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QUOTIENTS OF FINITE W^*-ALGEBRAS[1]

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1. In this note we present results concerning the following problem. Suppose M is a W^*-algebra and $J \subset M$ a uniformly closed two-sided ideal. Then the quotient algebra M/J is a C^*-algebra, and the problem is: What are the conditions that M/J be a W^*-algebra?

Since we can write M as a direct sum of a finite and a properly infinite W^*-algebra, we can discuss the two cases separately. In [3] and [4] Takemoto solved the problem for a properly infinite W^*-algebra, that can be represented on a separable space. His theorem states that M/J is a W^*-algebra, if and only if J is ultra-weakly closed.

2. If M is finite the situation is quite different. There are "many" non-ultra-weakly closed ideals J for which the quotient M/J is a W^*-algebra. Indeed, Wright [5] and Feldman [1] proved that if J is a maximal ideal, M/J is a finite factor. This result was proved by a different method by Sakai in [2]. The following theorem generalizes that result.

THEOREM 1. *Let M be a finite and σ-finite W^*-algebra with center Z. Let J be a uniformly closed two-sided ideal satisfying the following conditions:*

(i) *J is an intersection of maximal ideals,*
(ii) *$Z/Z \cap J$ is a W^*-algebra,*
(iii) *$Z/Z \cap J$ is σ-finite.*

Then M/J is a W^*-algebra.

As a partial converse we have

THEOREM 2. *If J is a uniformly closed two-sided ideal of the finite and σ-finite W^*-algebra M and M/J is a W^*-algebra, then J satisfies the conditions* (i) *and* (ii) *of Theorem* 1.

REMARK. If we assume that M can be represented on a separable

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[1] The results are contained in the author's doctoral dissertation at the University of Pennsylvania, and his supervisor is S. Sakai. The author is a research fellow at the University of Aarhus, Denmark.

space, and if we further assume the continuum hypothesis, the condition (iii) becomes necessary for M/J to be a W^*-algebra. Thus, under these conditions, (i), (ii), and (iii) are necessary and sufficient for M/J to be a W^*-algebra.

3. **Outline of proofs.** *The necessity.* If M/J is a W^*-algebra, it is necessarily a finite W^*-algebra. Since the intersection of maximal ideals in the finite W^*-algebra M/J is $\{0\}$, J is an intersection of maximal ideals. Moreover, $Z/Z\cap J$ is isomorphic to the center of M/J, and is therefore a W^*-algebra. Under the continuum hypothesis, a simple cardinality argument shows, that $Z/Z\cap J$ must be σ-finite, if we also assume that M can be represented on a separable space.

The sufficiency. Suppose that (i), (ii), and (iii) are satisfied. We consider the Banach Z-module $\mathfrak{L}$ generated by the maps $x\in M \to (ax)^\# \in Z$, where $a\in M$. $\#$ is the canonical center valued trace on M. Since J is an intersection of maximal ideals, it is invariant under $\mathfrak{L}$, and we can factor $\mathfrak{L}$ to $\tilde{\mathfrak{L}}$, consisting of linear maps $M/J\to Z/Z\cap J$. By conditions (ii) and (iii) there is a normal faithful state μ on $Z/Z\cap J$. Let F be the set of all linear functionals of the form $\mu\circ\Phi$, where $\Phi\in\tilde{\mathfrak{L}}$. By Sakai's criterion it suffices to prove that M/J is a dual space of F. Let E be the completion of F. By a theorem in [2], it suffices to prove that for every $f\in E$, there is an $x\in M/J$ with $\|x\|=1$ and $f(x)=\|f\|$. By means of polar decomposition for elements of $\tilde{\mathfrak{L}}$ this is easily proved, if $f\in F$. In order to extend this result to E, we decompose the functionals in E over $Z/Z\cap J$, and by applying a technique similar to that of standard measure theory, we obtain the desired result for $f\in E$.

4. Thus the problem of finding the W^*-quotients of the finite W^*-algebra M (which is supposed to act on a separable space), is reduced to the corresponding problem for the center Z. Since we assume that M acts on a separable space we may assume that $Z=L_m^\infty(I)$, the space of essentially bounded Lebesgue measurable functions on the unit interval, or $Z=l^\infty(N)$, the set of bounded complex sequences.

THEOREM 3. *There exist countably many surjective nonnormal *-homomorphisms $\Phi: Z_1\to Z_2$ with pairwise different kernels in each of the following three cases:*

(i) $Z_1=L_m^\infty(I)$, $Z_2=L_m^\infty(I)$,
(ii) $Z_1=L_m^\infty(I)$, $Z_2=l^\infty(N)$,
(iii) $Z_1=l^\infty(N)$, $Z_2=l^\infty(N)$.

Thus we see that in general there are many non-ultra-weakly closed ideals I for which Z/I is a W^*-algebra. To give nontrivial examples in the noncommutative case we need only consider the finite W^*-algebra $F\otimes Z$ where F is a finite factor. The center of $F\otimes Z$ is Z, and if $I\subset Z$ is an ideal, let J be the ideal of M which is the intersection of the maximal ideals that contain I. If Z/I is a σ-finite W^*-algebra, then so is $F\otimes Z/J$, and J is ultra-weakly closed if and only if I is.

5. As to the existence of non W^*-quotients we have the following theorem:

THEOREM 4. *Let Z be an infinite dimensional abelian W^*-algebra, which can be represented on a separable space. Then Z admits a C^*-quotient, which is not a W^*-algebra.*

PROOF. If $Z=l^\infty(N)$, $l^\infty(N)/c_0(N)$ will do. If $Z=L^\infty_m(I)$ we apply Theorem 3, part (ii).

THEOREM 5. (i) *There exists a pair J, M such that the center of M/J is a σ-finite W^*-algebra, but J is not an intersection of maximal ideals.*

(ii) *There exists a pair J, M such that J is an intersection of maximal ideals, but the center of (M/J) is not a W^*-algebra.*

In the proof of Theorem 5 (ii) we apply Theorem 4. To prove Theorem 5 (i) we need the following considerations. Let M be a finite W^*-algebra. The maximal ideal space $\mathrm{Max}(M)$ considered as a subset of the primitive ideal space $\mathrm{Prim}(M)$ is homeomorphic to $\mathrm{Max}(Z)$. Let $x(J)$ be the image of $x\in M$ in M/J under $M\to M/J$, if $J\in\mathrm{Max}(M)$. Then the following is applicable in the proof of Theorem 5 (i).

THEOREM 6. *Let $J_0\in\mathrm{Max}(M)$. Then the following two conditions are equivalent*:

(i) *There is no primitive ideal J such that $J\neq J_0$ and $J\cap Z=J_0\cap Z$.*

(ii) *The functions $J\in\mathrm{Max}(M)\to\|x(J)\|$, $x\in M$, are continuous at J_0.*

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Detailed proofs will appear elsewhere.

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