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ON FOURIER COEFFICIENTS OF SIEGEL MODULAR FORMS OF DEGREE TWO

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ABSTRACT. Siegel modular forms f of degree two are considered which satisfy: (1) the Fourier coefficients $b_f(R)$, for R a positive definite, semi-integral, primitive matrix, are solely a function of $\det(R)$; and (2) f is an eigenform for the Hecke algebra whose eigenvalues satisfy certain relationships. For such forms, results about multiplicative relationships and asymptotic growth are given, and formulae are given for $b_f(R)$ with R arbitrary in terms of $b_f(T)$ with $\det(2T)$ square-free.

Hecke operators play a vital role in investigating multiplicative relations among Fourier coefficients of modular forms of one complex variable. In this paper, we show that, for a certain class of Siegel modular forms of degree two, Hecke operators play a similar role in determining relations among Fourier coefficients.

Let $f(Z)$ be a Siegel modular form of degree two and weight w. Then $f(Z)$ has a Fourier expansion of the form $f(Z) = \sum_{R \geqq 0} b_f(R) e(RZ)$, where Z is a point in the Siegel upper half plane of degree two, R runs through all positive semidefinite, semi-integral 2×2 matrices, and $e(RZ) = \exp[2\pi i \cdot \mathrm{Trace}(RZ)]$. If $R = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, with a, c, $2b$ integers, then we set $\gcd(R) = \gcd(a, c, 2b)$. We will denote the determinant of a matrix A by $|A|$.

We now define the Hecke operators (degree two) on the space $\mathscr{F}_w$ of all Siegel modular forms of degree two and weight w. Let

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \qquad \mathscr{L}(n) = \{M \in GL(4, \mathbf{Z}) \colon M^t J M = nJ\}.$$

For f in $\mathscr{F}_w$, n a positive integer, and M in $\mathscr{L}(n)$, we write M in blocks of 2×2 matrices as $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and define

$$(f|_M)(Z) = |M|^{w/2}\,|CZ + D|^{-w} f[(AZ + B)(CZ + D)^{-1}].$$

Noting that one can write $\mathscr{L}(n) = \bigcup_A \mathscr{L}(1)A$, a finite, disjoint union, we define the unnormalized Hecke operator $T(n)\colon \mathscr{F}_w \to \mathscr{F}_w$ as $f|_{T(n)} = \sum_A f|_A$.

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Formally one has [**3**, Satz 2],

$$\sum_{n=1}^{\infty} T(n)n^{-s} = \prod_{p} T(p; p^{-s}),$$

where the product is over all primes. In [**2**, Theorem 2] we show that formally

$$T(p; x) = (1 - p^2x^2) \times [(1 - p^2x^2)(1 - p^4x^2) - T(p)(x + p^3x^3) + pW(p)x^2]^{-1},$$

where $W(p) = (1/p)[T(p)^2 - T(p^2)] + p^3$. (See also [**8**, Theorem 2] and [**7**, Appendix I, Equation 21].)

We shall be considering f in $\mathscr{F}_w$ which are eigenforms for all the $T(n)$. We shall denote its eigenvalues under $T(p)$ and $W(p)$ by $\tau_{p,f}$ and $\lambda_{p,f}$, respectively. In particular, we shall be interested in forms f satisfying two additional properties, *viz.*:

Property I. For R primitive (i.e. $\gcd(R) = 1$) $b_f(R)$ is solely a function of $|R|$ and

Property II(p). For the prime p, $\lambda_{p,f} = (p + 1)\tau_{p,f}$.

Property I is true for Eisenstein series [**4**, Satz 1] and appears to be probable for the unique cusp form of weight 10 on the basis of numerical data [**6**, p. 30] and for the unique cusp form of weight 12 for theoretical considerations. Property II(p) is true for all primes p for Eisenstein series and we shall show (Theorem 4) that, in a number of cases, Property I implies Property II(p). (This is true in particular for all p for the cusp form of weight 12.)

