ON THE DECOMPOSITION OF MODULES

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Let R be a commutative ring with $1\in R$, A and R-algebra—not necessarily commutative—and let M, N be two A-left-modules. We write $N-\mathrm{rk}(M)\geqq s$, if $M\cong sN\oplus M'$ for some A-left-module M' with $s\cdot N$ short for $N\oplus N\oplus\cdots\oplus N$, s-times.

Then one can prove the following generalization of a theorem of Serre (cf. [1] or [4]).

THEOREM 1. *Assumptions.*

(i) *N is finitely presented as A-left-module,* $\mathrm{End}_A(N)$ *finitely generated as R-module and M a direct summand in a direct sum of finitely presented A-modules*;

(ii) *the maximal ideal spectrum of R is noetherian of dimension d*;

(iii) *for any maximal ideal $\mathfrak{m}$ in R we have $N_\mathfrak{m}-\mathrm{rk}(M_\mathfrak{m})\geqq d+s$ with $N_\mathfrak{m}$, resp. $M_\mathfrak{m}$ the $A_\mathfrak{m}=R_\mathfrak{m}\otimes_R A$-module $R_\mathfrak{m}\otimes_R N$, resp. $R_\mathfrak{m}\otimes M$.*

Then $N-\mathrm{rk}(M)\geqq s$.

Moreover, if R is noetherian, $\hat{R}_\mathfrak{m}$ the $\mathfrak{m}$-adic completion of R for some maximal ideal $\mathfrak{m}$ and $\hat{N}_\mathfrak{m}$, resp. $\hat{M}_\mathfrak{m}$ the $\hat{A}_\mathfrak{m}=\hat{R}_\mathfrak{m}\otimes_R A$-module $\hat{R}_\mathfrak{m}\otimes_R N$, resp. $\hat{R}_\mathfrak{m}\otimes_R M$, then

$$N_\mathfrak{m}-\mathrm{rk}(M_\mathfrak{m})\geqq d+s\Leftrightarrow \hat{N}_\mathfrak{m}-\mathrm{rk}(\hat{M}_\mathfrak{m})\geqq d+s.$$

One can also prove the following generalization of the Cancellation Theorem of Bass (cf. [1]).

THEOREM 2. *Assumptions.*

(i) *and* (ii) *as in Theorem* 1;

(iii) *M contains a direct summand P with $N-\mathrm{rk}_\mathfrak{m}(P)>d$ for all maximal ideals $\mathfrak{m}$ in R, which is a direct summand in some $s\cdot N$*;

(iv) *Q is an A-left-module, which is also a direct summand in some $s\cdot N$, and M' is some A-left-module with $Q\oplus M\cong Q\oplus M'$.*

Then $M\cong M'$.

The proof follows closely those of Serre and Bass [1], [4], once the following observations have been made:

(1) If N is any A-left-module and if $B=\mathrm{End}_A(N)$—acting from the right on N—then the contravariant functor $\mathrm{Hom}_A(\cdot, N)$ from A-left-modules to B-right-modules defines a contravariant equivalence between the category $[N]$ of those A-left-modules P, which are a direct summand in some $s\cdot N$ (and all possible A-homomorphisms as morph-

isms) and the category of finitely generated projective B-right-modules. The functor $\operatorname{Hom}_A(N, \cdot)$ defines thus an equivalence between $[N]$ and the category of finitely generated projective B-left-modules.

(2) If N is a finitely presented A-module, $\rho: R\to\hat{R}$ a ring-homomorphism of R into some commutative ring $\hat{R}$ (with $1\in\hat{R}$ and $\rho(1)=1$), such that $\hat{R}$ becomes a flat R-module, and if $\hat{M}$, resp. $\hat{N}$ stands for the $\hat{A}=\hat{R}\otimes_R A$-module $\hat{R}\otimes_R M$, then the natural homomorphism $\hat{R}\otimes_R \operatorname{Hom}_A(M, N)\to\operatorname{Hom}_{\hat{A}}(\hat{M}, \hat{N})$ is an isomorphism. (Cf. N. Bourbaki, *Algèbre commutative*, Chapter 1.)

(3) If M, N are any two A-modules and $\phi: M\to N$ an A-homomorphism, define

$$\begin{aligned} P(\phi) &= \{\phi\psi\in\operatorname{End}_A(N)\mid\psi\in\operatorname{Hom}_A(M,N)\},\\ I(\phi) &= \{\psi\phi\in\operatorname{End}_A(M)\mid\psi\in\operatorname{Hom}_A(M,N)\},\\ P_0(\phi) &= \{r\in R\mid r\cdot\operatorname{Id}_N\in P(\phi)\},\\ I_0(\phi) &= \{r\in R\mid r\cdot\operatorname{Id}_M\in I(\phi)\}.\end{aligned}$$

$P(\phi)$ is a right $\operatorname{End}_A(N)$-ideal,

$I(\phi)$ is a left $\operatorname{End}_A(M)$-ideal,

$P_0(\phi)$ and $I_0(\phi)$ are R-ideals.

The following statements follow easily from (2):
With $\rho: R\to\hat{R}$ as in (2) and $\hat{\phi}=Id_{\hat{R}}\otimes\phi:\hat{M}\to\hat{N}$ we have
If N is finitely presented, then

$$P(\hat{\phi})=\widehat{P(\phi)},\qquad P_0(\hat{\phi})=\widehat{P_0(\phi)}.$$

If M is finitely presented and N a direct summand in a direct sum of finitely presented A-modules, then

$$I(\hat{\phi})=\widehat{I(\phi)},\qquad I_0(\hat{\phi})=\widehat{I_0(\phi)}.$$

There are many interesting applications of these observations and the two theorems above. For instance, in the case where A is a separable order over a Dedekind ring R, all this specializes to something closely related to the results of Jacobinski [**2**], [**3**]. We mention some immediate consequences of (3): If $\rho: R\to\hat{R}$ is faithfully flat and N finitely presented, then $\phi: M\to N$ is split-surjective if and only if $\hat{\phi}:\hat{M}\to\hat{N}$ is split-surjective.

If N is finitely presented, then $\phi: M\to N$ is split-surjective if and only if $\phi_{\mathfrak{m}}: M_{\mathfrak{m}}\to N_{\mathfrak{m}}$ is split-surjective for all maximal ideals $\mathfrak{m}$ in R.

If M is finitely presented and N a direct summand in a direct sum of finitely presented modules, then $\phi: M\to N$ is split-injective if and only if all $\phi_{\mathfrak{m}}: M_{\mathfrak{m}}\to N_{\mathfrak{m}}$ are split-injective.

It is now not too difficult to go through with the proof of Serre [4], to get Theorem 1, whereas Theorem 2 is now a corollary to Bass's Cancellation Theorem using (1).

REFERENCES

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