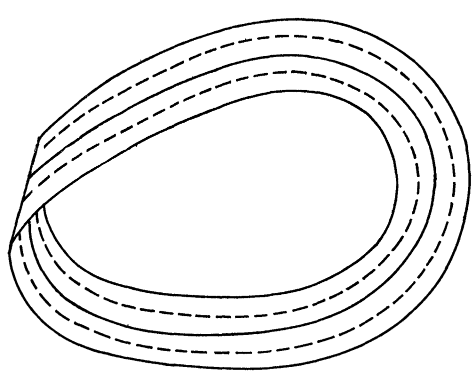

FIGURE 2

The simplest example of a bundle of this sort is the Möbius band, $M = \boldsymbol{R}\times\boldsymbol{R}/\boldsymbol{Z}$ where $\boldsymbol{Z}$ is generated by the map $f(x, y)=(x+1, -y)$. The lines y=constant project to a foliation of M by circles as in Figure 2. Note that the circles corresponding to $y=c$ for $c\neq 0$ must go around the band twice before closing.

The most common examples of bundles of this sort are flat vector bundles. In fact, any principal G-bundle with a flat connection (cf. **[KN]**) is a bundle with discrete structure group. The vanishing of curvature is exactly the condition that the horizontal planes be tangent to a foliation.

In a semilocal sense every foliation is a foliation of this sort. Specifically, the normal bundle to a leaf inherits a natural flat connection and corresponding discrete structure group. The resulting foliation of the normal bundle is the "first order part" of the foliation in a neighborhood of the leaf. In particular, the holonomy of this flat connection is the linear part of the "holonomy" of the foliation along the leaf (an important concept due originally to Ehresmann, cf. §8).

C. *Group actions.* Let G be a Lie group acting differentiably on a manifold M. If we assume the action is *locally free*, that is, for each $x \in M$ the isotropy subgroup $G_x=\{g \in G : g(x)=x\}$ is discrete, then the orbits of G form a foliation of M. When G is not compact these foliations can be quite complicated.

A simple case of this type arises when G is a subgroup of a Lie group $G'=M$, and the action is left multiplication. The leaves are then the left cosets of G in G'. If, for example, we let $G=\boldsymbol{R}$ be a noncompact 1-parameter subgroup of a torus (a line of irrational slope), then every leaf of the resulting foliation is dense.

Related to this discussion is the notion in differential topology of the *rank* of a manifold. This is defined as the largest n such that there exists a locally free action of $\boldsymbol{R}^n$ on the manifold. (Alternatively, for compact manifolds, it is the maximum number of pointwise independent, commuting vector fields.) The determination of this invariant generally depends on a deep study of foliations.

D. *Differential equations.* A foliation always appears as the family of solutions to some nonsingular system of differential equations. To study the foliation is to study the global behavior of the solutions. For instance a nonsingular system of ordinary differential equation, when reduced to first order, becomes a nonvanishing vector field. The local solutions (orbits of the local flow generated by the vector field) fit together to form a 1-dimensional foliation. The study of the global aspects of this foliation go back to Poincaré.

One can analogously consider ordinary differential equations in the complex case (where dependence on the variables is holomorphic). One obtains nonsingular holomorphic vector fields and corresponding foliations by complex curves. The first approach to this subject from the point of view of foliations was made by Painlevé who considered the important equation: $y'=R(x, y)$ where R is a rational function in y with coefficients holomorphic in x. An exposition of this work and later generalizations can be found in **[R3]**.

While foliations are themselves solutions to differential equations of a particular sort, they also occur in the intermediate stages of solving more complicated systems, where the leaves appear as characteristic manifolds. (See, for example, **[STG**, p. 135].) They also appear in the famous study, made by Anosov, of the general structure of certain systems of ordinary differential equations. (See **[ANO]**, **[AR]**.)

E. *Transversal mappings.* Within the general category of foliated manifolds there is a class of natural mappings.

DEFINITION 2. Let M be a manifold with a codimension-q, C^r foliation $\mathscr{F}$, and suppose $f:N\rightarrow M$ is a mapping of class C^s, $1\leqq s\leqq r$, of a manifold N into M. Then f is said to be *transverse* to $\mathscr{F}$ if for all $x\in N$, there exists a system of distinguished coordinates $(x^1, \cdots, x^m)$ at $f(x)$ on M such that the map $\varphi=(x^{m-q+1}\circ f, \cdots, x^m\circ f)$ is a submersion in a neighborhood of x.

The above condition is independent of the distinguished coordinates chosen at $y=f(x)$. In fact, if $\tau_y(\mathscr{F})$ denotes the vectors in $T_y(M)$ tangent to the foliation, and if $f_*:T_x(N)\rightarrow T_y(M)$ denotes the linear map on tangent vectors induced by f, then the condition of transversality at x is that:

$$T_y(M) = \tau_y(\mathscr{F}) + f_*T_x(N).$$

It follows immediately from the definitions that if $f:N\rightarrow M$ is transverse to a foliation $\mathscr{F}=\{\mathscr{L}_\alpha\}_{\alpha\in A}$ on M as above, then f induces a class C^s foliation $f^*\mathscr{F}$ on N where the leaves are defined as the components of $f^{-1}(\mathscr{L}_\alpha)$ for $\alpha\in A$. Note that $\operatorname{codim}(f^*\mathscr{F})=\operatorname{codim}(\mathscr{F})$.

In the special case that f is a submersion, we can consider f to be transverse to the trivial foliation of M by points. The induced foliation is the one described in part A above.

More generally, suppose $f:N\rightarrow M$ is a submersion and $\mathscr{F}$ is any foliation of M. Then f is transverse to $\mathscr{F}$ and $f^*\mathscr{F}$ is defined. Thus any codimension-q foliation of M can be lifted to codimension-q foliations of manifolds which fiber over M.

We shall say that a submanifold N of M is transverse to a foliation $\mathscr{F}$ if the inclusion map $i:N\subsetneq M$ is transverse to $\mathscr{F}$.

To simplify language in the subsequent discussion, we make the convention that the word "smooth" means "of class C^r" where r is an integer $\geqq 1$ which is fixed in context.

2. **Foliations of dimension one.** If a manifold M admits a foliation of dimension one, then the tangents to the leaves form a differentiable field of line elements on M. Conversely, every smooth line field on M is tangent to a one-dimensional foliation. To see this, observe that in a neighborhood of any point there is a smooth, nonvanishing vector field V which generates the line field. The integral curves of V (forgetting the parameter) give the foliation in this neighborhood. For future reference, we state this fact explicitly.

LEMMA 1. *The one-dimensional C^∞-foliations of a manifold M are in a natural one-to-one correspondence with the set of C^∞ line fields on M.*

COROLLARY 1. *Every open manifold ("open" means no component is compact) has a one-dimensional foliation.*

COROLLARY 2. *A compact manifold has a one-dimensional foliation if and only if its Euler characteristic is zero.*

Thus, the only compact surfaces with foliations are the torus and the Klein bottle. However, on these surfaces there is a rich variety of possibilities. There are the foliations of $T^2=\boldsymbol{R}^2/\boldsymbol{Z}^2$ coming from families of parallel lines in $\boldsymbol{R}^2$ (cf. §1, C). More generally, if $f: S^1 \to S^1$ is any orientation preserving diffeomorphism of the circle, we can consider the torus as a quotient $T^2=\boldsymbol{R}\times S^1/\boldsymbol{Z}$ where $\boldsymbol{Z}$ is generated by the diffeomorphism $(t, \theta)\to(t+1, f(\theta))$. The foliation $\{\boldsymbol{R}\times\{\theta\}\}_{\theta\in S^1}$ of $\boldsymbol{R}\times S^1$ projects to a foliation of T^2. (When f is a rotation, we obtain the linear foliation above.) We can then modify this foliation by introducing a Reeb component at each fixed point of f (cf. Figure 3 and §4 below). These constructions

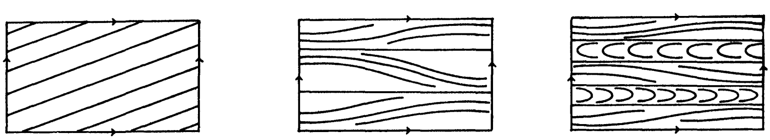

FIGURE 3

produce essentially all foliations of T^2 up to diffeomorphism. Thus, the study of these foliations is reduced to the study of $\mathrm{Diff}^r_+(S^1)$. It turns out that there is a radical difference here between the differentiability classes $r=1$ and $r\geqq 2$ [**DE**]. This difference is reflected throughout the study of foliations.

As we shall soon see, there is a major distinction between foliations of dimension 1 and those of higher dimension. It is basically the distinction between ordinary and partial differential equations. The one-dimensional case has inherently more structure and is more properly studied from the viewpoint of dynamical systems (cf. **[SM]**). Nonetheless, the deep theorems in this special case often lead to important general results. This has been particularly true of the Poincaré-Bendixson theorem and the Seifert conjecture as we shall see in §8.

3. **Higher dimensional foliations; integrability criteria.** Suppose that $\mathscr{F}=\{\mathscr{L}_\alpha\}_{\alpha\in A}$ is a foliation of dimension $p>1$ on an m-manifold M. Then associated to $\mathscr{F}$ is a smooth field of p-planes tangent to the leaves, which we denote $\tau(\mathscr{F})$. Consequently, in order that a manifold admit a foliation of dimension p, it must first admit a continuous field of p-planes (or, by duality, a field of $(m-p)$-planes). This is, of course, a nontrivial topological requirement. For example, S^5 admits no continuous 2-plane (or 3-plane) fields.

However, one might ask whether this is the only obstruction to finding foliations. If we are given a smooth field of r-planes τ on M, can we, in analogy with Lemma 1, find a foliation $\mathscr{F}$ such that $\tau=\tau(\mathscr{F})$? The answer in general is no. In order to solve the resulting system of partial differential equations, certain compatibility conditions (which result from the commutativity of second partial derivatives) must be satisfied. The condition can be stated as follows. Let $\mathscr{X}(\tau)$ denote the set of vector fields V on M such that $V_x\in\tau_x$ for all $x\in M$. If for all $V, W\in\mathscr{X}(\tau)$ we also have the Lie bracket $[V, W]\in\mathscr{X}(\tau)$, then τ is called *integrable*. This is equivalent to the condition that the ideal $\mathscr{I}(\tau)$ of exterior differential forms which vanish on τ is closed under exterior differentiation. One of the classical theorems of analysis is the following.

THEOREM 1. *τ is the field of tangent planes to a foliation if and only if it is integrable.*

This theorem is generally ascribed to Frobenius **[F]**, although it has been pointed out by Milnor **[M4]** (and Frobenius, himself) that it can be found in the earlier work of A. Clebsch and F. Deahna.

Observe that if $\mathscr{F}$ is a foliation of class C^r, then $\tau(\mathscr{F})$ is of class C^{r-1}. Unfortunately, if τ is an integrable plane field of class C^{r-1}, the associated foliation is, in general, only of class C^{r-1}. (When integrating one does not increase differentiability in the normal direction.)

It is easy to see that most plane fields are not integrable. So if we are given an r-plane field τ, the natural question (cf. Haefliger **[H2]**) is whether τ can be deformed to an integrable one. That is, does there exist a continuous family of r-plane fields τ_t, $0\leqq t\leqq 1$, such that $\tau_0=\tau$ and $\tau_1=\tau(\mathscr{F})$ for some foliation $\mathscr{F}$? The answer, as pointed out by Bott, is no. To see why we must consider the *normal bundle* $\nu(\mathscr{F})=\tau^\perp(\mathscr{F})$ of the foliation, which is defined as the bundle of cotangent vectors which vanish on τ.

(Introducing a riemannian metric, ν can be identified with the field of $(m-p)$-planes perpendicular to τ.)

THEOREM 2 (BOTT [**B1**]). *Let ν be the field of normal planes to a foliation of codimension-q on M, and denote by* $\mathrm{Pont}^*(\nu) \subset H^*(M; \boldsymbol{R})$ *the subring generated by the Pontryagin classes of ν. Then*

$$\mathrm{Pont}^k(\nu) = 0 \quad \textit{for } k > 2q. \tag{2.1}$$

Condition (2.1) depends only on the homotopy class of ν (in fact, only on its stable equivalence class as an abstract bundle), and must therefore hold for any bundle which can be deformed to the normal bundle of a foliation. The condition is nontrivial. As Bott shows, complex projective n-space $\boldsymbol{P}^n(\boldsymbol{C})$, for n odd, admits a plane field of codimension 2. However, an argument using (2.1) and the ring structure of $H^*(\boldsymbol{P}^n(\boldsymbol{C}); \boldsymbol{R})$ shows that no such plane field can be integrable.

The proof of Theorem 2 proceeds by constructing on ν a connection which is flat along the leaves. Condition (2.1) then follows directly from the Chern-Weil homomorphism.

A number of results similar to Theorem 2 have been proven. For example, Bott also has a version for the complex analytic case, with a resulting condition on Chern classes. Joel Pasternak [**P1**], [**P2**] has improved the vanishing criterion (2.1) to the range $k>q$ for foliations with certain metric properties. (Roughly, one assumes the leaves are locally a constant distance apart.) As a corollary, Pasternak concludes that *if an m-manifold M admits an almost-free action of a compact p-dimensional Lie group, then the real Pontryagin classes of M must vanish in dimensions greater then $m-p$.* This result is surprisingly false for integral Pontryagin classes.

Recently, Herbert Shulman [**SH**] has shown that there are secondary obstructions to integrability which are independent of Bott's primary ones. He proves that *if ν is the normal bundle to a foliation of codimension-q, then for all $\alpha, \beta, \gamma \in \mathrm{Pont}^*(\nu)$ (real classes!) with $\deg(\alpha \cdot \beta) > 2q$ and $\deg(\beta \cdot \gamma) > 2q$, the Massey triple product $\langle \alpha, \beta, \gamma \rangle = 0$.*

In view of these results one might wonder whether there exist reasonable conditions sufficient to guarantee that a plane field is homotopic to a foliation. This problem will be discussed, at least for open manifolds, in §7.

4. **Foliations of codimension-one.** It should be noted that none of the sundry versions of Theorem 2 apply to foliations of codimension-one. In fact, by generalizing the immersion theory of Hirsch and Smale, A. Phillips proved the following striking result.

THEOREM 3 (PHILLIPS [**PH1**], [**PH2**]). *On an open manifold every codimension-one plane field is homotopic to a smooth foliation.*

Therefore, in analogy with Corollary 1 above, every open manifold admits a smooth, codimension-one foliation. For compact manifolds this is not true since a compact manifold admits a continuous codimension-one plane field if and only if its Euler characteristic vanishes. (This rules out, for example, all even-dimensional spheres but says nothing about odd-dimensional manifolds.) The principal conjecture, made by Emery Thomas **[TS1]**, is the following analogue of Corollary 2 above.

CONJECTURE 1. *A compact manifold admits a smooth, codimension-one foliation if and only if its Euler characteristic vanishes.*

As we shall see there is now much more evidence to support this conjecture than existed when it was first made. It is not unreasonable therefore to make the following stronger conjecture (cf. Theorem 3).

CONJECTURE 2. *On a compact manifold every codimension-one plane field is homotopic to a smooth foliation.*

The remainder of this section will be devoted to these two questions.

In the above conjectures the foliations were asked to be differentiable but not necessarily analytic. This is for a good reason.