The appearance of Property II(p) is related to the factorization of $T(p)$ and $W(p)$ as [**2**, p. 28 and Lemma 9]:

$$(p + 1)T(p) = L(p) \cdot M(p) \quad \text{and} \quad W(p) = L(p) \cdot V(p) \cdot M(p),$$

where $L(p)$, $V(p)$, and $M(p)$ are Hecke operators of the Koecher type [**1**, p. 361]. Here, $V(p)$ is an involution.

We prove:

THEOREM 1. *Let f in $\mathscr{F}_w$ be an eigenform for the Hecke algebra and suppose f has Property* I. *Then $b_f(R)$ depends solely on* $\gcd(R)$ *and* $|R|$.

Note. Theorem 1 for Eisenstein series follows from a correction of [**4**, Satz 2] in [**5**].

Let

$$b_f[a(r_1, r_2, r_3)] = b_f\left[a\begin{pmatrix} r_1 & r_3/2 \\ r_3/2 & r_2 \end{pmatrix}\right],$$

for a, r_1, r_2, r_3 integers; p and q denote distinct primes; y, z, s, and t denote formal power series variables; $x_{p,w} = p^{w-2}$; $(A/B) \equiv$ Jacobi symbol,

$((2A+1)/2) \equiv 1$, for integers A and B;

$$\Delta_f(p;t) = 1 + (p + 1 - p^{-1}\tau_{p,f})x_{p,w}t + px_{p,w}^2t^2;$$

$$D_f(p;y,z) = (1 - px_{p,w}y)\,\Delta_f(p;y)\,\Delta_f(p;z^2);$$

$$\beta_w(p;y,z) = (1 - px_{p,w}^2yz^2);$$

$$\gamma_f(M,p;y,z) = [1 - (-M/p)x_{p,w}y]\beta_w(p;y,z) - (-M/p)x_{p,w}z^2\,\Delta_f(p;y);$$

and for positive integers C and T with $(CT,pq) = 1$, we set

$$H_f(p,q;C,T;y,z,s,t) = \sum_{m,n,a,b=0}^{\infty} b_f[Cp^nq^a(1,p^mq^bT,0)]y^nz^ms^at^bD_f(p;y,z)D_f(q;s,t).$$

We prove:

THEOREM 2. *Let p and q be distinct odd primes, C and T positive integers such that $(CT,pq) = 1$. Let f in $\mathscr{F}_w$ be an eigenform for the Hecke algebra and suppose f has Properties* I, II(p), *and* II(q). Then

$$\begin{aligned} H_f(p,q;C,T;y,z,s,t) = {} & \gamma_f(T,p;y,z)\gamma_f(T,q;s,t)b_f[C(1,T,0)] \\ & +\gamma_f(pT,q;s,t)\beta_w(p;y,z)zb_f[C(1,pT,0)] \\ & +\gamma_f(qT,p;y,z)\beta_w(q;s,t)tb_f[C(1,qT,0)] \\ & +\beta_w(p;y,z)\beta_w(q;s,t)ztb_f[C(1,pqT,0)]. \end{aligned}$$

THEOREM 3. *Let p be an odd prime. Let f in $\mathscr{F}_w$ be an eigenform for the Hecke algebra and suppose f has Properties* I, II(2), *and* II(p). *Let C and T be positive integers such that $(CT,2p) = 1$. Then*

