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PFISTER FORMS AND K-THEORY OF FIELDS[1]

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Let F be a field of characteristic different from two. We shall write $W(F)$ to denote the Witt ring of F, and $I(F)$ to denote the ideal of all even dimensional forms in $W(F)$. Also, let K_nF $(n \geqq 1)$ be the higher K-groups of F defined by Milnor in [**3**], and $k_nF = K_nF/2K_nF$. The elements $l(a_1) \cdots l(a_n) \in k_nF$ will be called the *generators* of k_nF. If $(a_1, \cdots, a_n)$ is an n-tuple of nonzero elements of F, we shall write $\langle\langle a_1, \cdots, a_n\rangle\rangle$ for the 2^n-dimensional quadratic form $\otimes_{i=1}^n \langle a_i, 1\rangle$, and refer to it as an *n-fold Pfister form*. Clearly, these n-fold Pfister forms give a system of generators of I^nF as an ideal in $W(F)$. In [**3**], Milnor showed that

$$s_n(l(a_1) \cdots l(a_n)) = \langle\langle -a_1, \cdots, -a_n\rangle\rangle \pmod{I^{n+1}F}$$

gives a well-defined epimorphism from k_nF onto $I^nF/I^{n+1}F$. In §4 of [**3**], Milnor raised the question whether these maps are isomorphisms.

In studying this problem, the Pfister forms clearly play a crucial role. In this note, we announce the following results.

THEOREM 1. *The following statements are equivalent:*

(1) $\langle\langle -a_1, -a_2\rangle\rangle$, $\langle\langle -b_1, -b_2\rangle\rangle$ *are isometric* (*Pfister*) *forms.*

(2) $\langle\langle -a_1, -a_2\rangle\rangle \equiv \langle\langle -b_1, -b_2\rangle\rangle \pmod{I^3F}$.

(3) $\left(\frac{a_1, a_2}{F}\right)$, $\left(\frac{b_1, b_2}{F}\right)$ *are isomorphic* (*quaternion*) *algebras.*

(4) $l(a_1)l(a_2) = l(b_1)l(b_2)$ *in* k_2F.

This theorem shows that $l(a_1)l(a_2) \in k_2F$ or the quaternion algebra

$$\left(\frac{a_1, a_2}{F}\right)$$

(in the Brauer group of F) gives a complete invariant for the isometry class of a 2-fold Pfister form $\langle\langle -a_1, -a_2\rangle\rangle$. One therefore naturally asks if an analogous result will hold for the isometry class of an

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n-fold Pfister form. Motivated by this, one is led to the following definition. Let $\langle\langle a_1, \cdots, a_n\rangle\rangle$ and $\langle\langle b_1, \cdots, b_n\rangle\rangle$ be two n-fold Pfister forms ($n \geqq 2$). We shall say that they are *simply-p-equivalent* if there exist two distinct indices i and j such that

(a) $\langle\langle a_i, a_j\rangle\rangle \cong \langle\langle b_i, b_j\rangle\rangle$ (see Theorem 1 for equivalent conditions), and

(b) $a_k = b_k$ whenever k is different from i and j.

More generally, we say that two n-fold Pfister forms ϕ and ϕ' are *chain-p-equivalent* if there exists a sequence of n-fold Pfister forms $\phi_0, \phi_1, \cdots, \phi_m$ such that $\phi_0 = \phi$, $\phi_m = \phi'$ and that each ϕ_i is simply-p-equivalent to ϕ_{i+1} ($0 \leqq i \leqq m-1$). This is clearly an equivalence relation on all n-fold Pfister forms, and Theorem 1 shows if $\langle\langle -a_1, \cdots, -a_n\rangle\rangle$ is chain-p-equivalent to $\langle\langle -b_1, \cdots, -b_n\rangle\rangle$, then $l(a_1) \cdots l(a_n)$ equals $l(b_1) \cdots l(b_n)$ in k_nF. Further, it turns out that *chain-p-equivalence* of n-fold Pfister forms coincides with ordinary *isometry* of such forms. This may be viewed as an analog of Witt's classical chain equivalence theorem [5].

