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BOOK REVIEWS

Perceptrons, An Introduction to Computational Geometry, by Marvin Minsky and Seymour Papert. MIT Press, Cambridge, Mass., 1969.

This book is a very interesting and penetrating study of the power of expression of perceptrons and some other mathematical problems concerning memory and learning. This subject is still quite new and hence at a stage of development in which the most important discoveries are being done. It seems to differ from the theory of automata in its greater relevance to our ideas about the organization of the brain and the construction of "such" machines.

The main positive result which motivates the research presented in this book is *the perceptron learning theorem* (proved in Chapter 11). This theorem was discovered as a property of Rosenblatt's perceptrons (the history is given in the book) but in fact it belongs to general linear approximation theory and can be stated as follows.

Let H be a real Hilbert space, $F^+, F^- \subseteq H, F^+ \cap F^- = \emptyset$ and for every $x \in F^+ \cup F^-$ we have $\delta \leqq \|x\| \leqq M$, where $\delta > 0$. We assume moreover that F^+ and F^- are *linearly separable with resolution* δ, i.e. there exists an $a^* \in H$ with $\|a^*\| = 1$ such that

$$(a^*, x) \geqq \delta \quad \text{if } x \in F^+ \quad \text{and} \quad (a^*, x) \leqq -\delta \quad \text{if } x \in F^-.$$

Given any sequence $x_1, x_2, \ldots \in F^+ \cup F^-$ and any $\varepsilon \geqq 0$ we define by induction a sequence of vectors $a_0, a_1, \ldots \in H$ and a sequence of integers $k_1, k_2, \ldots$. We put $a_0 = 0$ and

$$(1) \qquad a_{i+1} = a_i + k_{i+1}x_{i+1},$$

where k_{i+1} is the unique integer with minimal absolute value such that if $x_{i+1} \in F^+$ then

$$(a_i, x_{i+1}) + k_{i+1}\|x_{i+1}\|^2 > \varepsilon.$$

and if $x_{i+1} \in F^-$ then

$$(a_i, x_{i+1}) + k_{i+1}\|x_{i+1}\|^2 < -\varepsilon.$$

Then the sequence $k_1, k_2, \ldots$ *has finitely many terms different from* 0 *and moreover*

$$\sum_{i=1}^{\infty} |k_i| \leqq (M^2 + 2\varepsilon)/\delta^2.$$

(In the book $\varepsilon = 0$ and $M = 1$, but the proof is almost the same for all $\varepsilon \geqq 0$ and $M \geqq \delta$.)

The meaning of this theorem (which is a prototype result for a not yet existing mathematical theory of learning) is the following. We think of $x_1, x_2, \ldots$ as of a sequence of lessons while $a_0, a_1, \ldots$ as the consecutive stages of a search for a vector a such that

$$(2) \qquad (a, x) > \varepsilon \quad \text{for all } x \in F^+,$$

$$(3) \qquad (a, x) < -\varepsilon \quad \text{for all } x \in F^-.$$

The theorem guarantees that the procedure defined by (1) is successful inasmuch as the sequence of lessons is reasonable; moreover once $\sum_{i=1}^{N} |k_i| = [(M^2 + 2\varepsilon)/\delta^2]$ (although this may never happen) we know that $a = a_N$ satisfies (2) and (3).

Variants of this theorem have led to the invention of machines called equalizers which have tripled the capacity of some communication channels, but this is another story (see [**1**], [**2**], [**4**], [**5**]).

This book is mainly a study of the problem of expressing various properties P of a pattern, i.e. picture X by means of inequalities

$$(4) \qquad \alpha_0 + \sum_{i=1}^{m} \alpha_i \varphi_i (X) > 0,$$

where α_i are real numbers and φ_i are "simple" 01-valued functions of X. "Simple" means that each φ_i depends only on a small part of X. The inequality (4) is called a *perceptron* or a *linear threshold predicate*. Clearly if the space of patterns is finite, the φ_i are given and P is a property for which there exist α_i such that $P(X)$ holds iff (4) holds, then, (1) is a method for finding such α_i (with the help of a teacher who knows P).

There were great hopes about the applications of this method for pattern recognition problems but the main results proved in this book show that properties expressible in the form (4) are rather special, and hence the method is by no means "universal" as one may have thought.

In this book usually $X = (x_{rs})$ is a 01-matrix of fixed size $n \times n$, thus X is the characteristic function of a pattern, i.e. of a subset of an $n \times n$ square lattice. One of the first remarks is that for such X the power of expression of (4) does not diminish if we require that the 01-valued functions φ_i are of the form

$$(5) \qquad \varphi_i(X) = \prod_{(r,s) \in S_i} x_{rs}.$$

A perceptron of *degree d* is an inequality (4) with arbitrary m, φ_i as in (5) and max $\overline{\overline{S_i}} = d$.

