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FINITELY GENERATED NILPOTENT GROUPS WITH ISOMORPHIC FINITE QUOTIENTS

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Let G be a finitely generated nilpotent group and let $\mathfrak{F}(G)$ denote the set of isomorphism classes of finite homomorphic images of G. If H is another finitely generated nilpotent group, we will say that G and H have isomorphic finite quotients if $\mathfrak{F}(G)=\mathfrak{F}(H)$. The finite quotients of a finitely generated nilpotent group provide much information about the structure of the group [3], [5], although they do not determine the group up to isomorphism [8, and G. Higman, unpublished]. The following result shows, however, that a finitely generated nilpotent group is determined to a large extent by its finite quotients.

THEOREM. *Let G be a finitely generated nilpotent group. Then the finitely generated nilpotent groups H, for which $\mathfrak{F}(G)=\mathfrak{F}(H)$, lie in only finitely many isomorphism classes.*

This theorem, which is a much stronger version of an unpublished result of A. Borel, is proved by using the Lie algebras of the respective nilpotent groups [6], [7] to apply some finiteness results for arithmetic subgroups of algebraic groups [4], a technique introduced by Auslander and Baumslag [1], [2].

OUTLINE OF THE PROOF. We first give some necessary notation and state a few fundamental lemmas. If G is a finitely generated nilpotent group, we can define a p-adic topology on G for which a neighborhood basis of the identity is given by the groups $G^{p^i}=\text{gp}\{x^{p^i}\mid x\in G\}$. The completion of G in this topology will be denoted Z_pG. The connection between these completions and finite quotients is given by the following lemma of Borel:

LEMMA 1. *If G and H are finitely generated nilpotent groups, then $\mathfrak{F}(G)=\mathfrak{F}(H)$ iff Z_pG and Z_pH are isomorphic for each prime p.*

If N is a subgroup of G, then Z_pN may be considered to be a sub-

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group of Z_pG and if N is normal in G, Z_pN is normal in Z_pG and $Z_p(G/N)\cong Z_pG/Z_pN$. Using this and Lemma 1, it is easy to prove the following lemma which allows us to reduce to the torsion free case.

LEMMA 2. *If two finitely generated nilpotent groups G and H have isomorphic finite quotients, then their torsion subgroups τG and τH are isomorphic and the respective torsion free factor groups $G^*=G/\tau G$ and $H^*=H/\tau H$ have isomorphic finite quotients.*

If G is a torsion free finitely generated nilpotent group (an N-group for short), then Z_pG is easily seen to be torsion free so we may form the Mal′cev completions QG of G and Q_pG of Z_pG [11, p. 256]. G and H are commensurable N-groups if and only if QG and QH are isomorphic [7]. It is not too hard to check that if ϕ is an isomorphism of G onto H, ϕ extends uniquely to isomorphisms $\phi_p: Z_pG\to Z_pH$ and $\bar{\phi}: QG\to QH$. Also isomorphisms $\phi_p: Z_pG\to Z_pH$ extend uniquely to isomorphisms $\phi_p: Q_pG\to Q_pH$ and an isomorphism $\psi: QG\to QH$ extends uniquely to isomorphisms $\psi_p: Q_pG\to Q_pH$. We then may show:

LEMMA 3. *If G and H are N-groups and $\psi: QG\to QH$ is an isomorphism, then for all but a finite number of primes p, the extension $\psi: Q_pG\to Q_pH$ sends Z_pG isomorphically onto Z_pH.*

If H is a subgroup of finite index in an N-group G, it is easy to see that Z_pH is of finite index in Z_pG. This fact is used in a rather long argument following [9] to show:

LEMMA 4. *If H is a subgroup of finite index in an N-group G, then there is a subgroup K of* $\mathrm{Aut}(Q_pG)$ *of finite index in each of* $\mathrm{stab}(Z_pG, \mathrm{Aut}(Q_pG))$ *and* $\mathrm{stab}(Z_pH, \mathrm{Aut}(Q_pG))$.

If R is the rational group ring of an N-group G and B is its augmentation ideal, B is residually nilpotent [6] so we may form the completion $R^\wedge$ of R in a Hausdorff B-adic topology. If x is an element of $B^\wedge$, the usual power series for $\exp(x)$ and $\log(1+x)$ converge in $R^\wedge$ giving inverse maps between the sets $B^\wedge$ and $1+B^\wedge$. Since G is contained in $1+B^\wedge$, we may consider the Q-vector space Λ spanned by $\log(G)$ in $B^\wedge$. Λ is a finite dimensional nilpotent Lie subalgebra of the commutation Lie algebra on R [6]. We call Λ the Lie algebra of G. If $\log(G)$ is an additive lattice in Λ, we say G is lattice nilpotent [12].