THEOREM 4 (A. HAEFLIGER **[H1]**). *A compact manifold with finite fundamental group has no real-analytic foliations of codimension-one.*

This was one of the earliest results on the question of existence of foliations. The proof, in outline, is as follows. Suppose M is simply-connected with an analytic, codimension-one foliation. Let $\gamma \subset M$ be a closed curve transversal to the foliation. (See Lemma 4 below.) Then γ is the boundary of a mapping $f: D^2 \to M$ which by general position arguments can be assumed to be "Morse regular" with respect to the foliation. That is, at any point $p \in D^2$ consider a system of distinguished local coordinates $(x^1, \cdots, x^n)$ where $x^n = \text{constant}$ defines the foliation. Then $x^n \circ f$ has nondegenerate (Morse-type) singular points. It can now be shown that the induced foliation (with singularities) on D^2 contradicts analyticity. In particular, the nonanalytic phenomenon of "one-sided holonomy," pictured in Figure 4, must occur.

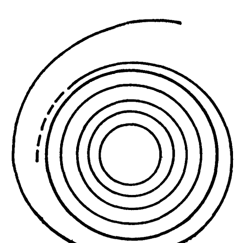

FIGURE 4

Theorem 4 says nothing about C^∞ foliations, and indeed in 1944 (long before Haefliger's paper) G. Reeb had constructed a codimension-one C^∞-foliation of S^3 as follows. Consider the C^∞-foliation of the (x, y)-plane given by the lines $x=c$ for $|c|\geqq 1$ together with the graphs of the functions $y=f(x)+c'$, $-1<x<1$ and $c'\in \boldsymbol{R}$, where f has the property that $\lim_{|x|\to 1} f^{(k)}(x)=\infty$ for all k.

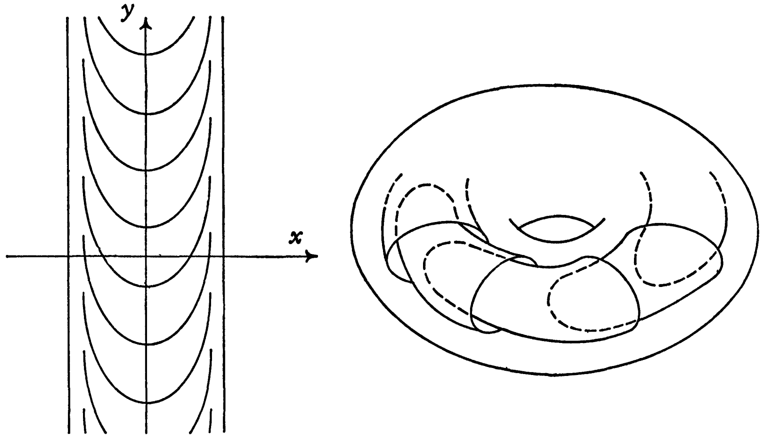

FIGURE 5

Consider now the foliation of the solid cylinder obtained by rotating the strip $\{(x, y)\in \boldsymbol{R}^2: -1\leqq x\leqq 1\}$ about the y-axis in 3-space. This foliation is invariant by vertical translations, and so we can obtain a foliation of the solid torus where each noncompact leaf has the form of a snake eternally eating its tail.

The 3-sphere can be decomposed as two solid tori joined along their common 2-torus boundary. Indeed, if one removes the solid torus of rotation from $\boldsymbol{R}^3=S^3\sim\{\infty\}$ (cf. Figure 5), what remains is homeomorphic to a solid torus minus an interior point. (Consider the vertical coordinate axis as the core circle.) Gluing together two copies of our foliated solid torus gives a *Reeb foliation* of the 3-sphere.

Note that gluing together the two solid tori by different diffeomorphisms of T^2 gives codimension-one foliations of all the 3-dimensional lens spaces. Actually much more is true. Combining this idea of Reeb with a paper written by J. Alexander in 1923 proves the following.

THEOREM 5 (ALEXANDER, LICKORISH, NOVIKOV, REEB, AND ZEISCHANG). *Every compact orientable 3-manifold has a smooth codimension-one foliation.*

Before entering a discussion of this theorem we shall make some useful general remarks. We begin by introducing the following important notion.

DEFINITION 2. A compact manifold is said to have an *Alexander decomposition* if there exists a differentiable map $F:M\to \mathbf{C}$ such that:

(i) The origin $0\in \mathbf{C}$ is a regular value, i.e., there is a neighborhood of $A=F^{-1}(0)$ in which F is a submersion.

(ii) The map $f=F/|F|:M-A\to S^1$ is a submersion.

The submanifold $A=F^{-1}(0)$ is called the *axis* of the decomposition. It is not difficult to see (cf. [M2]) that for $\varepsilon>0$ sufficiently small, the (tubular) neighborhood $T(A)=F^{-1}(\{z:|z|\leqq\varepsilon\})$ of the axis is diffeomorphic to $A\times D^2$. In particular A has a trivial normal bundle. Furthermore, the map $f:M-A\to S^1$ is a fiber bundle whose fiber $\mathscr{A}=f^{-1}(1)$ is called the *generator* of the decomposition. One can check that for each θ, the closure of $\mathscr{A}_\theta=f^{-1}(e^{i\theta})$ is a compact manifold with boundary $\partial\bar{\mathscr{A}}_\theta=A$.

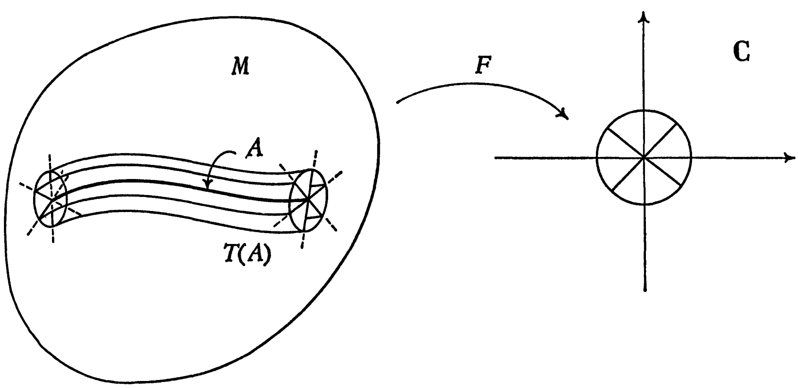

FIGURE 6

We can therefore think of Alexander decompositions as obtained by "spinning" the generator $\mathscr{A}$ about its boundary $\partial\mathscr{A}=A$ (the axis). In fact, all manifolds with Alexander decompositions can be constructed as follows. Let $\mathscr{A}$ be a compact manifold with boundary $\partial\mathscr{A}$ and let $d:\mathscr{A}\to\mathscr{A}$ be a diffeomorphism which is the identity in a neighborhood $N(\partial\mathscr{A})$ of $\partial\mathscr{A}$. The manifold M is then obtained from $\mathscr{A}\times[0,2\pi]$ by first identifying $(x,0)$ with $(d(x),2\pi)$ for $x\in\mathscr{A}$, and then for each $x\in\partial\mathscr{A}$ identifying all the points (x,θ), $0\leqq\theta\leqq 2\pi$. M has a natural smooth structure and an Alexander decomposition with function F defined as follows. Let $\varphi:\mathscr{A}\to \mathbf{R}^+\cup\{0\}$ be a smooth function such that $\varphi\equiv 1$ outside $N(\partial\mathscr{A})$ and $\varphi(x)=$ distance$(x,\partial\mathscr{A})$ near $\partial\mathscr{A}$ (where distance is defined with respect to some riemannian metric on $\mathscr{A}$). Then we set $F(x,\theta)=\varphi(x)e^{i\theta}$.

Two important examples of Alexander decompositions are the following.

EXAMPLE 4A. Let $M=S^{2n+1}=\{Z\in \boldsymbol{C}^{n+1}:|Z|=1\}$ and let $p(Z)$ be a polynomial such that $\nabla p=(\partial p/\partial Z_0,\cdots,\partial p/\partial Z_n)\neq 0$ for $Z\neq 0$. Then $F=p|S^{2n+1}$ gives an Alexander decomposition of S^{2n+1}. It is straightforward to see that 0 is a regular value of F. Therefore we need only check that $f=F/|F|:S^{2n+1}-A\to S^1$ is a submersion. When p is homogeneous of some degree $\nu>0$, this is easy. At any $Z\in S^{2n+1}-A$ consider the curve $t\mapsto e^{it}Z$. Then $f(e^{it}Z)=e^{i\nu t}f(Z)$ and so f maps the velocity vector of this curve at $t=0$ to a nonzero tangent vector to S^1 at $f(Z)$. Thus, f is a submersion. In the general case we must appeal to a theorem of Milnor **[M3]**.

EXAMPLE 4B. Let $\pi:E\to S^1$ be a smooth, orientable fiber bundle with connected fiber $\mathscr{A}_0$. Let $\sigma:S^1\to E$ be a smooth cross-section and denote by $T(\sigma)$ a tubular neighborhood of $\sigma(S^1)$ in E. Then $\mathscr{E}=E-T(\sigma)$ is a manifold with boundary $\partial\mathscr{E}\approx S^1\times S^{n-2}$ where $\pi|\{e^{i\theta}\}\times S^{n-2}\equiv e^{i\theta}$. We then construct a compact manifold M by gluing $\mathscr{E}$ to $D^2\times S^{n-2}$ along their common boundary by a diffeomorphism which sends $\{e^{i\theta}\}\times S^{n-2}$ to itself for each θ. An Alexander decomposition $F:M\to \boldsymbol{C}$ is given by

$$F(x)=\begin{cases}\pi(x), & \text{for } x\in\mathscr{E},\\ \varphi(z)z, & \text{for } x=(z,x')\in D^2\times S^{n-2},\end{cases}$$

where $\varphi:D^2\to\boldsymbol{R}^+$ is a smooth function which $\equiv 1$ in a neighborhood of zero and $=1/|z|$ near ∂D^2. In this case the generator $\mathscr{A}$ is diffeomorphic to the fiber $\mathscr{A}_0$ with a disk removed, and the axis A is diffeomorphic to S^{n-2}. In fact $A=\{0\}\times S^{n-2}\subset D^2\times S^{n-2}$.

The following is a sometimes useful observation.

LEMMA 2. *Let M_k, $k=0,1$, be compact m-manifolds with Alexander decompositions having axes A_k and generators $\mathscr{A}_k$ respectively. Then the connected sum $M_0\# M_1$ has an Alexander decomposition with axis $A_0\# A_1$ and generator $\mathscr{A}_0\natural\mathscr{A}_1$ (connected sum at the boundary).*

The proof is straightforward (cf. **[DL]**).

The relevance of Alexander decompositions to foliations is given in the next proposition. We say that a codimension-one C^r-foliation of a manifold M is *trivial at the boundary* if each component of the boundary is a leaf and if the foliation extends to a C^r-collaring of M by defining the leaves in the collar $\partial M\times[0,1]$ to be the components of $\partial M\times\{t\}$ for $0\leqq t\leqq 1$. (See **[W1]**, **[L]**.)

PROPOSITION 1. *Let M be a compact manifold with an Alexander decomposition F. If the manifold $F^{-1}(0)\times D^2$ has a smooth, codimension-one foliation which is trivial at the boundary, then M has a smooth, codimension-one foliation.*

PROOF. As above we set $A=F^{-1}(0)$ and choose $\varepsilon>0$ sufficiently small that:

(i) $T_r(A)=F^{-1}(D_r^2)\approx A\times D^2$, where $D_r^2=\{z:|z|\leqq r\}$, for $0<r\leqq 2\varepsilon$.

(ii) F is a submersion in $T_{2\varepsilon}(A)$.

We now consider a foliation $\mathscr{F}$ of $C-\{0\}$ defined as follows. Let V be a smooth vector field in $C-\{0\}$ such that $V=-\partial/\partial r$ outside $D_{2\varepsilon}^2$ and $V=\partial/\partial\theta$ inside D_ε^2, and let the foliation $\mathscr{F}$ correspond to the orbits of the flow. Observe now that the map $F|M-A$ is transverse to $\mathscr{F}$. In particular, F lifts $\mathscr{F}$ to a smooth codimension-one foliation of $M-T_\varepsilon(A)^0$ which is trivial at the boundary. By assumption we have a foliation of $T_\varepsilon(A)$ which

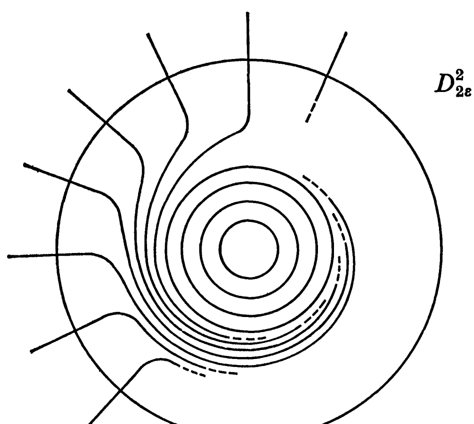

FIGURE 7

is trivial at the boundary. Consequently, we may join the two to obtain a foliation of M.

We are now in a position to discuss Theorem 5. The principal step in its proof is:

ALEXANDER'S THEOREM [A]. *Every compact orientable* 3-*manifold has an Alexander decomposition.*

The axis A of any such decomposition is a disjoint union of circles. Thus, $A\times D^2$ is a disjoint union of solid tori which, by Reeb's construction above, have codimension-one foliations which are trivial at the boundary. Applying Proposition 1 then proves Theorem 5.

The case of nonorientable 3-manifolds is somewhat more complicated and has been done by John Wood. (See [W1] for details.)

THEOREM 5′ (ALEXANDER, LICKORISH, ETC. AND WOOD). *Every compact* 3-*manifold has a smooth, codimension-one foliation.*

Curiously, the relevance of Alexander's Theorem to foliations has been noticed only recently. Proofs of Theorem 5 using surgery methods have been given by Lickorish [LI], Novikov, and Zeischang.

The first major use of Alexander decompositions was made when the geometry of isolated singularities of algebraic hypersurfaces was used to construct foliations of higher dimensional spheres.

THEOREM 6 (LAWSON [L]). *There exist smooth, codimension-one foliations of* S^{2^k+3} *for* $k=1, 2, 3, \cdots$.

PROOF. We first consider the Alexander decomposition F of S^5 given by setting $F=p|S^5$ where p is the polynomial

$$p(Z_0, Z_1, Z_2) = Z_0^3 + Z_1^3 + Z_2^3.$$

(See Example 4A above.) The axis $A=F^{-1}(0)$ is the inverse image under the Hopf map $\pi: S^5 \to P^2(C)$ of the nonsingular algebraic curve defined by p in the complex projective plane $P^2(C)$. By a classical formula of algebraic geometry (genus$=\frac{1}{2}(d-1)(d-2)$ where $d=$degree of p) this curve must be a torus. Projecting this torus onto one of its factors, we get a submersion $A \to S^1$ and, thus, a submersion $A \times D^2 \to S^1 \times D^2$. We may now lift the Reeb foliation of $S^1 \times D^2$ to a codimension-one foliation of $A \times D^2$ which is trivial at the boundary. By Proposition 1 we then have a foliation of S^5.

Note. It has been pointed out by D. Tischler and A. Verjovsky that this foliation of S^5 can be generalized as follows. Let $i: S^1 \times S^1 \to P^2(C)$ be an embedding with homology degree $m \neq 0$. (That is, $[i(S^1 \times S^1)] = m[P^1(C)]$ in $H_2(P^2(C); Z) \cong Z$.) Let ξ denote the restriction of the Hopf bundle to the complement of $i(S^1 \times S^1)$. It is straightforward to see that the Chern class of ξ is a torsion class of order m. Thus, ξ has a finite structure group, and the total space of ξ has a foliation given by a closed 1-form. The rest of the construction is as before. In the special case $m=1$, $\xi \approx (P^2(C) - i(S^1 \times S^1)) \times S^1$. When $m=3$ we get the foliation above.

