# AN $n$-DIMENSIONAL EXTENSION OF PICARD'S THEOREM

BY H. WU[1]

Communicated by M. H. Protter, March 10, 1969

We will prove the following theorem:

*For each positive integer $n$, let*

$$\rho(n) = \left(\frac{n}{2}+1\right)^2 + 1 \qquad \textit{if } n \textit{ is even,}$$

$$= \left(\frac{n+1}{2}\right)\left(\frac{n+3}{2}\right) + 1 \quad \textit{if } n \textit{ is odd.}$$

*Let $C^m$ be the $m$-dimensional complex euclidean space and $P_nC$ the $n$-dimensional complex projective space. Then every holomorphic mapping $x:C^m \rightarrow P_nC$ such that $x(C^m)$ omits $\rho(n)$ hyperplanes in general position must reduce to a constant.*

Note that $\rho(1)=3$ and $P_1C$ is just the Riemann sphere. So when $m=n=1$, this is exactly the classical theorem of Emile Picard. The first few values of $\rho$ are: $\rho(2)=5, \rho(3)=7, \rho(4)=10, \rho(5)=13, \rho(6)=17, \rho(7)=21, \rho(8)=26, \rho(9)=31, \rho(10)=37$.

Recently, Kobayashi introduced the notion of a hyperbolic (complex) manifold, [3], [4] (cf. also Definition 1.3 of [6]). A hyperbolic manifold has most of the useful analytic properties of a bounded domain in $C^n$, including the fact that every holomorphic mapping of $C$ into it reduces to a constant. There is the natural question of obtaining hyperbolic manifolds by removing suitable subsets of $P_nC$. P. A. Griffiths suggested that $P_nC$ minus a *singular* hypersurface of a very high degree might be hyperbolic. On the other hand, P. J. Kiernan has proved in [2] that $P_nC$ minus $2n$ hyperplanes in general position is never hyperbolic. The above result suggests the

CONJECTURE. $P_nC$ minus $\rho(n)$ hyperplanes in general position is hyperbolic.

The determination of the smallest value of $\rho(n)$ for which the above theorem and conjecture remain valid is probably a difficult problem.

The Picard theorem may be generalized in yet another way. We formulate a second

[1] Research partially supported by the National Science Foundation. I wish to thank Professor S. S. Chern for his encouragement.

CONJECTURE. If $x$: $C^n \to P_n C$ is a holomorphic mapping whose differential is nonsingular somewhere, then $x(C^n)$ must intersect one of $(n+2)$ hyperplanes in general position.

This conjecture has been verified for $n \leqq 4$. See [5, Chapter V, §5].

PROOF OF THE THEOREM. We shall make use of the following lemma whose proof is given at the end.

LEMMA. *Let $H_1, \cdots, H_{\rho(n)}$ be $\rho(n)$ hyperplanes in $P_n C$ in general position, and let $A^k$ be a $k$-dimensional projective subspace of $P_n C$ not contained in any of $H_1, \cdots, H_{\rho(n)}$, $(1 \leqq k \leqq n-1)$. Then the set $\{A^k \cap H_i: i=1, \cdots, \rho(n)\}$ contains at least $(k+2)$ hyperplanes of $A^k$ in general position.*

We will also need the following well-known consequence of the defect relations of Ahlfors ([1] and [5, Chapter V, §5]): If $x: C \to P_n C$ is a holomorphic mapping which avoids $(n+2)$ hyperplanes of $P_n C$ in general position, then $x(C)$ lies in a proper projective subspace of $P_n C$.

Now it suffices to prove the theorem for a holomorphic mapping $x$: $C \to P_n C$. Since by assumption, $x(C)$ avoids $\rho(n)$ $(\geqq n+2)$ hyperplanes in general position, $x(C)$ must lie in a proper projective subspace of $P_n C$. If we suppose $x$ is not constant, then there is a $k$-dimensional projective subspace $A^k$, $1 \leqq k \leqq n-1$, such that $x(C)$ is contained in $A^k$, but not in a proper projective subspace of $A^k$. So $x$: $C \to A^k$ is holomorphic and avoids the following hyperplanes of $A^k$: $A^k \cap H_1, \cdots, A^k \cap H_{\rho(n)}$. But there are at least $(k+2)$ of the latter in general position, and so the contradiction completes the proof.

It remains to prove the lemma. We lift the problem to $C^{n+1}$, where it assumes this form: for $1 \leqq k \leqq n-1$, let $E^{k+1}$ be a vector subspace of $C^{n+1}$ of dimension $k+1$, and let $H_1, \cdots, H_{\rho(n)}$ be $\rho(n)$ hyperplanes of $C^{n+1}$ (=vector subspaces of codimension one) in general position. Then $\{E^{k+1} \cap H_i: i=1, \cdots, \rho(n)\}$ contains at least $(k+2)$ hyperplanes of $E^{k+1}$ in general position.

