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APPROXIMATION AND WEAK-STAR APPROXIMATION IN BANACH SPACES[1]

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ABSTRACT. If X^* has a weak-star basis and if X is separable, then X has a basis. If X^* has the weak-star λ-m.a.p. [a weak-star π_λ-decomposition], then X has the λ-m.a.p. [a $\pi_{\lambda+\lambda^2+\varepsilon}$-decomposition]. If X^* has a weak-star π_λ-decomposition and if X is separable, then X has a finite dimensional decomposition.

The problem of whether X is separable if X^* has a weak-star basis [**8**, p. 151] is unsolved, though there are candidates for a counterexample [**3**, 3.1], [**6**, pp. 243, 244]. In this note techniques developed in [**5**] are used, together with certain properties of weak convergence, to show that weak-star approximation methods in X^* will yield approximation properties in X.

In [**5**] the authors established very deep relationships between approximation methods in a Banach space X and its dual X^*. In particular they proved that if X^* has a basis, then X has a shrinking basis; and if X^* is a π_λ-space, then X is a π_δ-space for some $\delta > 1$. A fundamental tool in this work was the "principle of local reflexivity" [**5**], [**6**]. The basic corollary needed below is the following Theorem A [**5**, 3.1] or [**4**, p. 482], where $\mathscr{L}(B)$ is the space of bounded linear operators from B to B.

THEOREM A. *Let T be a finite rank operator in $\mathscr{L}(X^*)$ and let $F \subset X^*$ have* $\dim F < \infty$. *Let $\varepsilon > 0$. Then there is an S in $\mathscr{L}(X)$ such that $S^*(X^*) = T(X^*)$, $f(Sx) = Tf(x)$ for each f in F, x in X, and $\|S\| \leqq (1 + \varepsilon)\|T\|$. If T is a projection, then taking F to include $T(X^*)$, S is a projection.*

THEOREM 1. *Let (T_α) be a net of finite rank operators in $\mathscr{L}(X^*)$ such that $\|T_\alpha\| \leqq \lambda$ for all α and* $\lim T_\alpha f(x) = f(x)$ *for each f in X^*, x in X. Then there is a net of finite rank operators (S_β) in $\mathscr{L}(X)$ such that* $\lim S_\beta x = x$ *for each x, $\|S_\beta\| \leqq \lambda$ for each β.*

PROOF. For each finite-dimensional subspace F of X^*, use Theorem A to find $S_{\alpha,F}$ such that $f(S_{\alpha,F}x) = T_\alpha f(x)$ for every f in F, x in X, $S^*_{\alpha,F}(X^*) = T_\alpha(X^*)$ and $\|S_{\alpha,F}\| \leqq \lambda(1 + 1/(1 + \dim F))$. Let $(\alpha_1, F_1) \geqq (\alpha_2, F_2)$, if $\alpha_1 \geqq \alpha_2$, $F_1 \supset F_2$. Then $(1 + \dim(F))S_{\alpha,F}/(2 + \dim(F)) = R_{\alpha,F}$ has norm $\leqq \lambda$ and $\lim f(R_{\alpha,F}x) = f(x)$ for every f in X^*, x in X. Then a net(P_β) of convex combinations of $(R_{\alpha,F})$ has the property that $\lim P_\beta x = x$ for

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each x (using [**2**, p. 477], for example).

REMARK 1. If X is separable and if (x_n) is dense in X, then choosing α_1 such that $\|P_{\alpha_1}x_1 - x_1\| < 1$ and $\alpha_{n+1} > \alpha_n$ such that $\|P_{\alpha_{n+1}}x_i - x_i\| < 1/(n+1)$ when $i \leqq n+1$, one constructs a sequence $S_n = P_{\alpha_n}$ such that $S_n x \to x$ for each x.

THEOREM 2. *Let (T_α) be a net of finite rank projections in $\mathscr{L}(X^*)$ such that $T_\alpha(X^*) \supset T_\beta(X^*)$, if $\alpha > \beta$ and $\lim T_\alpha f(x) = f(x)$ for each f in X^*, x in X. Let X be separable. Then X has a finite-dimensional decomposition.*

PROOF. If (T_α) is a sequence (T_n), the proof proceeds easily from [**5**, 4.1] as follows. Set $Y = \overline{\bigcup T_n(X^*)}$. Then $T_n y \to y$ for each y in Y and, choosing S_n as in Remark 1, the conditions of 4.1(c) in [**5**] are satisfied. For the general case choose (S_n) $(=P_{\alpha_n})$, as in Remark 1 above, such that (S_n) has the following property: If

$$P_{\alpha_n} = \sum_1^{k_n} a_i R_{(\alpha_i, F_i)} \quad \text{and} \quad P_{\alpha_{n+1}} = \sum_1^{k_{n+1}} b_i R_{(\beta_i, G_i)},$$

then

$$(\alpha_i, F_i) \leqq (\beta_j, G_j) \quad \text{for each } i, j, n$$

(e.g. [**2**, p. 477] or [**1**, p. 40]). Further, choose the $R_{\alpha,F}$ to be projections such that $R^*_{\alpha,F}(X^*) = T_\alpha(X^*)$, as promised in Theorem A above. Let $R_n = R_{(\alpha_i, F_i)}$, where (α_i, F_i) is larger than the indices in $\sum_1^{k_n} a_j R_{(\alpha_j, F_j)}$. Then $Q_n = R_n + P_{\alpha_n} - R_n P_{\alpha_n}$ is a projection such that $Q_n^*(X^*) = R_n^*(X^*) = R^*_{(\alpha_i, F_i)}(X^*) = T_{\alpha_i}(X^*)$, and $Q^*_{n+1}(X^*) \supset Q_n^*(X^*)$ for each n. This is computed in the proof of the Theorem 3 below using the method in [**5**, Lemma 4.3]. Set $Y = \overline{\bigcup Q_n^*(X^*)}$ and apply [**5**, 4.1(c)].