To get the result for higher dimensional spheres, we establish an induction procedure. We begin by considering the Alexander decomposition of S^{2n+1} given by the complex polynomial

$$q_n(Z) = Z_0^2 + \cdots + Z_n^2.$$

The axis $A_n = S^{2n+1} \cap \{q_n = 0\}$ is diffeomorphic to the bundle of unit tangent vectors to the n-sphere. To see this write $Z \in S^{2n+1} \subset C^{n+1}$ as $Z = X + iY$ where $|X|^2 + |Y|^2 = 1$. Then, $p(Z) = |X|^2 - |Y|^2 + 2i\langle X, Y\rangle$, and so $A_n = \{(X, Y): |X|^2 = \frac{1}{2},\ |Y|^2 = \frac{1}{2}$ and $\langle Y, X\rangle = 0\}$. The diffeomorphism can now be explicitly constructed. However, all we need observe for our purposes is that the map $\pi: A_n \to S^n$ given by $\pi(X, Y) = X$ is a submersion. This is easy to check.

According to Proposition 1 the sphere S^{2n+1} has a foliation if there is a foliation of $A_n \times D^2$ (trivial at the boundary). Since there is a submersion $\pi\times 1: A_n\times D^2 \to S^n \times D^2$, it is sufficient to find a foliation of $S^n \times D^2$. However, the following was pointed out to me by Alberto Verjovsky.

LEMMA 3. *$S^n \times D^2$ has a smooth codimension-one foliation which is trivial at the boundary if and only if S^{n+2} does.*

To see this note that $S^{n+2} = S^n \times D^2 \cup D^{n+1} \times S^1$ where the two pieces are joined along their boundary. If there is a foliation of $S^n \times D^2$, we construct a Reeb foliation of $D^{n+1}\times S^1$ and, putting them together, get a foliation of S^{n+2}. Conversely, suppose S^{n+2} has a codimension-one foliation. Then by Lemma 4 below, there is a circle $S^1 \subset S^{n+2}$ embedded transversely to the foliation. There is then a tubular neighborhood $T(S^1)$ of S^1 in S^{n+2} and a diffeomorphism $d: T(S^1) \to D^{n+1} \times S^1$ which carries the foliation of $T(S^1)$ to the natural foliation of $D^{n+1}\times S^1$ by $D^{n+1} \times \{e^{i\theta}\}$ for $0 \leqq \theta \leqq 2\pi$. (See [W1] for details.) We now modify the foliation in the interior of $T(S^1)$ by introducing a *Reeb component*. That is, we construct a new foliation of $D^{n+1} \times S^1$ which agrees with the old one near the boundary and which contains a copy of $D^{n+1}\times S^1$ with the Reeb foliation in its interior. This new foliation is carried to $T(S^1)$ by d. Removing the Reeb

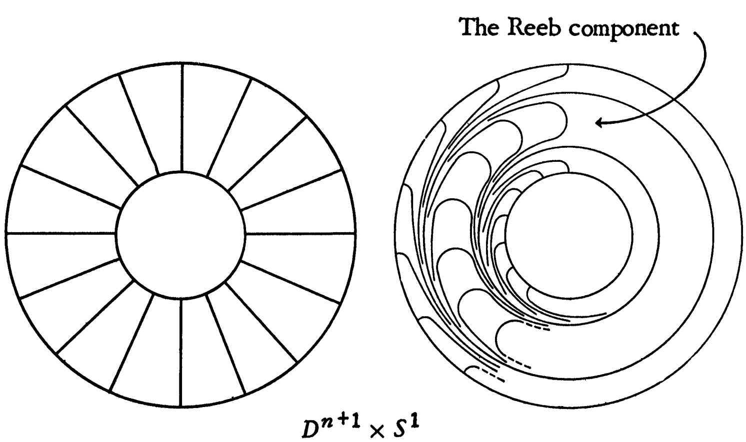

FIGURE 8

component gives us a foliation of $S^{n+2} - D^{n+1}\times S^1 \approx S^n \times D^2$ which is trivial at the boundary.

We now proceed inductively as follows. $\exists$ fol on $S^5 \Rightarrow \exists$ fol on $S^3\times D^2 \Rightarrow$ $\exists$ fol on $A_3 \times D^2 \Rightarrow \exists$ fol on $S^7 \Rightarrow \exists$ fol on $S^5 \times D^2 \Rightarrow \cdots$. In general, we see that *if there is a smooth codimension-one foliation of S^n, then there is a* *smooth codimension-one foliation of* $S^{2^k(n-3)+3}$, *for* $k=0, 1, 2, \cdots$. This completes the proof of Theorem 6.

We now turn our attention to the general case of odd-dimensional spheres. To handle this we must strengthen the induction argument. Let us suppose that there exist smooth, codimension-one foliations of S^{2k+1} for $2 \leqq k < n$. The above arguments show that *if n is odd*, there is a codimension-one foliation of S^{2n+1}. If n is even, the Alexander decomposition given by q_n fails because the axis A_n fibers over an even-dimensional sphere. Hence, the key to doing the general case is to find an Alexander decomposition of S^{4m+1} for $m \geqq 2$, with axis which fibers over an odd-dimensional sphere. This was achieved simultaneously and independently by A. Durfee [**D**] and I. Tamura [**T1**]. While their methods were quite different, the results were remarkably similar. They both constructed Alexander decompositions of S^{4m+1}, $m \geqq 2$, having as axis $S^{2m} \times S^{2m-1}$. Briefly, Durfee's argument is as follows. Consider the Alexander decompositions of S^{4m+1} coming from the polynomials

$$p_0(Z) = (Z_0 + Z_1^2)(Z_0^2 + Z_1^5) + \sum_{k=2}^{2m} Z_k^2,$$

$$p_1(Z) = Z_0^5 + Z_1^3 + \sum_{k=2}^{2m} Z_k^2.$$

By resolving the singularity and using some results from his thesis, Durfee shows that the axis $A_0 = S^{4m+1} \cap \{p_0 = 0\} \approx S^{2m} \times S^{2m-1} \# \Sigma^{4m+1}$ where Σ^{4m+1} is the Milnor exotic sphere. It is well known that the axis $A_1 = S^{4m+1} \cap \{p_1 = 0\} \approx -\Sigma^{4m+1}$ (if orientations are chosen properly). Therefore, taking connected sums of the Alexander decompositions (cf. Lemma 1) gives the result. It is interesting to note that while Tamura used methods purely from differential topology, he also needed to cancel a Milnor exotic sphere from the product in the same way.

We can now complete the induction begun above. Suppose there are foliations of S^{2k+1} for $2 \leqq k < n$. Then there are foliations of $S^{2k-1} \times D^2$ for $0 \leqq k < n$. There is an Alexander decomposition of S^{2n+1} with axis A admitting a submersion $A \to S^{2k-1}$ (where $2k-1 = n$ or $n-1$). Lifting the foliation of $S^{2k-1} \times D^2$ to $A \times D^2$ and applying Proposition 1 gives a foliation of S^{2n+1}. Since there is a foliation of S^5, we get

THEOREM 7 (DURFEE, TAMURA). *Every odd-dimensional sphere admits a smooth codimension-one foliation.*

It has been remarked by Milnor that if S^{2n+1} has a codimension-one foliation, so does every exotic $(2n+1)$-sphere. (See [**L**, Corollary 6].)

The natural question at this point is what can be said about other odd-dimensional manifolds. The first result in this direction was the following.

THEOREM 8 (N. A'CAMPO). *Every compact simply-connected 5-manifold has a smooth codimension-one foliation.*

The proof goes roughly as follows. In **[BA]** Barden constructs a sequence $\{M_k\}_{k=0}^{\infty}$ of compact simply-connected 5-manifolds, where $M_0=S^2\times S^3$, and shows that every other compact, simply-connected 5-manifold is a connected sum of a finite number of M_k's. A'Campo **[AC]** shows that each M_k for $k\geqq 1$ has an Alexander decomposition of type 4B above. Hence, any connected sum of these has an Alexander decomposition with axis S^3, and since there is a foliation of $S^3\times D^2$, this connected sum has a codimension-one foliation.

Suppose now that $M=S^2\times S^3\# M'$ where M' has a codimension-one foliation $\mathscr{F}'$. Then M is obtained from M' by removing a tubular neighborhood $\mathscr{T}$ of a smoothly embedded curve $\gamma: S^1\subset M'$, and sewing in a (foliated) copy of $S^3\times D^2$. If M' is simply-connected, we may assume γ to be transverse to $\mathscr{F}'$ and then modify the foliation in $M'-\mathscr{T}$ to be trivial at the boundary, as we did in proving Proposition 1. This produces a codimension-one foliation of M. Applying this procedure inductively handles the remaining cases.

We note that this result was obtained for simply-connected, spin 5-manifolds also by H. Shulman.

One is now led to consider $(n-1)$-connected $(2n+1)$-manifolds. Using the geometry of algebraic hypersurface singularities and classification theorems, Durfee and Lawson **[DL]** proved the existence of foliations on a class of these which bound parallelizable manifolds.

These results fairly well exhausted the theorems possible from known classification results, and some classification-free methods were needed. In particular it was natural to investigate the higher-dimensional versions of Alexander's Theorem. An extensive study of this was made by E. H. Winkelnkemper **[WI]**, and we mention the relevant aspects of his work here.

WINKELNKEMPER'S THEOREM. *Let M be a compact simply-connected manifold of dimension $n>5$. If $n\not\equiv 0 \bmod 4$, then M has an Alexander decomposition. If $n\equiv 0 \bmod 4$, M has an Alexander decomposition if and only if the signature of M is zero. Furthermore, the Alexander decompositions can always be chosen to have a simply-connected generator $\mathscr{A}$ where $H_i(\mathscr{A}; \mathbf{Z})=0$ for $i>[n/2]$ and where the natural map $H_i(\mathscr{A}; \mathbf{Z})\to H_i(M; \mathbf{Z})$ is an isomorphism for $i<[n/2]$.*

This is a beautiful and quite powerful result. It essentially gives an inductive method for constructing manifolds, and it should prove to be a basic tool in differential topology. The application of this result to foliations was made by Tamura who obtained independently some partial results of the above type **[T3]**.

Suppose $n=2k+1\geqq 7$. By Lemma 3 and Theorem 7, if M^n has an Alexander decomposition with axis S^{n-2}, then M^n has a codimension-one foliation. Unfortunately, we cannot make this requirement on the axis in general. (It cannot be done on $S^n\times S^2$ for $n>3$, for example.) However, when M^{2k+1} is $(k-1)$-connected, the generator $\mathscr{A}$ of the Alexander decomposition is a $2k$-disk with k-handles attached. By attacking these handles one at a time with clever surgery techniques, Tamura **[T4]** succeeded in changing the Alexander decomposition into one with S^{2k-1} as axis. This gave the following very pretty generalization of Theorems 5′ and 8.

THEOREM 9 (TAMURA). *Every compact $(k-1)$-connected, $(2k+1)$-manifold has a smooth codimension-one foliation.*

I have learned that, using Winkelnkemper's method of proof, M. Freedman **[FR]** gave an independent proof of this theorem at about the same time as Tamura.

We note incidentally, that these last methods apply only to the cases $2k+1>5$. For dimensions 3 and 5 we still need the old proofs.

Of course, this theorem immediately implies that there are foliations of a somewhat larger class of manifolds.

COROLLARY 3. *Any manifold which fibers over an $(n-1)$-connected, $(2n+1)$-manifold, for example, all the classical groups and their associated Stiefel manifolds, have smooth codimension-one foliations.*

The proof of Theorem 9 actually shows that every $(n-1)$-connected, $(2n+1)$-manifold is obtained from a bundle over a circle by performing a surgery on a cross-section. (See Example 4B.) Actually, a much larger class of manifolds (but not all) can be constructed this way, and every such manifold has a codimension-one foliation.

It is interesting to note at this point that by returning to polynomials in C^{n+1} we can construct large numbers of distinct foliations on $(n-1)$-connected $(2n+1)$-manifolds. Consider for example the Alexander decompositions on S^3 given by

$$p(Z_0, Z_1) = Z_0^r + Z_1^s.$$

When r and s are relatively prime, the axis is a torus knot of type (r, s), and the generator is a compact, orientable surface of genus $(r-1)(s-1)$ punctured at one point. By taking connected sums, we get an infinite family of interesting Alexander decompositions (and, therefore, foliations) of any compact, orientable 3-manifold.

The analogous remarks apply in higher dimensions where one considers the Brieskorn polynomials $p(z)=z_0^d+z_1^2+\cdots+z_n^2$. For appropriate d, the axis is a sphere S^{2n-1} knotted in S^{2n+1}. (See **[L]** for details.)

At the moment Theorem 9 and its corollaries represent the state of our knowledge concerning Conjecture 1. However, one remark is in order here. Note that in the Winkelnkemper Theorem a requirement for a $4k$-manifold to have an Alexander decomposition is that its signature be zero. This requirement is independent of the requirement that the Euler characteristic vanish. For example, $P^2(C)\#P^2(C)\#S^1\times S^3\#S^1\times S^3$ has signature 2 and Euler number 0. Thus we pose the following

Problem 1. Does there exist a $4k$-manifold of nonzero signature which admits a codimension-one foliation?

An example of such an animal was constructed in **[RI4]**, but an error has been found in the proof, so the question is still open.

We turn our attention now to Conjecture 2. Most of the work on this problem has been done by John Wood **[W1]**. To state his theorem we need the following definition. A codimension-one plane field τ on a manifold M is said to be *transversely orientable* if there is a (nowhere zero) vector field V on M transverse to τ, i.e., $V_p \notin \tau_p$ for all $p \in M$.

THEOREM 10 (WOOD). *Every transversely orientable* 2-*plane field on a compact* 3-*manifold is homotopic to a foliation.*

This theorem is impressive for two reasons. There are an infinite number of distinct homotopy classes of such 2-plane fields on a compact 3-manifold. Furthermore, as one can check locally, most 2-plane fields cannot be C^0-approximated by integrable ones. In fact, there are 2-plane fields which cannot lie within 90° of an integrable one. For example, let V be any nonvanishing, divergence-free vector on S^3 (e.g. the Hopf field $V(Z)=iZ$ for $Z\in S^3\subset C^2$). Then V is not transverse to any codimension-one foliation. In general, V cannot be transverse to any compact surface Σ embedded in S^3, for if φ_t is the flow generated by V and if $\mathscr{D}^+$ is the component of $S^3-\Sigma$ having V as interior normal at $\partial\mathscr{D}^+=\Sigma$, then $\varphi_t(\mathscr{D}^+)\subsetneqq\mathscr{D}^+$ for $t>0$. Since φ_t preserves volume this is impossible. However, by Novikov (see §8) every codimension-one foliation of S^3 has a compact leaf. It follows that the 2-plane field $\tau=V^\perp$ cannot lie uniformly within 90° of the plane field to a foliation.

In higher dimensions Wood has proved the following result **[W1]**.