If Γ is any nilpotent Lie algebra over a field F of characteristic zero, we may define a multiplication $*$ on Γ using the Baker-Campbell-Hausdorff formula so that the group $(\Gamma, *)$ is a nilpotent group admitting the action of F (as a set) [6]. If Λ is the Lie algebra of the N-group G, exp gives an isomorphism of the subgroup $(\log G, *)$ with

G and exp takes $(\Lambda, *)$ isomorphically onto a Mal'cev completion of G contained in $R^{\wedge}$ [6], [2]. In general, if Λ and Γ are nilpotent Lie algebras over a field F (char 0) and ϕ is a bijective map of F-sets, a tedious double induction shows that ϕ is a Lie algebra isomorphism iff ϕ is a group theoretic isomorphism of $(\Lambda, *)$ onto $(\Gamma, *)$. Using this we may prove the following crucial propositions.

PROPOSITION 5 (BOREL). *Suppose G and H are N-groups with Lie algebras Γ and Λ respectively. Then $QG \cong QH$ iff $\Gamma \cong \Lambda$ and $Q_pG \cong Q_pH$ iff $Q_p \otimes_Q \Gamma \cong Q_p \otimes_Q \Lambda$.*

PROPOSITION 6. *If G is a lattice nilpotent group, there is an algebraic matric group $\mathfrak{G}$ [4, p. 10] and, for each prime p, isomorphisms δ_p:* $\mathrm{Aut}(Q_pG) \to \mathfrak{G}_{Q_p}$ *which take* $\mathrm{stab}(Z_pG, \mathrm{Aut}(Q_pG))$ *to* $\mathfrak{G}_{Z_p}$, $\mathrm{stab}(QG, \mathrm{Aut}(Q_pG))$ *to* $\mathfrak{G}_Q$ *and* $\mathrm{stab}(G, \mathrm{Aut}\ Q_pG)$ *to* $\mathfrak{G}_Z$ *in such a way that for ϕ in* $\mathrm{Aut}(QG)$, *$\delta_p(\phi)$ is independent of p in $\mathfrak{G}_Q$. ($\mathfrak{G}_Q$ is* $\mathrm{Aut}(\Lambda)$.)

The following unpublished theorem of A. Borel, which was the major motivation for this work, may be proved using Lemma 1, Proposition 5, and Theorem 7.11 of [10].

THEOREM 7 (BOREL). *Let G be an N-group. Then the N-groups H for which $\mathfrak{F}(G) = \mathfrak{F}(H)$ are contained in finitely many commensurability classes.*

For any N-group G, we define $\mathcal{G}_A$ to be the subgroup of $\Pi_p \mathrm{Aut}(Q_pG)$ consisting of elements $\Pi(\alpha_p)$ for which $\alpha_p \in \mathrm{stab}(Z_pG, \mathrm{Aut}(Q_pG))$ for all but a finite number of primes p. We define $\mathcal{G}_A^\infty$ to be the subgroup of $\mathcal{G}_A$ consisting of elements $\Pi(\alpha_p)$ such that $\alpha_p \in \mathrm{stab}(Z_pG, \mathrm{Aut}(Q_pG))$ for all primes p. By Lemma 3 we may embed $\mathrm{Aut}(QG)$ diagonally as a subgroup $\mathcal{G}_Q$ of $\mathcal{G}_A$.

If H and G are N-groups, we say H is in the genus of G if $\mathfrak{F}(G) = \mathfrak{F}(H)$ and H is commensurable with G. If so, we have isomorphisms $\phi_p: Z_pG \to Z_pH$ and $\psi: QG \to QH$ which extend to $\bar{\phi}_p$, $\psi_p: Q_pG \to Q_pH$. By Lemma 3, $\Pi(\bar{\phi}_p^{-1} \circ \psi_p)$ is in $\mathcal{G}_A$. Using this construction and following the idea of Proposition 2.3 of [4] we may show:

PROPOSITION 8. *The isomorphism classes in the genus of an N-group G are in 1-1 correspondence with a subset of the set of double cosets $\mathcal{G}_A^\infty \backslash \mathcal{G}_A / \mathcal{G}_Q$.*

If G is lattice nilpotent, Proposition 6 and Theorem 5.1 of [4] imply that the number of double cosets is finite. If not, G is a subgroup of finite index in a lattice nilpotent group H [12, Theorem 2]. We then have $\mathcal{G}_A = \mathcal{H}_A$, $\mathcal{G}_Q = \mathcal{H}_Q$ and by Lemmas 3 and 4, $\mathcal{G}_A^\infty$ and $\mathcal{H}_A^\infty$ are commensurable. Thus in this case the number of double cosets is again finite so we have:

THEOREM 9. *There are only finitely many isomorphism classes of N-groups in the genus of an N-group G.*

The main theorem then follows from Lemma 2, Theorem 7 and Theorem 9.

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