COROLLARY 1. *Let $X^* = \sum_1^\infty Y_i$, where each Y_i is finite dimensional, and for x^* in X^* there is a unique sequence (f_i), $f_i \in Y_i$, such that $\lim_n \sum_1^n f_i(x) = x^*(x)$ for every x in X. If X is separable, then X has a finite-dimensional decomposition.*

PROOF. The partial sum projections $V_n(X^*) = \sum_1^n Y_i$ are uniformly bounded [**8**, pp. 147–149]. Set $Y = \overline{\bigcup V_n(X^*)}$. Then Y is separable and, by Theorem 2, X has a finite-dimensional decomposition.

COROLLARY 2. *Let X^* have w^*-basis (f_n) and suppose X is separable. Then X has a basis.*

PROOF. By hypothesis each f in X^* has an expansion $\sum_1^\infty a_n f_n$ where the convergence is in the w^*-topology. Set $V_n f = \sum_1^n a_n f_n$. Let $R_{n,F}$ be a projection on X as in Theorem A, such that $R^*_{n,F}(X^*) = V_n(X^*) = [f_1, \ldots,$

$f_n]$. Then $\lim_{(n,F)} f(R_{n,F}x) = f(x)$ for every f in X^*. Since X is separable, a sequence of convex combinations (P_n) of $(R_{n,F})$ converges strongly ($\lim P_n x = x$) to the identity, and $P_j^*(X^*) \subset [f_n]$. Since $V_n y \to y$ for all y in $[f_n]$ and $P_n x \to x$ for all x in X, Theorem 4.1 in [**5**] yields that X has a finite-dimensional decomposition given, say, by (Q_n). Moreover, $Q_n^*(X^*)$ is ε-close to some $V_{k(n)}(X^*)$ [**5**, 4.9]. This assures that $\{(Q_{n+1}^* - Q_n^*)(X^*)\}$ have bases with uniformly bounded basis constants [**5**, p. 501], and so $\{(Q_{n+1} - Q_n)X\}$ have bases with uniformly bounded basis constants [**5**, p. 502]. Thus, X has a basis (e.g., [**5**, Lemma 2.2]).

W. B. Johnson, in conversation with the author, observed that the methods above, together with the proof of Lemma 4.3 in [**5**], yield the following theorem.

THEOREM 3. *Let (T_α) be a net of finite rank projections such that $\|T_\alpha\| \leqq \lambda$ for every α and such that if $\alpha > \beta$, then $T_\alpha X^* \supset T_\beta X^*$. Suppose further that $\lim_\alpha T_\alpha f(x) = f(x)$ for each f in X^* and x in X. Then X is a $\pi_{\lambda^2+2\lambda+\delta}$ space for each $\delta > 0$.*

PROOF. If the (T_α) of Theorem 1 are projections with $T_\alpha(X^*) \supset T_\beta(X^*)$ when $\alpha > \beta$, and if the $R_{\alpha,F}$ in the proof of Theorem 1 are chosen to be projections such that $R_{\alpha,F}^*(X^*) = T_\alpha(X^*)$, let (U_β) be the corresponding net of finite rank operators such that $\lim U_\beta x = x$ for every x. If $U_\beta = \sum_1^n a_i R_{\alpha_i,F_i}$, where $(\alpha_1, F_1) < \cdots < (\alpha_n, F_n)$, let $S_\beta = R_{\alpha_n,F_n} + U_\beta - R_{\alpha_n,F_n} U_\beta$. Then

$$S_\beta^*(X^*) = R_{\alpha_n,F_n}^*(X^*) + U_\beta^*(X^*) - U_\beta^* R_{\alpha_n,F_n}^*(X^*) \subset T_{\alpha_n}(X^*)$$

and

$$S_\beta^* T_{\alpha_n} x^* = T_{\alpha_n} x^*$$

for each x^* in X^*. Thus, S_β^* is a projection onto $T_{\alpha_n}(X^*)$. It follows that (S_β) is a net of projections. Moreover, if $\|U_\beta x - x\| < \delta$, then

$$\begin{aligned}\|S_\beta x - x\| &= \|R_{\alpha_n,F_n} x + U_\beta x - R_{\alpha_n,F_n} U_\beta x - x\| \\ &\leqq \|R_{\alpha_n,F_n}\| \, \|x - U_\beta x\| + \|U_\beta x - x\| \leqq (\lambda + \varepsilon + 1)\delta\end{aligned}$$

so that $\lim S_\beta x = x$.

If X has a finite-dimensional decomposition [basis], then X^* has a weak-star finite-dimensional decomposition [weak-star basis]. If X has a π_λ-decomposition [λ-m.a.p.], then X^* has the weak-star λ-m.a.p. It is not known to the author if X^* has a weak-star π_λ-decomposition. An answer to this question will answer, via Theorem 2, whether X has a finite-dimensional decomposition if X is a separable π_λ-space.

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