THEOREM 11 (WOOD). *Let M be a compact manifold with a transversely orientable, codimension-one foliation $\mathscr{F}$. If it is possible to find a family of closed curves transverse to $\mathscr{F}$, which generate $H_1(M;\mathbf{Z})$, then every transversely orientable, codimension-one plane field is homotopic to a foliation.*

In particular this conclusion holds for all manifolds of the type $M=N\times S^1$.

Combining Theorems 7 and 11 gives the following.

COROLLARY. *Let M be a compact manifold which fibers over an odd-dimensional sphere. Then every transversely orientable, codimension-one plane field on M is homotopic to a foliation.*

There are also results for transversely unorientable plane fields. For details see **[W1]** and **[W2]**. However, Wood's general problem is still open.

Problem 2. Show that Conjecture 2 holds for any manifold which admits a codimension-one foliation.

Note. Since the writing of this article Paul Schweitzer has succeeded in removing the homology condition from Wood's Theorem 11, obtaining, however, only C^0 foliations. Thurston then modified Schweitzer's procedure to produce C^∞ foliations. Thus, Problem 2 is solved in the transversely orientable case.

Appendix to §4. We conclude this section with two items of separate interest.

It was mentioned above that the continuous, integrable plane fields on S^3 are not C^0-dense in the set of all plane fields. With our thoughts in this direction it is interesting to note that Rosenberg and Thurston **[RT]** have constructed a continuous integrable plane field on the 3-torus which cannot be approximated by an integrable plane field of class C^2. (More recently, examples of integrable C^1 fields not approximable by integrable C^2 fields have been found.)

We end by sketching the proof of the following useful fact (cf. **[H1]**).

LEMMA 4. *Let M be a compact manifold with a foliation $\mathscr{F}$ of codimension-one. Then there exists a smooth closed curve embedded in M transversely to $\mathscr{F}$.*

PROOF. By passing, if necessary, to a two-sheeted covering of M, choose a vector field V transverse to the foliation. Fix any point and consider the integral curve of V through it. Either it is closed or it eventually accumulates on itself. When the self-accumulation is sufficiently tight, the curve will return to the same distinguished coordinate neighborhood and can be easily closed while maintaining transversality.

5. **Notions of equivalence.** The collection of different foliations on a given manifold is, in general, tremendous, and in order to make a quantitative approach to the subject it is necessary to establish some broad notions of equivalence. We shall list some of the important ones here for future reference.

Let $\mathscr{F}_0$ and $\mathscr{F}_1$ be two codimension-q C^r-foliations on an m-manifold M.

A. $\mathscr{F}_0$ and $\mathscr{F}_1$ are said to be *C^s-conjugate* $(0\leqq s\leqq r)$ if there exists a diffeomorphism of M, of class C^s, which maps the leaves of $\mathscr{F}_0$ onto the leaves of $\mathscr{F}_1$. If $s=0$, $\mathscr{F}_0$ and $\mathscr{F}_1$ are *topologically conjugate*. If $s=r$ and the diffeomorphism is isotopic to the identity, the foliations are *completely equivalent*.

B. $\mathscr{F}_0$ and $\mathscr{F}_1$ are said to be *homotopic* if there is a continuous family of integrable plane fields τ_t, $0 \leqq t \leqq 1$, such that $\tau_0=\tau(\mathscr{F}_0)$ and $\tau_1=\tau(\mathscr{F}_1)$. (We assume that each $\mathscr{F}_t$ gives a foliation of class C^r.)

Problem 3. Do there exist nonhomotopic foliations with homotopic plane fields?

C. $\mathscr{F}_0$ and $\mathscr{F}_1$ are called *integrably homotopic* if there exists a codimension-q, C^r foliation $\mathscr{F}$ of the product $M \times [0, 1]$ which is transverse to the slice $M \times \{t\}$ for each $t \in [0, 1]$, and which induces the foliation $\mathscr{F}_0$ on $M \times \{0\}$ and $\mathscr{F}_1$ on $M \times \{1\}$ (by intersection of these slices with the leaves of $\mathscr{F}$). Note that if $\mathscr{F}_0$ and $\mathscr{F}_1$ are integrably homotopic, they are homotopic. The converse is not true. In fact, if M is compact, then $\mathscr{F}_0$ and $\mathscr{F}_1$ are integrably homotopic if and only if they are completely equivalent. (See **[M4]**.) Thus, any two foliations of the torus by "parallel lines" are homotopic but they are integrably homotopic iff they have the same slope.

If we drop the requirement that $\mathscr{F}$ be transverse to $M \times \{t\}$ for $0<t<1$ in the above definition, the foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ are then called *concordant*. A more general notion than this is the following.

D. Let M_0 and M_1 be two closed, oriented m-manifolds with codimension-q, C^r-foliations. Then these foliated manifolds are said to be *foliated cobordant* if there is a compact, oriented $(m+1)$-manifold $\mathscr{M}$ with boundary $\partial\mathscr{M}=M_1-M_0$ and with a codimension-q, C^r-foliation $\mathscr{F}$ transverse to the boundary and inducing the given foliation there. The resulting foliated cobordism classes form a group under disjoint union, which we denote $\mathscr{F}\Omega^r_{m,q}$.

In the case $q=1$ the group operation has an interpretation similar to that of connected sum. Let M_0 and M_1 be compact, oriented manifolds with codimension-1 foliations and choose embedded closed curves $\gamma_0 \subset M_0$, $\gamma_1 \subset M_1$ transverse to these foliations. (See Lemma 4 above.) Let $\mathscr{N}_0$ and $\mathscr{N}_1$ be tubular neighborhoods of γ_0 and γ_1 sufficiently thin that there exist diffeomorphisms $f_k:\mathscr{N}_k \to S^1 \times D^{m-1}$, $k=0, 1$, mapping γ_k to $S^1 \times 0$ and sending the foliation of $\mathscr{N}_k$ onto the foliation of $S^1 \times D^{m-1}$ by the disks $\{t\} \times D^{m-1}$. We now glue $M_0-\gamma_0$ to $M_1-\gamma_1$ along the sets $\mathscr{N}_k-\gamma_k$ by the diffeomorphism $d=f_1^{-1} \circ h \circ f_0:\mathscr{N}_0-\gamma_0 \to \mathscr{N}_1-\gamma_1$ where $h:S^1 \times D^{m-1} \to S^1 \times D^{m-1}$ is given by $h(t, x)=(t, (1-|x|)x/|x|)$ and $f_1^{-1} \circ f_0$ is assumed to be orientation reversing. The resulting manifold, which we denote $M_0 \bigstar M_1$, is again compact, oriented and has a codimension-1 foliation.

Of course, the manifold $M_0 \bigstar M_1$ depends on many choices (of curves, diffeomorphisms, etc.), however, *the resulting foliated cobordism class is independent of these choices*. In fact, $M_0 \bigstar M_1$ is foliated cobordant to the disjoint union; so for any $\bigstar$ operation, $[M_0 \bigstar M_1] = [M_0] + [M_1]$, in $\mathscr{F}\Omega^r_{m,1}$.

As a first approximation to $\mathscr{F}\Omega$ one can study the bordism groups of q-plane fields. For some nice results in this direction see Koschorke **[KO]**.

It has been proven by W. Thurston that $\mathscr{F}\Omega^r_{2,1} \cong \Gamma^r/[\Gamma^r, \Gamma^r]$ where $\Gamma^r = \mathrm{Diff}^r_+(S^1)$ is the group of orientation preserving C^r diffeomorphisms of the circle. Hence, by a result of M. Herman **[HER]**, $\mathscr{F}\Omega^\infty_{2,1} = 0$.

In contrast it has also been proven by Thurston **[Th2]**, using the characteristic class of Godbillon-Vey (see §5), that for $r \geqq 2$ there is a surjective homomorphism

$$\mathscr{F}\Omega^r_{3,1} \to \boldsymbol{R}. \tag{5.1}$$

Hence, even with this highly indiscriminant notion of equivalence there exist uncountably many distinct codimension-1 foliations of three-manifolds. In fact, Thurston constructs these noncobordant foliations all on S^3.

Problem 4. Determine the kernel of the homomorphism (5.1). In particular, decide whether it is zero.

It has been shown by Rosenberg and Thurston **[RT]** that the natural map

$$\mathscr{F}\Omega^\infty_{3,1} \to \mathscr{F}\Omega^0_{3,1} \tag{5.2}$$

has a nonzero kernel. In light of the results of Mather on $B\Gamma^0_1$ (see next section), it is plausible to conjecture that $\mathscr{F}\Omega^0_{3,1} = 0$.

Problem 5. Determine whether $\mathscr{F}\Omega^0_{3,1} = 0$, or at least whether the map (5.2) is surjective.

We finally mention the general problem.

Problem 6. Determine the structure of $\mathscr{F}\Omega^r_{m,1}$ for $m > 3$.

6. **The general theory.** One very effective technique of modern mathematics is that of reformulating a given problem in the category of topological spaces and continuous maps where one has available the machinery of algebraic topology. This approach has enjoyed a certain amount of success in the study of foliations, and has led to constructions which should be of central importance for many questions in the study of analysis on manifolds. The work is due to Haefliger, and the primary reference for details omitted in the following discussion is his article **[H4]**.

To make a homotopy-theoretic approach to foliations it is necessary to generalize (or soften) the definition. Observe that any codimension-q, C^r foliation (for $r > 0$) on a manifold M can be presented in the following way. There is an open covering $\{\mathscr{O}_i\}_{i \in I}$ of M and a family of C^r maps $\{f_i\}_{i \in I}$, where $f_i : \mathscr{O}_i \to \boldsymbol{R}^q$ is a submersion, with the following property. For each $i, j \in I$ and each $x \in \mathscr{O}_i \cap \mathscr{O}_j$ there is a C^r diffeomorphism γ^x_{ji} from a neighborhood of $f_i(x)$ to a neighborhood of $f_j(x)$ such that

$$f_j = \gamma_{ji}^x f_i \tag{6.1}$$

in a neighborhood of x. Furthermore, for any $x \in \mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k$ we have

$$\gamma_{ki}^x = \gamma_{kj}^x \circ \gamma_{ji}^x \tag{6.2}$$

in a neighborhood of $f_i(x)$.

Our "cocycle" conditions (6.1) and (6.2) guarantee that foliations defined locally by the submersions f_i piece together to give a global foliation of M. If we begin with a foliation presented as in Definition 1, we obtain functions f_i as above by projecting the local distinguished coordinate maps onto the last q coordinates.

Note that if we set

$$g_{ij}^x = d\gamma_{ij}^x \tag{6.3}$$

for all $i, j \in I$, we obtain the transition functions for the normal bundle to the foliation (cf. Steenrod [ST]).

Note furthermore that if each f_i above is a local diffeomorphism (i.e., the foliation has dimension 0) then the foliated structure defined is just the differentiable structure of M, and the bundle given by (6.3) is the tangent bundle of M (thought of as the normal bundle to the "point" foliation).

We now broaden the definition of a foliation by allowing the local submersions f_i to be arbitrary continuous maps. In this way we can define foliations on general topological spaces. Before stating this explicitly we make one observation (following Haefliger) for the sake of elegance. For each $x \in \mathcal{O}_i$, γ_{ii}^x is the identity map on a neighborhood of $f_i(x)$. Hence, if we know the map $x \to [\gamma_{ii}^x]$ (=germ of the identity map at $f_i(x)$), we certainly know $f_i(x)$; and we may replace $f_i(x)$ by γ_{ii}^x. Doing this reduces (6.1) and (6.2) to the single cocycle condition (6.2).

Denote by Γ_q^r the set of germs of local C^r diffeomorphisms of $\boldsymbol{R}^q$ (homeomorphisms, if $r=0$). For $\gamma \in \Gamma_q^r$ let $\sigma(\gamma) \in \boldsymbol{R}^q$ denote the source of γ, and $\tau(\gamma)$ its target. Whenever, $\sigma(\gamma_2)=\tau(\gamma_1)$, the composition $\gamma_2 \circ \gamma_1$ is defined. There is a natural topology on Γ_q^r such that the inverse map, $\gamma \mapsto \gamma^{-1}$, and the composition map, where it is defined, are continuous. (The topology is the usual germ, or "sheaf-like", topology.) Identifying a point $x \in \boldsymbol{R}^q$ with the germ of the identity map at x gives a topological embedding of $\boldsymbol{R}^q$ into Γ_q^r.

DEFINITION 3. Let X be a topological space. A codimension-q, C^r *Haefliger cocycle* over an open covering $\mathcal{U}=\{\mathcal{O}_i\}_{i\in I}$ of X is an assignment to each pair $i, j \in I$ of a continuous map $\gamma_{ij}: \mathcal{O}_i \cap \mathcal{O}_j \to \Gamma_q^r$ such that for all $i, j, k \in I$

$$\gamma_{ki}(x) = \gamma_{kj}(x) \circ \gamma_{ji}(x)$$

for $x \in \mathcal{O}_i \cap \mathcal{O}_j \cap \mathcal{O}_k$.

Two such cocycles over open coverings $\mathscr{U}$ and $\mathscr{U}'$ are said to be *equivalent* if they extend to a cocycle on the disjoint union $\mathscr{U}\coprod\mathscr{U}'$. (We allow $\mathscr{O}_i=\mathscr{O}_j$ for $i\neq j$ in the definition above.) An equivalence class of cocycles is called a codimension-q, C^r *Haefliger structure* on X.

Note that if $\mathscr{H}$ is a Haefliger structure on X and f: $Y\to X$ is a continuous map, then f induces a natural Haefliger structure $f^*\mathscr{H}$ on Y.

Clearly there are many uninteresting Haefliger structures on any given space, so we introduce an equivalence relation which is sensitive to certain important properties. Two Haefliger structures $\mathscr{H}_0$ and $\mathscr{H}_1$ on X are said to be *concordant* if there is a Haefliger structure $\mathscr{H}$ on $X\times[0, 1]$ such that $\mathscr{H}_k=i_k^*\mathscr{H}$ where $i_k: X\to X\times[0, 1]$ by $i_k(x)=(x, k)$ for $k=0, 1$.

Suppose that $\mathscr{H}$ is a codimension-q, C^r ($r>0$) Haefliger structure on X, represented by the cocycle $\{\gamma_{ij}\}$ over $\mathscr{U}=\{\mathscr{O}_i\}_{i\in I}$. Then we can associate to $\mathscr{H}$ a q-dimensional vector bundle $\nu(\mathscr{H})$ over X given by the local transition functions $g_{ij}(x)=d\gamma_{ij}(x)$ in a neighborhood of $x\in\mathscr{O}_i\cap\mathscr{O}_j$. $\nu(\mathscr{H})$ is called the *normal bundle* of $\mathscr{H}$. (For $r=0$, one gets normal microbundles.) One can easily check that concordant structures have the same normal bundle.

Before proceeding let us examine some examples of codimension-1, C^∞ Haefliger structures on a manifold M.

EXAMPLE 6A. Any codimension-1, C^∞ foliation of M.

EXAMPLE 6B. Any continuous function $f: M\to \boldsymbol{R}$. ($\mathscr{U}=\{M\}$ and $\gamma=i\circ f$ where $i:\boldsymbol{R}\to\Gamma_1^\infty$ is the natural embedding.) We may approximate f by a smooth Morse function $f': M\to\boldsymbol{R}$ so that the resulting structure is concordant to the one given by f.

EXAMPLE 6C. Consider a foliation with singularities given by functions $f_i: \mathscr{O}_i\to\boldsymbol{R}$, as in the beginning of the section, except that each f_i is allowed isolated nondegenerate critical points. This gives a Haefliger structure in the obvious way.

We remark that every codimension-1, C^∞ Haefliger structure on M is concordant to one of type 6C.