Theorem 1 now generalizes as follows:

THEOREM 2. *The following statements are equivalent:*

(1) $\langle\langle -a_1, \cdots, -a_n\rangle\rangle$ *and* $\langle\langle -b_1, \cdots, -b_n\rangle\rangle$ *are chain-p-equivalent.*

(2) $l(a_1) \cdots l(a_n) = l(b_1) \cdots l(b_n)$ *in* k_nF.

(3) $\langle\langle -a_1, \cdots, -a_n\rangle\rangle \equiv \langle\langle -b_1, \cdots, -b_n\rangle\rangle \pmod{I^{n+1}F}$.

(4) $\langle\langle -a_1, \cdots, -a_n\rangle\rangle$ *and* $\langle\langle -b_1, \cdots, -b_n\rangle\rangle$ *are isometric.*

(5) *There exist nonzero elements a and b in F such that* $\langle a\rangle \otimes \langle\langle -a_1, \cdots, -a_n\rangle\rangle$ *is isometric to* $\langle b\rangle \otimes \langle\langle -b_1, \cdots, -b_n\rangle\rangle$.

In particular $l(a_1)l(a_2) \cdots l(a_n) \in k_nF$ is a complete invariant for the isometry class of a Pfister form $\langle\langle -a_1, \cdots, -a_n\rangle\rangle$.

COROLLARY 1. $\langle\langle -a_1, \cdots, -a_n\rangle\rangle$ *is hyperbolic if and only if* $l(a_1) \cdots l(a_n) = 0$ *in* k_nF. *In particular* $k_nF = 0$ *if and only if* $I^nF = 0$, *if and only if every n-fold Pfister form is hyperbolic.*

COROLLARY 2. *If the level of F is* $s = 2^m$, *then m is precisely the smallest integer such that* $l(-1)^{m+1} = 0$ *in* $k_{m+1}F$. (*This refines Milnor's result in* [3, *Theorem* 1.4].) *Furthermore,* $\dim_{Z_2} k_{m-r+1} F \geqq r(r+1)/2$ $(1 \leqq r \leqq m)$, *and* $|W(F)| \geqq 2 \cdot 2^{m(m+1)(m+2)/6}$.

Theorem 2 is proved by showing the chain of implications (1)$\Rightarrow$(2) $\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(1) ((4) and (5) are easily seen to be equivalent). The implication (3)$\Rightarrow$(4) is proved by extending the techniques of Arason and Pfister [1], while (4)⇒(1) is based on the following Proposition.

PROPOSITION. *If* $\tau = \langle\langle a_1, \cdots, a_r\rangle\rangle$ *and* $\sigma = \langle\langle b_1, \cdots, b_s\rangle\rangle = \langle 1\rangle \perp \sigma'$, *and if* $c_1 \neq 0$ *is represented by the form* $\tau \otimes \sigma'$, *then there exist nonzero* $c_2, \cdots, c_s$ *in* F *such that* $= \langle\langle a_1, \cdots, a_r, b_1, \cdots, b_s\rangle\rangle$ *is chain-p-equivalent to* $\langle\langle a_1, \cdots, a_r, c_1, \cdots, c_s\rangle\rangle$.

Note that the Proposition also gives a proof of the following well-known fact: *if a k-fold Pfister form* ϕ_1 *is a subform of an n-fold Pfister form* γ, *then there exists an* $(n-k)$*-fold Pfister form* ϕ_2 *such that* $\gamma \cong \phi_1 \otimes \phi_2$. The known proof of this utilizes Pfister's theory of strongly multiplicative forms (see [4]).