In the dual space $(C^{n+1})^*$ of $C^{n+1}$, $E^{k+1}$ corresponds to a vector subspace $F$ of dimension $n-k$ (i.e. $F$ is the set of $f$ such that kernel $f \supseteq E^{k+1}$), and each $H_i$ corresponds to a nonzero vector $P_i$ (i.e. kernel $P_i = H_i$). $F$ contains none of the $P_i$'s because $H_i$ does not contain $E^{k+1}$ for each $i$. Furthermore, the fact that $H_1, \cdots, H_{\rho(n)}$ are in general position is equivalent to saying that any $(n+1)$ of $P_1, \cdots, P_{\rho(n)}$ are linearly independent. Choose a basis $Q_1, \cdots, Q_{n-k}$ of $F$. Since no $P_i$ is in $F$, $\{Q_1, \cdots, Q_{n-k}, P_1\}$ is a linearly independent set. Suppose $P_1, \cdots, P_l$ $(l \leqq k)$ have been so chosen that $\{Q_1, \cdots,$

$Q_{n-k}, P_1, \cdots, P_l\}$ is a linearly independent set; the linear span of the latter is of dimension $n-k+l \leqq n$ and so contains at most $n$ of $\{P_1, \cdots, P_{\rho(n)}\}$. Since $\rho(n) \geqq n+2$, we can assume that

$$P_{l+1} \notin \operatorname{span}\{Q_1, \cdots, Q_{n-k}, P_1, \cdots, P_l\}.$$

Hence $\{Q_1, \cdots, Q_{n-k}, P_1, \cdots, P_{l+1}\}$ is a linearly independent set. In this way, we can choose a basis $\{Q_1, \cdots, Q_{n-k}, P_1, \cdots, P_{k+1}\}$ of $(\mathbf{C}^{n+1})^*$.

Relative to this basis, the coordinates of $Q_1, \cdots, Q_{n-k}, P_1, \cdots, P_{k+1}$ are of course:

$$\begin{aligned} Q_1 &= (1, 0, \cdots\cdots\cdots, 0) \\ &\cdots\cdots\cdots\cdots\cdots \\ Q_{n-k} &= (0, \cdots, 0, 1, 0, \cdots, 0) \\ P_1 &= (0, \cdots\cdots, 0, 1, 0, \cdots, 0) \\ &\cdots\cdots\cdots\cdots\cdots \\ P_{k+1} &= (0, \cdots\cdots\cdots, 0, 1). \end{aligned}$$

Now let $P=(a_0, \cdots, a_n)$ be the coordinates of a hyperplane of $\mathbf{C}^{n+1}$ relative to $\{Q_1, \cdots, Q_{n-k}, P_1, \cdots, P_{k+1}\}$. If this hyperplane does not contain $E^{k+1}$, then its intersection with $E^{k+1}$ is a hyperplane of $E^{k+1}$ whose coordinates relative to $\{P_1, \cdots, P_{k+1}\}$ are obviously $P'=(a_{n-k}, \cdots, a_n)$. Let us fix this notation of passing from $P$ to $P'$. So if we write for each $i \in \{k+2, \cdots, \rho(n)\}$,

$$P_i = (a_{i0}, \cdots, a_{in})$$

relative to $\{Q_1, \cdots, Q_{n-k}, P_1, \cdots, P_{k+1}\}$, the coordinates of the hyperplane of $E^{k+1}$ given by $H_i \cap E^{k+1}$ are

$$P_i' = (a_{i,n-k}, \cdots, a_{in})$$

relative to $\{P_1, \cdots, P_{k+1}\}$. Now consider the matrix:

$$A' = \begin{bmatrix} a_{k+2,n-k} & \cdots & a_{k+2,n} \\ \cdots & \cdots & \cdots \\ a_{\rho(n),n-k} & \cdots & a_{\rho(n),n} \end{bmatrix}$$

whose rows are just the coordinates of $P'_{k+2}, \cdots, P'_{\rho(n)}$. We claim that there is at least one row such that none of its entries is zero. If we suppose not, then there is at least one zero in each row. Let us pick out exactly one zero from each row. We have then to consider the distribution of these $\{\rho(n)-(k+1)\}$ zeros in the $(k+1)$ columns of this matrix. We will show presently that at least one column has at least $(n-k+1)$ zeros. Assuming this for the moment, we may as well let the first column contain at least $(n-k+1)$ zeros. This means that there are at least $(n-k+1)$ of $P'_{k+2}, \cdots, P'_{\rho(n)}$ whose first coordinates are zero. But the first coordinates of $P'_2, \cdots, P'_{k+1}$ are also zero. So there are at least $(n+1)$ of $P'_2, \cdots, P'_{\rho(n)}$ whose first coordinates are zero, or equivalently, there are at least $(n+1)$ of $P_2, \cdots, P_{\rho(n)}$ in the hyperplane of $(C^{n+1})^*$ defined by setting the $(n-k+1)$th coordinate equal to zero. This contradicts the fact that any $(n+1)$ of $P_1, \cdots, P_{\rho(n)}$ are linearly independent. Hence we may assume that $P'_{k+2}=(a_{k+2,n-k}, \cdots, a_{k+2,n})$ has the property that none of its coordinates is equal to zero. This implies that $H_1 \cap E^{k+1}, \cdots, H_{k+2} \cap E^{k+1}$ are $(k+2)$ hyperplanes of $E^{k+1}$ in general position because now they have these coordinates relative to $\{P_1, \cdots, P_{k+1}\}$:

$$\begin{gathered}(1, 0, \cdots\cdots, 0)\\ \cdots\cdots\cdots\\ (0, \cdots\cdots, 0, 1)\\ (a_{k+2,n-k}, \cdots, a_{k+2,n}).\end{gathered}$$

It remains to show that at least one column of $A'$ has at least $(n-k+1)$ zeros if every row of $A'$ has a zero. If not, each column has at most $(n-k)$ zeros. So there are altogether at most $(n-k)(k+1)$ such zeros in $A'$. Since we chose exactly one zero from each row of $A'$, there are at most $(n-k)(k+1)$ rows of $A'$ or, what is the same thing, $\{P_{k+2}, \cdots, P_{\rho(n)}\}$ contains at most $(n-k)(k+1)$ elements. Hence

$$\rho(n) \leqq (n-k)(k+1)+(k+1)=(k+1)(n-k+1).$$

We now show that this is impossible.

Consider the function $f\colon [1, n-1] \to R$ such that

$$f(x)=(x+1)(n-x+1)+1.$$

Clearly, $f'(x)=n-2x$ so that the maximum of $f$ is assumed at $x=n/2$. We consider two cases separately.

*Case* 1. $n$ is even. Let $n=2p$. Then

$$f(n/2)=f(p)=(p+1)^2+1=\rho(n).$$

Hence $\rho(n) \geqq (n-k+1)(k+1)+1>(n-k+1)(k+1)$ for any integer $k$ such that $1 \leqq k \leqq n-1$. Contradiction.

*Case* 2. $n$ is odd. Let $n=2p+1$, then $n/2=p+\frac{1}{2}$. Now note that $f'>0$ in $[1, n/2)$ so that $f$ is strictly increasing in $[1, n/2)$. Since $p$ is the largest integer in $[1, n/2)$, $f(p) \geqq f(k)$ for any integer $k \leqq p$.

Similarly, $f'<0$ on $(n/2, n-1]$, so that since $(p+1)$ is the smallest integer in $(n/2, n-1]$, $f(p+1)\geqq f(k)$ for all integers $k\geqq p+1$. One checks easily that in this case, $f(p)=f(p+1)=(p+1)(p+2)+1=\rho(n)$. Hence $\rho(n)\geqq f(k)$ for all integers $k$ such that $1\leqq k\leqq n-1$. So

$$\rho(n) \geqq (n-k+1)(k+1)+1 > (n-k+1)(k+1).$$

Again a contradiction.

NOTE. After the completion of the paper, R. Osserman pointed out that the "well-known consequence of the defect relations of Ahlfors" above should be properly attributed to E. Borel (Acta Math. **20** (1897), 357–396). He further called my attention to H. Cartan's extension of Borel's theorem to holomorphic maps in the neighborhood of an isolated singularity. (Ann. École Norm. Sup. **45** (1928), 255–346). Cartan's result together with the lemma above lead to the following: given a holomorphic map of a punctured disk into $P_nC$, either it extends to a holomorphic map in the full disk, or else the image intersects infinitely often every hyperplane with at most $\rho(n)-1$ exceptions. It should be noted that Cartan's theorem is also a consequence of the Ahlfors-Weyl theory (cf. [5] and the Appendix in H. Wu, Proc. Sympos. Pure Math. Vol. XI pp. 480–532).

## REFERENCES


1. L. V. Ahlfors, *The theory of meromorphic curves*, Acta. Soc. Sci. Fenn. N. Ser. A **3** (1941), 1–31.

2. P. J. Kiernan, *Hyperbolic submanifolds of complex projective space*, Proc. Amer. Math. Soc. (to appear).

3. S. Kobayashi, *Monograph on hyperbolic manifolds*, Springer-Verlag (to appear).

4. ———, *Invariant distances on complex manifolds and holomorphic mappings*, J. Math. Soc. Japan **19** (1967), 460–480.

5. H. Wu, *The equidistribution theory of holomorphic curves*, Lecture notes, Ann. of Math. Studies, Princeton, N. J. (to appear).

6. ———, *Normal families of holomorphic mappings*, Acta Math. **119** (1967), 193–233.


UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA 94720