Note that the differentiable structure on M^m is itself a codimension-m, C^∞ Haefliger structure on M.

The notion of Haefliger structures fits into a beautiful, unified theory. Each Γ_q^r is an example of a topological groupoid. A *groupoid* is a category in which the morphisms are invertible (and, generally, the objects in the category are identified with the units). A *topological groupoid* is a groupoid with a topology in which the maps $\gamma\mapsto\gamma^{-1}$ and $(\gamma_1, \gamma_2)\mapsto\gamma_1\circ\gamma_2$ (on the set where it is defined) are continuous.

The following are two important classes of these objects.

EXAMPLE 6D. Every topological group is a topological groupoid. (Here the category has only one object.)

EXAMPLE 6E. Let $\hat{\Gamma}$ be a pseudogroup of local C^r diffeomorphisms of

$\boldsymbol{R}^q$, for example: the pseudogroup $\hat{\Gamma}_q(\boldsymbol{C})$ of local biholomorphisms in $\boldsymbol{C}^q \cong \boldsymbol{R}^{2q}$, the pseudogroup $\hat{\Gamma}_q^r(\sigma)$ preserving a fixed symplectic form σ on $\boldsymbol{R}^q$, etc. (See [**KN**, Chapter I].) Associated to $\hat{\Gamma}$ is a topological groupoid $\Gamma \subset \Gamma_q^r$, namely, the germs of elements in $\hat{\Gamma}$.

By replacing Γ_q^r with a general topological groupoid Γ in Definition 3 we obtain the notion of a *Γ-structure on X*.

If Γ is a topological group, a Γ-structure on X is (the equivalence class of) a principal Γ-bundle over X.

Suppose Γ comes from $\hat{\Gamma}$ as in Example 6E. Then every $\hat{\Gamma}$-structure is, in particular, a Γ-structure. For example, every complex structure on a manifold M^{2m} is a $\Gamma_m(\boldsymbol{C})$-structure. However, a Γ-structure, in this case, is best thought of as a generalized $\hat{\Gamma}$-foliation (a complex analytic foliation, a symplectic foliation, a foliated foliation or "multifoliation", etc.).

Note that if Γ is defined as in Example 6E, then there is associated to Γ a subgroup $G \subset GL_q(\boldsymbol{R})$, defined by taking the differentials of the elements in Γ which fix $0 \in \boldsymbol{R}^q$. To any Γ-cocycle $\{\gamma_{ij}\}$ we can associate a G-cocycle $\{g_{ij}\}$ by setting $g_{ij}(x) = d\gamma_{ij}(x)$ where each g_{ij} can be considered as acting at 0 by parallel translation in $\boldsymbol{R}^q$. This process associates to the Γ-structure a G-structure, that is, a principal G-bundle which is the normal bundle to the $\hat{\Gamma}$-foliation with structure group reduced to G.

Having introduced and motivated Γ-structures, we are in a position to state some results. Using categorical arguments (or, more directly, by using a generalization of Milnor's join construction for topological groups, cf. [**BL**]), Haefliger proves the following [**H4**].

PROPOSITION 2. *Associated to any topological groupoid Γ there is a topological space $B\Gamma$, equipped with a Γ-structure $\mathscr{H}_\Gamma$, such that:*

(i) *To each Γ-structure $\mathscr{H}$ on a paracompact X there is a continuous map $f: X \to B\Gamma$ such that $\mathscr{H} = f^*\mathscr{H}_\Gamma$.*

(ii) *Two Γ-structures $\mathscr{H}_0$ and $\mathscr{H}_1$ on a paracompact X are concordant if and only if the associated maps f_0 and f_1 are homotopic.*

$B\Gamma$ is called the *classifying space* for Γ, and a map $f: X \to B\Gamma$ inducing a Γ-structure $\mathscr{H}$ on X is called the *classifying map* for $\mathscr{H}$. The proposition states simply that concordance classes of Γ-structures on X correspond bijectively to $[X, B\Gamma]$, the homotopy classes of maps $X \to B\Gamma$.

For topological groups this theorem is classical.

Let us return to Haefliger structures. The space $B\Gamma_q^r$ has a Γ_q^r-structure and thus for $r > 0$ a normal bundle. Let

$$\nu: B\Gamma_q^r \to BGL_q$$

denote the map classifying the normal bundle. (This map can be obtained also by noting that taking the differential gives a continuous homomorphism $\Gamma_q^r \to GL_q$. ν is then the induced map on classifying spaces.) Let $\mathscr{H}$ be a Haefliger structure on a paracompact X with normal bundle $\nu(\mathscr{H})$. If f is the classifying map for $\mathscr{H}$ and $\nu(f)$ the classifying map for $\nu(\mathscr{H})$, then the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\nu(f)"'] & B\Gamma_q^r \arrow[d, "\nu"] \\ & BGL_q \end{tikzcd} \tag{6.4}$$

We can now formulate in a homotopy-theoretic setting the general question discussed in previous sections, namely:

Q. *When is a given plane field homotopic to a foliation?*

We replace this question with the following.

HQ. *When can a given map $n: X \to BGL_q$ be lifted to a map $f: X \to B\Gamma_q^r$ such that $n = \nu \circ f$?*

We shall see in the next section that for open manifolds an answer to HQ gives an answer to Q. Moreover, the question HQ can be dealt with by the methods of homotopy theory. To do this it is necessary to study the *homotopy-theoretic fiber* $F\Gamma_q^r$ of the map ν. (The space $F\Gamma_q^r$ can be thought of as the classifying space for Haefliger structures with a framing or trivialization of the normal bundle.) The obstructions to lifting the map n will be cohomology classes with (twisted) coefficients in the homotopy groups of $F\Gamma_q^r$. Any knowledge of the homotopy-type of these basic spaces $B\Gamma_q^r$ and $F\Gamma_q^r$ is of enormous interest in geometry. The first results of this type are the following.

THEOREM 12 (HAEFLIGER [H3]). *For* $1 \leqq r \leqq \infty$,

$$\pi_j(F\Gamma_q^r) = 0 \tag{6.5}$$

for $0 \leqq j \leqq q$. *In the real analytic case* ($r = \omega$), (6.5) *holds for* $0 \leqq j \leqq q-1$.

COROLLARY 4. *If a paracompact X has the homotopy type of a k-dimensional complex, where $k \leqq q+1$, then the concordance classes of Γ_q^r structures ($1 \leqq r \leqq \infty$) correspond bijectively to the equivalence classes of bundles over X.*

We recall that Bott has found necessary topological conditions for a bundle to be equivalent to an integrable one (Theorem 2). One might expect this condition to carry over to the homotopy-theoretic case, i.e. from question Q to HQ. Indeed, it does.

THEOREM H2. *The map $\nu^*: H^k(BGL_q; \mathbf{R}) \to H^k(B\Gamma_q; \mathbf{R})$ is zero for $k > 2q$.*

The fascinating thing is that over finite fields quite the reverse is true.

THEOREM 13 (BOTT AND HEITSCH [**BHI**], [**HI1**]). *For* $q \geqq 2$, *the map* $\nu^*: H^*(BGL_q; Z_p) \to H^*(B\Gamma_q; Z_p)$ *is injective for any prime p. Hence,* ν^* *is also injective on integral cohomology.*

Combining Theorems H2 and 13 gives

COROLLARY 5 (BOTT-HEITSCH). *For* $q \geqq 2$, *the group* $H_{4k-1}(B\Gamma_q; \boldsymbol{Z})$ *is not finitely generated for* $2k > q$. *Thus some homotopy group* $\pi_j(F\Gamma_q)$ *is not finitely generated.*

Thus, as one might have expected, the spaces $B\Gamma_q^r$ are tremendous.

The theorems above do not apply to the codimension-one case. There is much known, however, about this case, and the results vary drastically with the differentiability. The first theorem reflects the rigidity of analytic foliations that we witnessed in Theorem 4.

THEOREM 14 (HAEFLIGER [**H4**]). *The space* $F\Gamma_1^\omega$ *has the homotopy-type of an Eilenberg-Mac Lane space* $K(\pi, 1)$ *where* π *is an uncountable, torsion-free, perfect group (i.e.,* $\pi_1 F\Gamma_1^\omega = \pi$, $\pi_j F\Gamma_1^\omega = 0, j > 1$).

Note that $F\Gamma_1^\omega$ is the classifying space for transversely oriented, codimension-one analytic Haefliger structures. Therefore, the concordance classes of such structures on X correspond bijectively to the homomorphisms $\pi_1(X) \to \pi$. In particular, if $\pi_1(X)$ is a torsion group, then every Γ_1^ω-structure is analytically concordant to the trivial one.

Some of the deepest work in the codimension-one case is due to Mather [**MA1**], [**MA2**] who proved by means of explicit geometric constructions that the homology of $F\Gamma_1^r$, $0 \leqq r \leqq \infty$, is isomorphic to the homology of a double complex obtained by iterating the bar construction on the group $G^r = \mathrm{Diff}_K^r(\boldsymbol{R})$ of C^r-diffeomorphisms of $\boldsymbol{R}$ with compact support. From this he obtains a spectral sequence whose E^1 term can be expressed in terms of $H_*(G^r)$ and which converges to $H_*(F\Gamma_1^r; \mathbf{Z})$. For all the corollaries of this see the announcement [**MA1**]. We mention here two important consequences.

THEOREM 15 (MATHER). *The space* $F\Gamma_1^0$ *is contractible. Hence, any two codimension-one topological Haefliger structures are topologically concordant.*

THEOREM 16 (MATHER). *For* $1 \leqq r \leqq \infty$, *there is a natural isomorphism*

$$G^r/[G^r, G^r] \cong H_2(F\Gamma_1^r; \boldsymbol{Z}).$$

In particular,

$$H_2(F\Gamma_1^\infty; \boldsymbol{Z})\,(\approx \pi_2(F\Gamma_1^\infty)) = 0.$$

Together with Theorem 12, Theorem 16 shows that $F\Gamma_1^\infty$ is 2-connected.

Utilizing the new cohomology invariant of Godbillon-Vey, Thurston proved the following.

THEOREM 17 (THURSTON **[Th2]**). *For $r \geqq 2$, there is a surjective map*

$$\pi_3(F\Gamma_1^r) \to \boldsymbol{R}.$$

This contrasts dramatically with Theorems 14 and 15, and shows that even the spaces $B\Gamma_1^r$ are very large.

Note. This result has since been generalized by Thurston to give surjective homomorphisms $H_{2q+1}(F\Gamma_q^r; \boldsymbol{Z}) \to \boldsymbol{R}$ for all $q \geqq 1$, $r \geqq 2$.

Theorem 17 still leaves open the following.

Problem 7. Determine the homotopy groups $\pi_j(F\Gamma_1^r)$ for $j=4, 5, 6, \cdots$ and $2 \leqq r \leqq \infty$. To begin, determine whether the above foliations of S^{2n+1} can be extended as Haefliger structures over the ball D^{2n+2}.

Very recently Thurston has succeeded in extending Mather's constructions to foliations of general codimension. He has developed a theory which relates the classifying spaces for certain classical pseudogroups (Example 6E above) to diffeomorphism groups of manifolds. To state his results I will need some definitions. Let G be a topological group, and let G_δ denote the same group with the discrete topology. The identity homomorphism $G_\delta \to G$ is continuous and has a homotopy theoretic fiber denoted G^-. G^- in isfact a group. To see this we recall that

$$G^- = \{(g, \gamma) \in G_\delta \times G^I : \gamma(0) = g \text{ and } \gamma(1) = e\}$$

where $I=[0, 1]$ and $e \in G$ is the identity. The multiplication on G^- is the one inherited from $G_\delta \times G^I$. The classifying space for G^- arises in the long exact sequence of fibrations $\cdots \to G^- \to G_\delta \to G \to BG^- \to BG_\delta\ (=K(G_\delta, 1)) \to BG \to \cdots$.

We can now state one of the principal results. For a topological space X, let $\Omega^q X$ denote the q-fold loops on X.

THEOREM 18 (THURSTON **[TH4]**). *For $0 \leqq r \leqq \infty$ there exists a continuous map $B(\mathrm{Diff}_K^r(\boldsymbol{R}^q)^-) \to \Omega^q(F\Gamma_q^r)$ which induces an isomorphism on integral homology.*

This leads to generalizations of Theorems 15 and 16 above. Using the result of Mather **[MA3]** that $H_*(B\,\mathrm{Diff}_K^0(\boldsymbol{R}^q))=0$, Thurston obtains

COROLLARY 6. *For all q, the space $F\Gamma_q^0$ is contractible.*

By factoring the above map through $\mathrm{B}(\mathrm{Diff}^r(T^q)^-)$ and applying Herman's theorem **[HER]**, he obtains

COROLLARY 7. *For all $q \geqq 1$, $\pi_{q+1}(F\Gamma_q^\infty)=0$.*

This corollary shows that the dimension in Bott's vanishing result (Theorem H2 above) is best possible. Let K^4 be a 4-dimensional finite simplicial complex and $f: K^4 \to B0_2$ a map such that $f^*p_1 \neq 0$ in $H^4(K^4; \mathbf{Z})$ where p_1 denotes the first universal Pontryagin class. Since $\pi_3(F\Gamma_2^\infty)=0$, we know from obstruction theory that there exists a lifting of f, $\tilde{f}: K^4 \to B\Gamma_2^\infty$ such that $\nu \circ \tilde{f} = f$. Hence, $\nu^* p_1 \neq 0$. By considering products of K^4, we get that $\nu^*(p_1^k) \neq 0$ in $H^{4k}(B\Gamma_{2k}^\infty; \boldsymbol{R})$.

The results above leave some interesting questions.

Problem 8. Is the fiber $F\Gamma_q^\infty$ $2q$-connected?

Problem 9. What is the homotopy type of $F\Gamma_q^\omega$ for $q>1$?

It is interesting to note that the homology of $F\Gamma_q^r$ is also related to that of the diffeomorphism group of a manifold.

THEOREM 19 (THURSTON). *Let M^q be a compact differentiable manifold of dimension q, and let $i: \boldsymbol{R}^q \to M^q$ be an embedding given, say, by a local coordinate chart. Then for any r, $0 \leqq r \leqq \infty$, the map*

$$i_*: H_k(B(\mathrm{Diff}_K^r(\boldsymbol{R}^q)^-); \boldsymbol{Z}) \to H_k(B(\mathrm{Diff}^r(M^q)^-); \boldsymbol{Z})$$

is an isomorphism up to and including the first dimension k where the groups are nonzero.

Combining this result with a theorem of Epstein **[EP1]** gives

COROLLARY 8. *For any compact manifold M^q, the connected component of the identity in* $\mathrm{Diff}^\infty(M^q)$ *is a simple group.*

One of the first things that occurs to a topologist when classifying spaces are mentioned is characteristic classes. Given any topological groupoid Γ, the *ring of universal characteristic classes for* Γ is defined simply as $H^*(B\Gamma; \mathbf{Z})$. Note that for a given $\alpha \in H^*(B\Gamma; \mathbf{Z})$, every Γ-structure $\mathscr{H}$ on a paracompact X has associated a well-defined characteristic class $\alpha(\mathscr{H}) = f^*(\alpha)$ where $f: X \to B\Gamma$ classifies $\mathscr{H}$. From the above theorems we know there are rather a lot of universal characteristic classes for Haefliger structures.