The Proposition also yields the following:

THEOREM 3. *Suppose* ϕ, τ *are n-fold Pfister forms. Then* $\gamma = \phi \perp \langle -1\rangle \tau$ *contains a hyperbolic form* $2^r\boldsymbol{H}$ ($r \leqq n$ *and* $\boldsymbol{H}$ *denotes the hyperbolic plane* $\langle 1, -1\rangle$) *if and only if there exist an r-fold Pfister form* σ *and two* $(n-r)$*-fold Pfister forms* ϕ_0 *and* τ_0 *such that* $\phi \cong \sigma \otimes \phi_0$ *and* $\tau \cong \sigma \otimes \tau_0$. *Furthermore the Witt index of* γ *is a 2-power.*

COROLLARY 1. *Suppose every* 2^n*-dimensional form over* F *is universal. Then*

(1) $I^{n+1}F = 0$;

(2) s_n *is an isomorphism from* k_nF *to* I^nF;

(3) *every element of* I^nF *is represented by an n-fold Pfister form*;

(4) *every pair of n-fold Pfister forms* ϕ, τ *can be written as* $\sigma \otimes \phi_0$ *and* $\sigma \otimes \tau_0$, *where* σ *is an* $(n-1)$*-fold Pfister form, and* ϕ_0, τ_0 *are* 1*-fold Pfister forms.*

COROLLARY 2. *If* F *is a* C_3*-field (e.g. any function field of transcendence degree* 3 *over the complex numbers), then for all* n, $s_n: k_n \rightarrow I^n/I^{n+1}$ *is an isomorphism for* F, *and hence also for* $F(X)$, *by* [3, *Corollary* 5.8].

Using Theorems 2 and 3, we also obtain the following:

THEOREM 4. *Let* $\phi_i = \langle\langle a_{i1}, \cdots, a_{in}\rangle\rangle$ $(i = 1, 2, 3)$ *be three n-fold Pfister forms. If* $\phi_1 \perp \phi_2 \perp \phi_3 \in I^{n+1}(F)$, *then there exist an* $(n-1)$*-fold Pfister form* τ *and nonzero elements* x, y *in* F, *such that*

$$\phi_1 \cong \tau \otimes \langle -xy, 1\rangle, \quad \phi_2 \cong \tau \otimes \langle x, 1\rangle, \quad \text{and} \quad \phi_3 \cong \tau \otimes \langle y, 1\rangle.$$

In particular, there is an isometry $\langle -y\rangle \phi_1 \perp \phi_3 \cong \phi_2 \perp 2^{n-1}\boldsymbol{H}$. *Also, the summation* $\sum_{i=1}^{3} l(-a_{i1}) \cdots l(-a_{in})$ *vanishes in* k_nF.

COROLLARY 1. *If every element in* k_nF *can be expressed as a sum of at most three generators, then* $s_n: k_nF \rightarrow I^nF/I^{n+1}F$ *is an isomorphism. In particular, if* $|k_nF| \leqq 64$, *then* s_n *is an isomorphism.*

COROLLARY 2. *Let n be a fixed integer, and suppose that every element of k_nF is equal to a generator, then every element in k_mF ($m \geqq n$) is also equal to a generator of k_mF. In particular, $s_m: k_mF \to I^mF/I^{m+1}F$ is an isomorphism for all $m \geqq n$. Furthermore, given any pair of m-fold Pfister forms ϕ and τ ($m \geqq n$), there exist an $(m-1)$-fold Pfister form σ and nonzero elements x, y such that $\phi \cong \sigma \otimes \langle x, 1\rangle$ and $\tau \cong \sigma \otimes \langle y, 1\rangle$.*

We will demonstrate the usefulness of Corollary 2 by two examples:

EXAMPLE 1. Suppose k_2F has at most four elements. Then one can show: every element of k_2F is equal to a generator (in which case Corollary 2 applies).

EXAMPLE 2. Let F be a global field. Then one can show that every element of k_3F is equal to a generator (see [**3**, Appendix], or [**2**]). Hence the corollary also applies in this case. Actually, every element of k_2F is equal to a generator.

In these and some other examples, we have been able to determine all relations among Pfister forms.

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