$$\begin{aligned} H_f(p,2;C,T;y,z,s,t) = {} & \beta_w(2;s,t)\gamma_f(T,p;y,z)b_f[C(1,T,0)] \\ & + \beta_w(2;s,t)\gamma_f(2T,p;y,z)tb_f[C(1,2T,0)] \\ & + \beta_w(2;s,t)\beta_w(p;y,z)zb_f[C(1,pT,0)] \\ & + \beta_w(2;s,t)\beta_w(p;y,z)ztb_f[C(1,2pT,0)] \\ & + [1 - (-1/pT)]x_{2,w} \\ & \quad \cdot\{\gamma_f(1,2;s,t) - \beta_w(2;s,t)\} \\ & \quad \cdot\beta_w(p;y,z)zb_f[C(\tfrac{1}{4}(pT+1),1,1)] \\ & + [1 - (-1/T)]x_{2,w}\{\gamma_f(1,2;s,t) - \beta_w(2;s,t)\} \\ & \quad \gamma_f(T,p;y,z)b_f[C(\tfrac{1}{4}(T+1),1,1)]. \end{aligned}$$

If M and N are positive integers such that $M \equiv 3$ (mod 4) and $N \equiv 1$ (mod 2), then:

$$b_f[N(1,M,0)] = \{2^{-1}x_{2,w}\tau_{2,f} - [3 + (2/M)]x_{2,w}\}b_f[N(\tfrac{1}{4}(M+1),1,1)].$$

THEOREM 4. *Let f in $\mathscr{F}_w$ be an eigenform for the (degree two) Hecke algebra. Let*

$$f\left[\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\right] = \sum_i g_i(z_1)g_i(z_2),$$

where the g_i are modular forms of one complex variable of weight w which are eigenforms for the (degree one) Hecke algebra.

(1) *If f has Property* I *and $b_f(1, 1, 0) \neq 0$, then f has Property* II(2).

(2) *If p is an odd prime, f has Property* I, *and $[p - (-3/p)]b_f(1, 1, 0) + [p - (-1/p)]2b_f(1, 1, 1) \neq 0$, then f has Property* II(p).

Let

$$\begin{aligned} b_f(\langle R\rangle) &= b_f(1, |R|, 0), & |R| \in \mathbf{Z}, \\ &= b_f(|R| + \tfrac{1}{4}, 1, 1), & |R| \notin \mathbf{Z}. \end{aligned}$$

We shall prove the following theorem which includes some of the conjectures of [**6**]. The proof of Theorem 5, Parts 2a, b below for Eisenstein series will appear shortly in [**5**] via very different methods.

THEOREM 5. (1) *If f is the normalized Eisenstein series of weight w then f has Property* I *and Property* II(p) *for all primes p. In particular*

$$\tau_{p,f} = px_{p,w}^{-1}(1 + px_{p,w})(1 + x_{p,w}).$$

Also, for any $\varepsilon > 0$ and R positive definite,

$$b_f(R) = 0(|R|^{w-(3/2)+\varepsilon}),$$

where the implied constant depends only on w and ε.

(2) *Let f in $\mathscr{F}_w$ be an eigenform for the Hecke algebra and suppose f has Property* I *and Property* II(p) *for all primes p and $\tau_{2,f} \neq 4, 8$. Then, for semi-integral, positive definite R,*

(a)
$$b_f(R) = \sum_{0<d\,|\,\gcd(R)} d^{w-1} b_f(\langle R/d\rangle),$$

and

(b)
$$b_f(NS)b_f(MS) = b_f(NMS)b_f(S),$$

for S a primitive matrix and relatively prime positive integers M and N.

If, in addition, $|p + 1 - p^{-1}\tau_{p,f}| \leqq 2\sqrt{p}$ for all primes p, then for any $\varepsilon > 0$ and R positive definite

$$b_f(R) = 0([\gcd(R)]^{1/2}\,|R|^{(2w-3)/4+\varepsilon}),$$

where the implied constant depends only on f and ε.

Note. If χ_{12} and χ_{10} are the unique normalized cusp forms of weight 12 and 10, respectively, then [**6**, pp. 37–38] and Theorem 4 imply that *if* χ_{10} and χ_{12} have Property I, then:

(a) χ_{12} has Property II(p) for all primes p, and

(b) χ_{10} has Property II(p) whenever $(3/p) \neq 1$ or $p = 2$.

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