Theorem 10 states that up to dimension q the Γ_q^r-characteristic classes are exactly the Pontryagin classes of the normal bundle. (Thus, for differentiable structures on manifolds we are getting nothing new.) However, in higher dimensions, new classes occur. Of course, the mere existence of such classes has little value without some method of computing them for a given foliation. The first formulas of this sort were given by Godbillon and Vey **[GV]** and by Bott **[B3]**. The Godbillon-Vey construction goes as follows. Let $\mathscr{F}$ be a codimension-q foliation on a manifold M and suppose $\nu(\mathscr{F})$ is orientable. Then $\mathscr{F}$ is defined by a global decomposable q-form Ω. (Let $\{(\mathscr{O}_i, \chi_i)\}_{i \in I}$ be a locally finite cover of distinguished coordinate charts on M with a smooth partition of unity $\{\rho_i\}$. Then, set

$$\Omega = \sum_{i \in I} \rho_i \, dx_i^{m-q+1} \wedge \cdots \wedge dx_i^m.)$$

Since, Ω is integrable $d\Omega = \theta \wedge \Omega$ where θ is a one-form on M. The $(2q+1)$-form $\gamma = \theta \wedge d\theta^q$ is closed, and its de Rham cohomology class $[\gamma](\mathscr{F}) \in H^{2q+1}(M; \boldsymbol{R})$ is independent of all the choices involved in defining it. It depends only on $\mathscr{F}$. This construction can be generalized to arbitrary Γ_q^r-structures for $r \geqq 2$, as a mixed de Rham-Čech cohomology class, and thus gives an element in $H^{2q+1}(B\Gamma_q^r; \boldsymbol{R})$. If M is the unit tangent sphere bundle to a compact manifold of constant negative curvature, and if $\mathscr{F}$ is the codimension-q Anosov foliation arising from the geodesic flow (flow $\times$ stable foliation; see [**AR**]), then $[\gamma](\mathscr{F}) \neq 0$ in $H^{2q+1}(M; \boldsymbol{R})$. This was first noticed by Roussarie and Thurston. It follows that none of these classes is trivial.

Note that the Godbillon-Vey class is, in fact, a cobordism invariant of codimension-q foliations of compact $(2q+1)$-manifolds (cf. §5).

The Godbillon-Vey construction gives us one computable characteristic class for Γ_q^r. What we would like optimally is a generalization of the Chern-Weil construction for GL_q. (See [**KN**, vol. 2].) That is, we would like an abstract graded differential algebra with the property that for any codimension-q foliation $\mathscr{F}$ on a manifold M, there is a g.d.a. homomorphism into the de Rham algebra on M, defined in terms of $\mathscr{F}$, such that the induced map on cohomology factors through a universal map into $H^*(B\Gamma_q^r; \boldsymbol{R})$. This was already accomplished by Godbillon and Vey for $q=1$. The algebra they discovered was the Gelfand-Fuchs Lie coalgebra of formal vector fields in one variable. This construction has been generalized to arbitrary codimension by Bott and Haefliger as follows. Consider the graded differential algebras (over $\boldsymbol{R}$):

$$WO_q = \Lambda(u_1, u_3, \cdots, u_{2[q/2]-1}) \otimes P_q(c_1, \cdots, c_q)$$

with $du_i = c_i$ for odd i and $dc_i = 0$ for all i; and

$$W_q = \Lambda(u_1, u_2, \cdots, u_q) \otimes P_q(c_1, \cdots, c_q)$$

with $du_i = c_i$ and $dc_i = 0$ for $i = 1, \cdots, q$, where: $\deg u_i = 2i-1$; $\deg c_i = 2i$; Λ denotes exterior algebra; and P_q denotes the polynomial algebra in the c_i's modulo elements of total degree $>2q$. The cohomology of W_q is the Gelfand-Fuchs cohomology of the Lie algebra of formal vector fields in q variables. We note that the ring structure at the cohomology level is trivial, that is, all cup products are zero.

THEOREM 20 (BOTT-HAEFLIGER-GODBILLON-VEY [**BH**], [**GH**]). *There are homomorphisms*

$$\varphi: H^*(WO_q) \to H^*(B\Gamma_q^r; \boldsymbol{R}), \qquad \bar{\varphi}: H^*(W_q) \to H^*(F\Gamma_q^r; \boldsymbol{R})$$

for $r \geqq 2$ with the following property. If $\mathscr{F}$ is a codimension-q, C^r foliation of a manifold M, there is a g.d.a. homomorphism

$$\varphi_{\mathscr{F}}: WO_q \to \Lambda^*(M)$$

into the de Rham algebra on M, defined in terms of the differential geometry of $\mathscr{F}$ and unique up to chain homotopy, such that on cohomology we have $\varphi_{\mathscr{F}} = f^* \circ \varphi$ where $f: M \to B\Gamma_q^r$ classifies $\mathscr{F}$. If the normal bundle of $\mathscr{F}$ is trivial, there is a homomorphism

$$\bar{\varphi}_{\mathscr{F}}: W_q \to \Lambda^*(M)$$

with analogous properties.

For the explicit geometric construction of $\varphi_{\mathscr{F}}$ and $\bar{\varphi}_{\mathscr{F}}$ in terms of connections on $\nu(\mathscr{F})$, see Conlon's notes from Bott's lectures [**B2**].

The diagram

$$\begin{tikzcd} H^*(WO_q) \arrow[r, "\varphi"] \arrow[d, "p"'] & H^*(B\Gamma_q^r; \boldsymbol{R}) \arrow[d, "i^*"] \\ H^*(W_q) \arrow[r, "\bar{\varphi}"] & H^*(F\Gamma_q^r; \boldsymbol{R}) \end{tikzcd}$$

where p is the natural map and i^* is induced by inclusion, is commutative. If a foliation has trivial normal bundle, $\nu(\mathscr{F})$, the theorem says that $\varphi_{\mathscr{F}}$ factors through $H^*(W_q)$ (on the cohomology level). Note that if $\pi: P(\mathscr{F}) \to M$ is the principal bundle associated to $\nu(\mathscr{F})$, then $\pi^*\mathscr{F}$ on $P(\mathscr{F})$ always has a trivial normal bundle. Hence, there is always a diagram:

$$\begin{tikzcd} H^*(WO_q) \arrow[r, "\varphi_{\mathscr{F}}"] \arrow[d, "p"'] & H^*(M; \boldsymbol{R}) \arrow[d, "\pi^*"] \\ H^*(W_q) \arrow[r, "\bar{\varphi}_{\pi^*\mathscr{F}}"] & H^*(P(\mathscr{F}); \boldsymbol{R}) \end{tikzcd}$$

which factors through the diagram above.

The appearance of Gelfand-Fuchs cohomology has a direct analogy in the cohomology of topological groups. Bott and Haefliger have shown that the notion of continuous (or differential) cohomology for Lie groups has a natural generalization to the Lie groupoids coming from pseudogroups in $\boldsymbol{R}^q$ (cf. Example 6E), and that in analogy with the work of Van Est, this continuous cohomology is just the Gelfand-Fuchs cohomology. (See [**H6**] for an excellent exposition.) Thus, $H^*(W_q)$ is naturally the continuous cohomology of Γ_q^r, and $H^*(WO_q)$ is the continuous cohomology of (Γ_q^r, O_q) (i.e., the cohomology of continuous, O_q-invariant cochains). A fundamental consequence of this result is that any characteristic class of foliations which is constructed locally in the algebra of differential forms will be, universally, in the image of φ (or $\tilde{\varphi}$). However, $H^*(B\Gamma_q^r; \boldsymbol{R})$ in general contains much more than the image of φ. (See Theorem 17.) Some of the remaining, "discontinuous" cohomology is related to classes for bundles with discrete structure groups.

An alternative approach to the cohomology of $B\Gamma_q^r$ has been taken by J. Simons, who has avoided the business of passing to the normal bundle by using circle coefficients. Continuing his work with S. S. Chern **[CS]**, he has developed a theory which associates to a principal bundle with connection a family of characteristic homomorphisms from the integral cycles on a manifold to S^1. For the Bott connection on a normal bundle to a foliation, the Simons characters of degree $>2q$ define cohomology classes independent of the choice of Bott connection. His result can be described as follows. From the Bockstein sequence for the coefficient homomorphism $0\rightarrow\boldsymbol{Z}\rightarrow\boldsymbol{R}\rightarrow S^1\rightarrow 0$, we get the commutative diagram:

$$\begin{array}{ccccc} H^{2k-1}(BGL_q; S^1) & \xrightarrow{\delta} & H^{2k}(BGL_q; \boldsymbol{Z}) & \xrightarrow{i} & H^{2k}(BGL_q; \boldsymbol{R}) \\ \downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle 0} \\ H^{2k-1}(B\Gamma_q^r; S^1) & \xrightarrow{\delta} & H^{2k}(B\Gamma_q^r; \boldsymbol{Z}) & \xrightarrow{i} & H^{2k}(B\Gamma_q^r; \boldsymbol{R}). \end{array} \tag{6.6}$$

For $k>q$, the right vertical map is zero by Theorem H2, and the middle map s is an injection by Theorem 13. By exactness, $s(H^{2k}(BGL_q; \boldsymbol{Z}))\subset \delta(H^{2k-1}(B\Gamma_q; S^1))$. Simons defines an extension K_q^{2k} of $H^{2k}(BGL_q; \boldsymbol{Z})$ and a map S such that the diagram

$$\begin{array}{ccc} K_q^{2k} & \longrightarrow & H^{2k}(BGL_q; \boldsymbol{Z}) \\ {\scriptstyle S}\downarrow & & {\scriptstyle s}\downarrow \\ H^{2k-1}(B\Gamma_q^r; S^1) & \xrightarrow{\delta} & H^{2k}(B\Gamma_q^r; \boldsymbol{Z}) \end{array}$$

commutes. The extension is defined as follows. Let $I^{2k}(GL_q)$ denote the homogeneous polynomials of degree k on the Lie algebra $\mathfrak{gl}_q$ of GL_q, which are invariant by the adjoint representation. (Recall that $I^*(GL_q)\cong \boldsymbol{R}[c_1, \cdots, c_q]$ where $\det(\lambda I-A)=\lambda^q-c_1(A)\lambda^{q-1}+\cdots+(-1)^q c_q(A)$ for $A \in \mathfrak{gl}_q$.) There is a natural map

$$j: I^{2k}(GL_q) \rightarrow H^{2k}(BGL_q; \boldsymbol{R})$$

given by the Chern-Weil homomorphism. We define the *Simons ring* K_q^{2k} to be the kernel of the homomorphism:

$$H^{2k}(BGL_q; \boldsymbol{Z}) \times I^{2k}(GL_q) \xrightarrow{i-j} H^{2k}(BGL_q; \boldsymbol{R})$$

where i is given in (6.6). Note that if $p \in I^{2k}(GL_q)$ involves some c_m for m odd, then $(0, p) \in K_q^{2k}$.

THEOREM 21 (SIMONS [SI]). *For $k>q$ and $r\geqq 2$ there is a homomorphism*

$$S: K_q^{2k} \to H^{2k-1}(B\Gamma_q^r; S^1)$$

with the following property. If $\mathcal{F}$ is a codimension-q, C^r foliation of a manifold M, there is a homomorphism

$$S_{\mathcal{F}}: K_q^{2k} \to H^{2k-1}(M; S^1)$$

defined in terms of the geometry of $\mathcal{F}$, such that $S_{\mathcal{F}}=f^* \circ S$ where $f: M \to B\Gamma_q^r$ classifies $\mathcal{F}$.

The Godbillon-Vey class arises in both theorems above. It is given as the cohomology class of $u_1 \otimes c_1^q \in WO_q$ in Theorem 20. Reduced mod $\boldsymbol{Z}$, it is the class $S(0, c_1^{q+1})$ in Theorem 21.

It should be pointed out that, potentially, the homomorphisms φ and S carry some essentially distinct classes. That is, if both φ and S are injective, then neither of the subgroups image(S) or image($\rho \circ \varphi$), where ρ is reduction mod $\boldsymbol{Z}$, is contained in the other.

The extent to which these homomorphisms are nontrivial is largely an open question. Some things are known. For example, one can easily check that $H^*(WO_1)$ has only two nonzero elements, the usual one in dimension zero and the Godbillon-Vey class in dimension three. Hence, $\varphi: H^*(WO_1) \to H^*(B\Gamma_1^r; \boldsymbol{R})$ for $2 \leqq r \leqq \infty$ is injective. Bott and Haefliger have certain partial results in this direction for $q \geqq 2$. (For example, $u_1 c_1^2$ and $u_1 c_2$ are linearly independent in $H^5(B\Gamma_2^r; \boldsymbol{R})$.) Since the map s in (6.6) is injective and the projection $K_q^{2k} \to H^{2k}(B\Gamma_q^r; \boldsymbol{Z})$ is surjective, the map S is nonzero for even k. Furthermore, by computation of the Godbillon-Vey class, $S(0, c_1^{q+1})$ is not zero for all q. However, the following question remains open.

Problem 10. Are the maps φ, $\tilde{\varphi}$ and S defined in Theorems 20 and 21 injective?

ADDED IN PROOF. It has been brought to my attention that some of these results on $H^*(B\Gamma_q)$ are also contained in the work of Kamber and Tondeur [**KT**].

It is interesting to note that most of the classes of dimension $>2q+1$ discussed in these theorems depend only on the homotopy class of the foliation in the sense of §5B. (See [**HI2**].) For classes in dimension $2q+1$ this is not true.

Before leaving this subject we remark that, in general, whenever Γ comes from a pseudogroup acting on $\boldsymbol{R}^q$, as in Example 6E, the structure of $F\Gamma$ is of great interest. A number of people are currently studying these spaces. Pasternak [**P2**] has results on $FR\Gamma_q^r$, the classifying space for foliations with bundle-like metrics. Landweber [**LA**] has shown that $F\Gamma_q(\boldsymbol{C})$ is $(q-1)$-connected, thereby obtaining results on the question of when an almost complex structure is homotopic to an integrable one. Bott ([**B3**] and [**B4**]) has results on $H^*(B\Gamma_q(\boldsymbol{C}); \boldsymbol{R})$ and $\pi_{2q+1}(B\Gamma_q(\boldsymbol{C}))$. Thurston [**TH4**] has theorems similar to Theorems 18 and 19 above for volume-preserving diffeomorphisms and for diffeomorphisms preserving a symplectic form. The interested reader should also see [**H4**] for a discussion of further results on these spaces.

7. **Results on open manifolds.** There is a tremendous difference between the theory of foliations on open manifolds and that on closed manifolds. In the open case it is difficult to obtain qualitative results. One can easily appreciate this by observing that any two codimension-q foliations of $\boldsymbol{R}^n$ are integrably homotopic. To prove this it suffices to show that any $\mathscr{F}$ on $\boldsymbol{R}^n$ is integrably homotopic to the foliation of a distinguished neighborhood $U \subset \boldsymbol{R}^n$ where $U \approx \boldsymbol{R}^n$. Let $\boldsymbol{R}^n \times \boldsymbol{R}$ have the product foliation, and choose an embedding $\Phi: \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n \times \boldsymbol{R}$ such that $\Phi(x, t) = (\varphi_t(x), t)$ where $\varphi_0 =$ identity and $\varphi_1: \boldsymbol{R}^n \to U$ is a homeomorphism. Then $\boldsymbol{R}^n \times \boldsymbol{R}$, with the foliation $\Phi^*\mathscr{F}$, is the desired integrable homotopy. This construction illustrates a powerful technique which is not available on compact manifolds, namely, that of "pushing the difficulties out to infinity". This technique makes it possible to obtain beautiful quantitative results.