A pattern X is called *connected* (i.e. rookwise connected) if there exists a sequence $(r_1, s_1), \ldots, (r_u, s_u)$ such that $|r_i - r_{i+1}| + |s_i - s_{i+1}| = 1$ for $i = 1, \ldots, u - 1$, and $x_{rs} = 1$ iff $(r, s) \in \{(r_1, s_1), \ldots, (r_u, s_u)\}$.

One of the main results of this book is that *there is a constant* $c_1 > 0$ *such that the property "X is connected" cannot be expressed by a perceptron of any degree* $d < c_1 n$. A similar result is true for k-dimensional patterns $(x_{r_1 \dots r_k})$, $r_i = 1, \dots, n$ for $i = 1, \dots, k$, with $k > 2$. Here $d < c_2 n^k$ is already sufficient for the same conclusion. It is still an open problem if $d < cn^2$ is sufficient in the 2-dimensional case and the proof assuming $d < c_1 n$ is quite delicate.

A stronger result due to Michael Paterson for $k = 2$ and $d < c_3 n$ is also proved. First a class of properties of patterns is defined which are called "topological invariants"; also an "Euler characteristic" $\chi(X)$ which is a topological invariant is defined. Then it is shown that if (4) is a topological invariant then it depends only on $\chi(X)$ (for X different from the 0 matrix). More exactly, in the domain of all $X \neq 0$, perceptrons of degree d which are topological invariants can express a property P iff P is of the form $\chi(X) \in U$, where U is a union of s intervals (intervals $[a, \infty)$ and $(\infty, b]$ are allowed) and $s < c_4 d$. E.g. a topologically invariant perceptron cannot distinguish between two nonempty patterns having the same differences between the number of holes and the number of components. This result has not yet been extended to $k > 2$. One of the important steps in the proofs of these results is the following proposition. If $c(X)$ denotes the number of 1's in X, then $c(X) \in U$ is expressible by a perceptron of degree d iff U is as above and $s < c_5 d$. E.g. "$c(X)$ is even" is expressible only with d large (in fact $d = n^k$).

These results have inspired similar studies of the power of expression of other languages either related or quite different from inequalities of the form (4), see [**6**], [**8**].

Among other topics in this book let me still mention estimates for the size of coefficients α_i, a number of properties expressible or not expressible in the form (4) under various restrictions on the sets S_i in (5) (e.g. an upper bound on their diameters), many interesting remarks on mathematical "mechanisms" which analyze pictures, and more general remarks on complexity of computation and various problems of classification. (See also [**3**], [**7**] for related material concerning the organization of memory.)

A general ideology expressed in this book is very critical of "universal approaches" in the domain of artificial intelligence. The authors advocate analytic studies of various tasks performed by living organisms rather than a search for a "single" principle and a "simple" machine which could perform all such tasks. {The reviewers ideology is different. He thinks that the benefits of synthetic thinking are unpredictable. The range of applications of mathematical ideas and algorithms often completely surpasses their initial scope. The guessing-game variety of mathematical creativity is very attractive and is as strong a factor of progress as systematic analysis.

Nothing even remotely similar to the negative laws of thermodynamics has been discovered which would forbid the existence of a large but simply describable mechanism, perhaps realizable within the means of present or future technology, which would be as intelligent as an animal or man.}

The book is very well written. Its clear compact and very engaging style and its large number of ideas and open problems make it perfect material for study in seminars, and it should have a strong influence on future writers in this subject. It is also the first purely mathematical monograph about certain aspects of learning and perception, and this subject may become the most important theory of 20th century mathematics.

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Methods of Hilbert Spaces, by Krzysztof Maurin. PWN-Polish Scientific Publishers, 1967. Vol. 45 in the series Monografie Matematyczne of Polska Akademia Nauk. 554 pp. and *General Eigenfunction Expansions and Unitary Representations of Topological Groups*, by Krzysztof Maurin. PWN-Polish Scientific Publishers, 1968. Vol. 48 in the series Monografie Matematyczne of Polska Akademia Nauk. 367 pp.

Each of these books has a strong allure for the modern analyst, for Maurin has assembled unique collections of interesting topics. *Methods of Hilbert spaces* (hereafter MHS) begins with nine chapters which provide an introduction to the abstract theory, including the basic definitions and geometry of Hilbert spaces and locally convex spaces, the spectral theory of selfadjoint, unitary and compact operators, and the general spectral theory of commutative C^*-algebras via the Gelfand theory for commutative Banach algebras. In the next fourteen chapters this general theory is applied to one parameter semigroups, elliptic partial differential equations, the orthogonal projection method, Bochner's theorem on the existence of