The general theory which applies to quantitative questions on open manifolds had its inception with the immersion theory of Smale and Hirsch and has undergone a long development, culminating recently in the work of Gromov. (See [**PH1**], [**PH4**], [**PH5**], [**G**], [**H5**].) The principle is: "Questions in analysis involving open conditions on open manifolds are always solvable". The general procedure of proof is to decompose the manifold as a handlebody; prove the result first for the disk, and then proceed to attach handles and argue inductively.

We shall discuss how this theorem together with the results in §6 can be used to classify foliations on open manifolds. To begin it is necessary to present a foliation by an "open condition" as follows. Let N be a manifold with a smooth, codimension-q foliation $\mathscr{F}$. For any manifold M we define the space $\mathrm{Trans}(M, N_{\mathscr{F}})$ of smooth maps $f: M \to N$ which are transverse to $\mathscr{F}$. The space is given the usual topology for smooth maps. Of course, for $f \in \mathrm{Trans}(M, N_{\mathscr{F}})$, $f^*\mathscr{F}$ is a codimension-q foliation of M, and $\mathrm{Trans}(M, N_{\mathscr{F}})$ is open in the space of all maps from M to N.

We now consider the associated space $\mathrm{Epi}(T(M), \nu(\mathscr{F}))$ of continuous bundle maps from the tangent bundle $T(M)$ of M to the normal bundle $\nu(\mathscr{F}) \cong T(N)/\tau(\mathscr{F})$ of the foliation, which is an epimorphism on each fiber. This space is given the compact open topology. There is a natural continuous map

$$D: \mathrm{Trans}(M, N_{\mathscr{F}}) \to \mathrm{Epi}(T(M), \nu(\mathscr{F})) \tag{7.1}$$

given by $D(f)=p \circ df$ where $df: T(M) \to T(N)$ is the differential of f and $p: T(N) \to T(N)/\tau(\mathscr{F}) \cong \nu(\mathscr{F})$ is projection.

Note that the space Epi is considerably larger than Trans. If M is parallelizable, for example, there always exists a map in $\mathrm{Epi}(T(M), \nu(\mathscr{F}))$, where $\mathscr{F}$ is the foliation of $\boldsymbol{R}^q$ by points ($q \leqq \dim M$). However, if M is compact, a corresponding transverse map, i.e., a submersion $M \to \boldsymbol{R}^q$, does not exist. The startling fact is that for open manifolds it does exist.

THEOREM 22 (GROMOV-PHILLIPS **[G]**, **[PH4]**). *For any open manifold M the map* (7.1) *is a weak homotopy equivalence, that is, it induces isomorphisms on all homotopy and homology groups.*

In particular, D establishes a one-to-one correspondence between the connected components (i.e., π_0) of the two spaces. Thus, we conclude the following.

COROLLARY 9. *If M is an open manifold, then every codimension-q plane field $\tau=\ker(\beta)$ for $\beta \in \mathrm{Epi}(T(M), \nu(\mathscr{F}))$ is homotopic to a foliation.*

It should be noted that Theorem 22 contains the Smale-Hirsch immersion theorem and the Phillips submersion theorem. For the immersion theorem, let M be the normal bundle.

In light of Corollary 9, our problem now is: given τ on M, find N, $\mathscr{F}$ and β so that $\tau=\ker \beta$. Following an idea of Milnor **[M4]**, we can do this canonically by defining N to be the total space of the vector bundle $\nu= T(M)/\tau$. We say that a q-dimensional vector bundle $\pi: V \to M$ is of *foliated type* if there exists a smooth codimension-q foliation of V whose leaves are everywhere transverse to the fibers. (Thus for $\mathscr{L} \in \mathscr{F}$, $\pi|\mathscr{L}: \mathscr{L} \to M$ is a local diffeomorphism.)

COROLLARY 10. *Let M be an open manifold. Then a continuous codimension-q plane field τ on M is homotopic to a smooth foliation if and only if the vector bundle $\nu=T(M)/\tau$ is of foliated type.*

PROOF. If $\tau=\tau(\mathscr{F})$, choose a riemannian metric on M and identify ν with $\tau^{\perp}$. The exponential map $e: \nu \to M$ is transverse on a neighborhood U of the zero-section. Shrink ν into U and pull back the foliation $e^*\mathscr{F}$ to all of ν.

If ν is of foliated type with a foliation $\mathscr{F}$, consider the epimorphic bundle map $T(M) \to T(\nu)/\tau(\mathscr{F})$ obtained as the composition.

$$T(M) \xrightarrow{p} \nu \xrightarrow{z} T(\nu) \xrightarrow{p} T(\nu)/\tau(\mathscr{F})$$

where the p's are natural projections and z is the canonical embedding of ν along the zero-section. Applying Corollary 9 concludes the proof.

We are now in a position to describe how one passes from general

Haefliger structures on M to honest foliations. The key observation is the following.

LEMMA 4. *A vector bundle ν over a manifold M is the normal bundle to a Haefliger structure on M if and only if it is of foliated type.*

PROOF. If ν is of foliated type, consider the Haefliger structure $\mathscr{H}$ on M induced from the foliation on ν by the zero section $M \rightarrow \nu$. Clearly, $\nu = \nu(\mathscr{H})$.

Conversely, suppose ν is the normal bundle of a Haefliger structure $\mathscr{H}$ on M defined by a cocycle γ_{ij} over a cover $\mathscr{U} = \{O_i\}_{i \in I}$. We define a new manifold $\mathscr{N}(M)$, a normal thickening of M, as follows. Assume $\mathscr{U}$ is locally finite. Then for $i \in I$, define U_i to be a small neighborhood of the graph $\Gamma_i = \{x, f_i(x)\} \subset O_i \times \boldsymbol{R}^q$. We identify a neighborhood of $(x, f_i(x)) \in \Gamma_i \subset U_i$ with a neighborhood of $(x, f_j(x)) \in \Gamma_j \subset U_j$ by the diffeomorphism $(x, y) \rightarrow (x, \gamma_{ji} \cdot y)$. The foliations of the U_i's obtained by projection on $\boldsymbol{R}^q$ fit together to give a foliation $\mathscr{F}$ of $\mathscr{N}(M)$. Choosing one of the obvious diffeomorphisms $\nu \rightarrow \mathscr{N}(M)$ to lift $\mathscr{F}$ shows that ν is of foliated type.

Combining Corollary 10 with Lemma 4 shows that: a codimension-q plane field τ on M is homotopic to a foliation iff $\nu = T(M)/\tau$ is the normal bundle of a Haefliger structure on M, that is, iff the map $\varphi_\nu : M \rightarrow BGL_q$, classifying ν, lifts to a map $\varphi_{\mathscr{F}} : M \rightarrow B\Gamma_q^r$. It is, furthermore, possible to show that two foliations with the same normal bundle determine homotopic liftings if and only if they are integrably homotopic.

The general classification theorem can now be stated. For $p+q=m$ let $B\rho : BGL_p \times BGL_q \rightarrow BGL_m$ be the map induced by the standard homomorphism $\rho : GL_p \times GL_q \rightarrow GL_m$.

THEOREM 23 (HAEFLIGER [H3]). *Let M be an open manifold of dimension m. Then the integrable homotopy classes of codimension-q, C^r foliations are in one-to-one correspondence with homotopy classes of liftings $\varphi_{\mathscr{F}} \times \varphi_\tau$ of the map φ_T in the diagram*:

$$\begin{tikzcd}
 & B\Gamma_q \times BGL_{m-q} \arrow[d, "\nu \times \mathrm{Id}"] \\
M \arrow[ur, "\varphi_{\mathscr{F}} \times \varphi_\tau"] \arrow[r, "\varphi_\nu \times \varphi_\tau"] \arrow[dr, "\varphi_T"'] & BGL_q \times BGL_{m-q} \arrow[d, "B\rho"] \\
 & BGL_m
\end{tikzcd}$$

where φ_T classifies the tangent bundle of M.

Note that lifting φ_T over $B\rho$ to $\varphi_\nu \times \varphi_\tau$ corresponds to the topological problem of finding of codimension-q plane field on M. The second lifting, of φ_ν over ν to $\varphi_{\mathscr{F}}$, is the problem of integrating the given (homotopy class of the) plane field. Of course, obstructions to this second lifting will, in general, exist. However, in time they should become reasonably computable. The beauty of this theorem is that the general machinery developed to handle the homotopy question HQ in §6 can now be brought to bear on the analytical question Q for open manifolds. For example, in view of Theorem 12 and Corollary 7 we have the following.

COROLLARY 11. *Let M be an open manifold having the homotopy-type of a k-dimensional* CW *complex. Then if $k \leqq q+2$, every codimension-q plane field on M is homotopic to a foliation which is uniquely determined up to integrable homotopy.*

8. **Results on closed manifolds.** In studying foliations on open manifolds, one finds wonderful classification theorems of a general type, but practically no qualitative results. The situation for compact manifolds is exactly the reverse. There are few general existence theorems. However, there is a vast literature concerned with deep special results. I shall mention here some of the interesting theorems and open questions. The reader should also consult **[H2]**, **[R1]**, **[R3]**, **[RO3]** and **[TS1]**.

The existence of foliations of codimension-one has been discussed in §4. For foliations of higher codimension, very little is known. One exception is the sphere S^7. By Theorem 6 there is a codimension-one foliation. Alberto Verjovsky **[V2]** has constructed a foliation of codimension 3 and Jose Arraut **[AR]** has constructed one of codimension 5. The classical Hopf fiberings $S^7 \to S^4$ and $S^7 \to \boldsymbol{P}^3(\boldsymbol{C})$ have codimensions 4 and 6 respectively. It remains only to *find a foliation of codimension*-2. The existence of this foliation would answer for S^7 the following general question of Reeb.

Problem 11. Does every sphere that admits a codimension-q plane field admit a codimension-q foliation?

ADDED IN PROOF. Very recently Thurston has established a startling and beautiful quantitative theory for foliations on compact manifolds in codimension $\geqq 2$ (cf. *The theory of foliations of codimension greater than one*, Comment. Math. Helv. (to appear)). The fundamental result is the following.

THEOREM. *Let τ be a plane field of codimension $q \geqq 2$ on a compact manifold M. Then τ is homotopic to a foliation of class C^r $0 \leqq r \leqq \infty$, on M if and only if $\tau^\perp$ is the normal bundle to a Haefliger structure on M (i.e., if and only if the classifying map $n: M \to BGL_q$ for $\tau^\perp$ can be lifted to $B\Gamma_q^r$, cf.* §6). *Furthermore if τ is already integrable in a neighborhood of some compact set $K \subset M$, then the homotopy can be chosen to be constant on K.*

There are several immediate consequences of this theorem. From Corollary 7 we have:

(1) *Every 2-plane field on a manifold of dimension $\geqq 4$ is homotopic to a*

foliation. (In fact, Thurston has extended his methods to make this true also for 3-manifolds, thereby eliminating the condition of transverse orientability in Wood's Theorem.)

From Corollary 6 we have:

(2) *Every plane field of codimension* $\geqq 2$ *on a manifold is homotopic to a* C^0 *foliation* (which is actually Lipschitz with C^∞ leaves).

(3) *The span of any q-frame field* ($q \geqq 2$) *on a manifold is homotopic to the normal bundle of a foliation.*

In particular, this result answers Reeb's question for S^7 and gives the following partial answer for general spheres.

(4) *Every plane field on* S^n *of codimension* $\geqq n/2$ *is homotopic to a foliation.*

While on this subject we shall mention a closely related question. Following Kodaira and Spencer **[KS]** we can define a *multifoliation* of a manifold M to be a (finite) collection of foliations $\mathscr{F}_1, \cdots, \mathscr{F}_r$ on M such that all intersections are "transverse", that is, at any point of M, $\operatorname{codim}(\tau\mathscr{F}_{i_1} \cap \cdots \cap \tau\mathscr{F}_{i_s}) = \operatorname{codim}(\tau\mathscr{F}_{i_1}) + \cdots + \operatorname{codim}(\tau\mathscr{F}_{i_s})$ for all subsets $\{i_1, \cdots, i_s\} \subset \{1, \cdots, r\}$. (Thus, these intersections themselves determine foliations.) The multifoliation is called *total* if $r = \dim(M)$, and $\operatorname{codim}(\mathscr{F}_i) = r-1$ for each i. In this case, the tangent bundle of M is reduced to a sum of line bundles. D. Tischler **[TI1]** has proven that *every 3-manifold which is the total space of a principal circle bundle over an oriented surface admits a total multifoliation.* In particular, this includes S^3, so in analogy with Reeb's question above, we pose the following problem.

Problem 12. Does S^7 admit a total multifoliation?

Problems 11 and 12 can, of course, be restated for general manifolds. For a discussion of further problems of this type, see **[TS1]**.

One of the most fundamental classical results in the study of dynamical systems in the Poincaré-Bendixson Theorem which asserts that for a C^1 flow on S^2 every minimal closed invariant set is either a fixed point or a closed orbit. A natural generalization of this result to foliations would assert that for a C^1 foliation of S^n, every minimal, closed, invariant (a union of leaves) set is a closed leaf. This is far from true. (See, for example, **[S2]**, **[R3]**, **[RR1]**, and **[RA]**.) However, the question of the existence of a closed leaf remains open, and one of the most celebrated results in the theory of foliations is the following.

THEOREM 24 (NOVIKOV **[N]**). *Let M be a compact 3-dimensional manifold with a smooth codimension-one foliation $\mathscr{F}$, and suppose that any of the following conditions are satisfied:*

(1) $\pi_1(M)$ *is finite,*

(2) $\pi_2(M) \neq 0$,

(3) *there exists a closed curve in M transverse to* $\mathscr{F}$ *which is null-homotopic, or*

(4) *there exists a leaf* $\mathscr{L} \in \mathscr{F}$ *such that the map* $\pi_1(\mathscr{L}) \to \pi_1(M)$ *has nontrivial kernel.*

Then $\mathscr{F}$ *has a closed leaf. In fact, except for case* (2), *the foliation contains a Reeb component.*

In case (2) either there is a Reeb component or M is a finite quotient of a fiber bundle over S^1 (i.e., all leaves are compact and $=S^2$ or $\boldsymbol{P}^2(\boldsymbol{R})$).

COROLLARY 12. *Every smooth codimension-one foliation of* S^3 *has a Reeb component.*

This leads to the following natural question.

Problem 13. Does every smooth, codimension-one foliation of S^{2n+1}, for $n>1$, have a closed leaf?

One partial result of this type has been obtained by Joe Plante **[PL1]**. He shows that if the leaves satisfy certain intrinsic growth conditions, then the strong Poincaré-Benedixson Theorem holds, i.e., *every* minimal set is a closed leaf. His methods give nice results for actions of nilpotent Lie groups on simply-connected manifolds.

Sacksteder **[S3]** has given a condition in terms of linear holonomy groups, which guarantees the existence of a compact leaf.

Note that if a codimension-one foliation of S^n has a closed leaf, then it certainly does not have a dense leaf. However, it is not even known that foliations of S^n with dense leaves cannot exist.

CONJECTURE. Let M be a compact manifold which admits a smooth codimension-one foliation having a dense leaf. Then, $H_1(M; \boldsymbol{R}) \neq 0$.

It should be noted that Hector **[HEC]** has recently constructed a codimension-one foliation of euclidean space in which every leaf is dense.

ADDED IN PROOF. Paul Schweitzer has recently shown that if a manifold of dimension $\geqq 5$ admits a smooth, codimension-one foliation, then it also admits a codimension-one, C^0 foliation having no compact leaves.

One might naturally ask whether Novikov's theorem extends to a larger class of 3-manifolds. In particular, could it be true for a manifold which is a $K(\pi, 1)$? This question was recently answered by W. Thurston who considered 3-manifolds which are circle bundles over surfaces. Here it is sometimes possible to construct foliations without compact leaves by the methods of §1B. However, this is the only allowable procedure.

THEOREM 25 (W. THURSTON **[TH1]**). *Let M be the total space of an oriented circle bundle* ξ *over an oriented surface* $\Sigma \neq S^1 \times S^1$, *and let* $\mathscr{F}$ *be a* C^2, *transversely oriented foliation of M. Then either* $\mathscr{F}$ *is isotopic to a foliation transverse to the fibers of* ξ *or* $\mathscr{F}$ *has a compact leaf. In particular* *if* $|\chi(\xi)|>|\chi(\Sigma)|$, *where* χ *denotes Euler characteristic, then* $\mathcal{F}$ *must have a compact leaf.*

The second part of this theorem follows from a pretty result of John Wood [**W3**] that a circle bundle ξ over a compact surface Σ admits a foliation transverse to the fibers if and only if $|\chi(\xi)|\leqq\min\{0, -\chi(\Sigma)\}$. (Here the bundle must be "Σ orientable", i.e., the total space of ξ must be orientable.)

The question of closed leaves for foliations of higher codimension goes back to Seifert who posed the following well-known

SEIFERT CONJECTURE. Every nonsingular vector field on S^3 has a closed integral curve.

Seifert proved the conjecture for vector fields C^0-close to the Hopf vector field (or, more generally, C^0 close to the vector field of any Seifert fibering of S^3). Novikov [**N**] proved the conjecture for fields transverse to a codimension-one foliation of S^3. Pugh [**PU**] proved the conjecture for generic vector fields. However, in a remarkable paper [**SC**], Paul Schweitzer has recently shown the following.

THEOREM 26 (P. SCHWEITZER). *Every* C^1 *foliation of codimension* $\geqq 2$ *on a manifold is homotopic to a* C^1 *foliation with no compact leaves.*

COROLLARY 13. *Every homotopy class of nonvanishing* C^1 *vector fields on* S^3 *contains a counterexample to the Seifert conjecture.*

Schweitzer's foliations actually have tangent plane fields of class C^1 but definitely not of class C^2. Hence, the conjecture remains open for vector fields of class C^r where $r\geqq 2$ or $r=\omega$.

In the case that compact leaves do exist, it is natural to ask about their topological type. The most classical and important result of this type is the Reeb Stability Theorem ([**R1**]).

THEOREM 27 (G. REEB). *Let* $\mathcal{F}$ *be a* C^r *foliation on a manifold* M *where* $r\geqq 0$, *and suppose that* $\mathcal{F}$ *has a compact leaf* $\mathcal{L}$ *with finite fundamental group. Then every neighborhood of* $\mathcal{L}$ *contains an invariant neighborhood* U *with the property that each leaf* $\mathcal{L}'\subset U$ *is a finite covering space of* $\mathcal{L}$.

Thus, if $\mathcal{L}$ is simply connected, U has the form $\mathcal{L}\times D^q$ with the foliation $\{\mathcal{L}\times\{p\}\}_{p\in D}$: In general, a finite covering of U is of this form, where $\mathcal{L}$ is replaced by the universal covering of $\mathcal{L}$.

Note that M need not be compact, and the codimension does not matter. If we tighten these requirements we get stronger results.

COROLLARY 14. *Let* M *be a compact manifold with a smooth, codimension-one, transversely orientable foliation* $\mathcal{F}$. *If* $\mathcal{F}$ *has a compact leaf* $\mathcal{L}$

with finite fundamental group, then M is a fiber bundle over S^1 with fiber $\mathscr{L}$ (and $\mathscr{F}$ is the associated foliation).

Theorem 25 can be strengthened to include all nearby foliations.

THEOREM 27′ (G. REEB). *Let M, $\mathscr{F}$ and $\mathscr{L}$ be as in Theorem 25. Then given a neighborhood $\mathscr{V}$ of $\mathscr{L}$, there is a neighborhood $\mathscr{O}$ of $\mathscr{F}$ in the space of C^r foliations of M and a neighborhood $\mathscr{U}$ of $\mathscr{L}$ in M such that every $\mathscr{L}' \in \mathscr{F}' \in \mathscr{O}$ with $\mathscr{L}' \cap \mathscr{U} \neq \varnothing$ is a finite covering space of $\mathscr{L}$.*

There is a version of Theorem 25, due to B. Reinhart **[RI1]**, for foliations with metric properties, that is, for foliations whose holonomy (see below) is distance-preserving in some riemannian metric. Such a metric is called *bundle-like*. For these foliations there is also an associated global stability theorem, similar to Corollary 14, which holds in general codimensions **[RI3]**.

Related to these results is a deep theorem of Sacksteder. To state it, we need to introduce the important notion, due to Ehresmann, of *holonomy*. Let $\mathscr{F}$ be a codimension-q foliation on M, and at a fixed $p \in M$ choose an embedding $\varphi: D^q \to M$ transverse to $\mathscr{F}$ with $\varphi(0)=p$. To each oriented loop γ based at p and lying in the leaf $\mathscr{L}$ containing p, we associate a (germ of) a diffeomorphism h_γ of a neighborhood of 0 in D^q to another neighborhood of 0 as follows. Choose a transverse map $\Phi: D^q \times S^1 \to M$ with the property that $\Phi(0, \theta)=\gamma(\theta)$ and $\Phi|D^q \times \{0\}=\varphi$. For $x \in D^q$, follow the curve of the foliation induced on $D^q \times S^1$ as θ traverses S^1. For x sufficiently close to 0, it is possible to pass completely around S^1 and return to a new point $h_\gamma(x) \in D^q$. The map h_γ depends only on the homotopy class of γ in $\mathscr{L}$, and is called the *holonomy map along* γ. We get a homomorphism, $\pi_1(\mathscr{L}, p) \to \tilde{\Gamma}^r_q$, the germs of local diffeomorphisms of $\boldsymbol{R}^q$ which fix 0. The image is the *holonomy group* at p. If $\pi_1(\mathscr{L}, p)=\{1\}$, then the holonomy group at p is trivial.

THEOREM 28 (SACKSTEDER [S3]). *Let M be a compact manifold with a codimension-one, C^2 foliation $\mathscr{F}$, and suppose that all the holonomy groups of $\mathscr{F}$ are finite. Then there is a riemannian metric on M invariant under holonomy, and M is covered by $\boldsymbol{R} \times \mathscr{L}$ where $\mathscr{L}$ is either a leaf or a two-fold cover of a leaf in $\mathscr{F}$.*

In particular, if $\mathscr{F}$ is orientable and all the holonomy groups are trivial, then there is a fixed-point free flow on M which preserves $\mathscr{F}$, i.e., maps leaves to leaves for all time.

Sacksteder also shows in [S3] that if M^m has an orientable, codimension-1 foliation $\mathscr{F}$, given by the orbits of a locally free $\boldsymbol{R}^{m-1}$ action, and if no leaves of $\mathscr{F}$ are compact, then every leaf is dense and $H^1(M:\boldsymbol{R}) \neq 0$.

Note. There was, incidentally, a minor error in Lemma 14.1(ii) of **[S3]**. It was pointed out by E. Vidal and A. Vasquez that under the assumptions of Theorem 28, one can conclude that $H^1(M; \boldsymbol{R}) \neq 0$ only if $\mathscr{F}$ is orientable. Note, however that if $\mathscr{F}$ is not orientable, its lift to an appropriate 2-sheeted covering surface is. The example of Vasquez is $M = S^1 \times S^2/\boldsymbol{Z}_2$ where $\boldsymbol{Z}_2$ is generated by $\alpha(z, x) = (\bar{z}, -x)$ for $(z, x) \in S^1 \times S^2 \subset \boldsymbol{C} \times \boldsymbol{R}^3$ and where $\mathscr{F}$ lifts to the foliation $\{\{z\} \times S^2\}_{z \in S^1}$ of $S^1 \times S^2$.

It is important to point out that neither the metric nor the flow in Theorem 28 are, in general, smooth. One must first change the differentiable structure of M. Once the change is made, the flow can be viewed as coming from a smooth vector field transverse to $\mathscr{F}$. The existence of such a vector field V means that *$\mathscr{F}$ can be defined by a closed one-form.* Let ω be the 1-form which vanishes on $\mathscr{F}$ and has the property $\omega(V) \equiv 1$. Then one can check that $d\omega = \omega \wedge (L_V \omega)$ where L denotes Lie derivative. Since $\mathscr{F}$ is invariant by V, $L_V \omega = 0$.

If $\mathscr{F}$ comes from a submersion $M \to S^1$ it is defined by a closed 1-form. The converse is not true. (Consider parallel foliations on a torus.) However, it is nearly true.

THEOREM 29 (TISCHLER **[TI2]**). *If M admits a codimension-one foliation defined by a closed form, then there is a fibering $M \to S^1$ which is close to $\mathscr{F}$.*

Much of the discussion in this chapter is concerned with when a foliation must have a compact leaf. One might equally well ask what can be said of a foliation *all* of whose leaves are compact. In codimension-one, the holonomy groups must all be finite, and by Sacksteder (or, in this case, the more elementary arguments of Reeb) some 2-sheeted cover of the foliation is a fibration over S^1. In higher codimensions the problem is much harder. For example, Reeb **[R1**, pp. 113–115] has given an example of an open manifold with a foliation by compact leaves such that the union of leaves meeting a certain compact set K is not compact. It is reasonable therefore to assume the manifold is also compact before asking for strong consequences. The best result of this type is the following deep theorem.

THEOREM 30 (EPSTEIN **[EP2]**). *Every C^r foliation $(1 \leqq r \leqq \infty)$ of a compact orientable three-manifold, possibly with boundary, by circles is C^r-conjugate to a Seifert fibration. That is, every leaf has a saturated neighborhood $U \approx S^1 \times D^2$ where the foliation is C^r-conjugate to the orbits of the action*

$$\varphi_t(e^{i\theta}, z) = (e^{i(t+\theta)}, e^{i(p/q)t} \cdot z)$$

on $S^1 \times D^2$ for relatively prime integers p and q.

Outside this result little is known about the general problem.

Problem 14. Characterize the foliations of compact manifolds in which every leaf is compact.

A fundamental area of study in the field of foliations surrounds the notion of stability. A smooth foliation $\mathscr{F}$ is said to be *C^r-stable* if every foliation $\mathscr{F}'$ such that $\tau(\mathscr{F}')$ is sufficiently close to $\tau(\mathscr{F})$ in the C^r-topology, is C^0-conjugate to $\mathscr{F}$. Unstable foliations exist. (Again consider parallel foliations on a torus.) The interesting problem is the following.

Problem 15. Find conditions which guarantee that a foliation is stable.

Reeb's Theorem 25 together with Corollary 14 gives a result of this type. Furthermore, Rosenberg and Roussarie have recently shown the following.

THEOREM 31 (ROSENBERG AND ROUSSARIE **[RR4]**, **[ROU]**). *No codimension-one foliation of S^3 is (C^∞) stable. The only stable foliations of $S^1 \times S^2$ are those conjugate to the foliation $\{\{\theta\} \times S^2 : \theta \in S^1\}$, modified by introducing a finite number of hyperbolic Reeb components along closed transversals.*

There are also results on stability of intrinsic components.

One could analogously ask for specific properties of a foliation to be stable, for example, the property of having a compact leaf. Theorem 27′ falls into this category and has been recently generalized by Hirsch.

THEOREM 32 (M. HIRSCH **[HR]**). *Let $\mathscr{F}$ be a C^r foliation ($r \geqq 1$) on a manifold M and suppose there is a compact leaf $\mathscr{L} \in \mathscr{F}$ and an element $\alpha \in \pi_1(\mathscr{L})$ such that*

(1) *α is in the center of $\pi_1(\mathscr{L})$.*

(2) *The differential of the holonomy map of α does not have* 1 *as an eigenvalue.*

Then for all ε sufficiently small, there is a neighborhood $\mathcal{O}$ of $\mathscr{F}$ in the space of C^r foliations of M, such that for each $\mathscr{F}' \in \mathcal{O}$, there is a compact leaf $\mathscr{L}' \in \mathscr{F}'$ and a map $f: \mathscr{L} \to \mathscr{L}'$ such that $\operatorname{dist}(x, f(x)) < \varepsilon$ *(in a fixed riemannian metric on M). Moreover, $\mathscr{L}'$ is unique and f is a homotopy equivalence.*

Condition (1) in this theorem can be replaced by: "α belongs to a nilpotent subgroup of finite index". A version of the result is also true for the case $r=0$. (See **[HR]**.)

There is an area in the study of foliations concerned with questions of the following general type. What are the topological conditions necessary for a manifold to admit a foliation of prescribed type? For example, Rosenberg **[RO2]** has shown that *a (not necessarily compact) 3-manifold, C^2 foliated by planes, has the property that any embedded 2-sphere bounds an embedded ball.* Moreover, from basic results of **[RO1]**, the following theorem (due to Rosenberg and Sondow **[RO2]** for $m=3$, Joubert and

Moussu [**JM**] for $m=4$, and Rosenberg [**RO2**] together with results of C. T. C. Wall for $m \geqq 5$) can be proved.

THEOREM 33. *If a closed m-manifold M admits a codimension-one, C^2 foliation where every leaf is homeomorphic to $\boldsymbol{R}^{m-1}$, then M is homeomorphic to an m-torus, $S^1 \times \cdots \times S^1$.*

Deep results in a similar vein have been obtained by Verjovsky [**V1**] for codimension-one, Anosov foliations. (These foliations are not C^2.)

Theorem 33 for the case $m=3$ has been extended by Rosenberg, Roussarie and Chatelet to foliations of 3-manifolds with boundary [**RR2**], [**RR3**], [**CR**]. This work includes a classification of Reeb foliations up to C^0 conjugation, and, in particular, the following generalization of the Denjoy theorem: *Every C^2 foliation of the 3-torus by 2-planes is C^0 conjugate to a linear foliation.* (M. Herman has shown this to be false for C^0 foliations. The C^1 case appears to be unknown.) This work, together with the results of Moussu and Roussaire [**MR1**], gives a classification of C^2 foliations of T^3 with no Reeb components as follows: For each such foliation there is a decomposition of T^3 into submanifolds with boundary $A_1, \cdots, A_p$, $B_1, \cdots, B_q$ where $A_i \approx B_j \approx T^2 \times I$ and where, up to C^0 conjugacy: the foliation on A_i is a suspension of the Reeb foliation of $S^1 \times I$, and the foliation on B_j is transverse to the fibers of the projection $T^2 \times I \to T^2$ (and, thus, given by a homomorphism $\pi_1(T^2) \to \mathrm{Diff}^+(I)$, cf. §1).

There is an entire literature on noncompact group actions (in particular, $\boldsymbol{R}^n$-actions) on manifolds, which we shall not discuss. However, for a presentation of the results of Lima, Novikov, and Rosenberg-Roussarie-Weil on the rank of 3-manifolds (cf. §1C), see [**TS1**].

We have also omitted mentioning questions of deformations of foliations (see [**KS**], [**H2**] and [**H4**]), and questions of the applications of elliptic operator theory (see Reinhart [**RI2**]).

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