REPRESENTATIONS OF COMPLEX SEMISIMPLE LIE GROUPS AND LIE ALGEBRAS[1]

BY K. R. PARTHASARATHY, R. RANGA RAO AND V. S. VARADARAJAN

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1. **Notation.** The object of this note is to announce some results on representations of complex semisimple Lie groups and Lie algebras.

$\mathfrak{G}$ is a semisimple Lie algebra over $\boldsymbol{C}$, the field of complex numbers. $\mathfrak{G}$, considered over $\boldsymbol{R}$, the field of real numbers, is denoted by $\mathfrak{G}_0$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{G}$, W, the Weyl group of $(\mathfrak{G}, \mathfrak{h})$. We use the standard terminology in the theory of semisimple Lie algebras (Jacobson [3] and Harish-Chandra [2(a)], [2(b)], [2(c)]). P_0 is a positive system of roots, fixed once for all and $S_0=\{\alpha_1, \cdots, \alpha_l\}$, the associated fundamental system, $\mathfrak{n}=\sum_{\alpha\in P_0}\mathfrak{G}^{-\alpha}$; $\mathfrak{n}$, considered as a Lie algebra over $\boldsymbol{R}$, is denoted by $\mathfrak{n}_0$. $\mathfrak{h}_0=\sum_\alpha \boldsymbol{R}\cdot H_\alpha$.

Fix a square root $(-1)^{1/2}$ of -1 in $\boldsymbol{C}$. $\mathfrak{k}_0$ is a compact form of $\mathfrak{G}$ containing $(-1)^{1/2}\mathfrak{h}_0$. $\mathfrak{G}_0=\mathfrak{k}_0+\mathfrak{h}_0+\mathfrak{n}_0$ is an Iwasawa decomposition of $\mathfrak{G}_0$ and $G=K\cdot A_+\cdot N$ the corresponding decomposition of G. $c(X\rightarrow X^c)$ is the conjugation of $\mathfrak{G}$ corresponding to the compact form $\mathfrak{k}_0$. Let $\hat{\mathfrak{G}}$ denote the Lie algebra $\mathfrak{G}\times\mathfrak{G}$ over $\boldsymbol{C}$, and let

$$i: X \rightarrow (X^c, X) \qquad (X \in \mathfrak{G}).$$

$(\hat{\mathfrak{G}}, i)$ is a complexification of $\mathfrak{G}_0$. For any $X\in\mathfrak{G}$ let $\overline{X}=(X, X)$, $\overline{\mathfrak{G}}=\{\overline{X}\colon \overline{\mathfrak{G}}\in\mathfrak{G}\}$. $\hat{\mathfrak{J}}(\mathfrak{J})$ is the universal enveloping algebra of $\hat{\mathfrak{G}}(\mathfrak{G})$ and $\overline{\mathfrak{J}}$ the subalgebra of $\hat{\mathfrak{J}}$ generated by $\overline{\mathfrak{G}}$. For any dominant integral λ, $\mu\in\mathfrak{h}^*$ π_λ denotes the associated irreducible representation of $\mathfrak{G}$ and π_λ^- that of $\overline{\mathfrak{G}}$ (under the isomorphism $X\rightarrow\overline{X}$); $\pi(\lambda, \mu)$ is the irreducible representation $\pi_\lambda\times\pi_\mu$ of $\hat{\mathfrak{G}}$ (Kronecker product).

2. **A theorem on finite dimensional representations.** We have:

THEOREM 1. *Let λ, $\mu\in\mathfrak{h}^*$ be dominant integral, $\nu=\lambda-\mu^*$ and ν^0 the unique dominant integral element in the orbit $w\cdot\nu$. Then the representation $\pi_{\nu^0}^-$ of $\overline{\mathfrak{G}}$ occurs exactly once in the restriction of $\pi(\lambda, \mu)$ to $\overline{\mathfrak{G}}$.*

3. **The homomorphisms h_Q.** For $X\in\mathfrak{G}$ and $a\in\mathfrak{J}$ we write $[X, a]=Xa-aX$. a is said to be of rank 0 if $[H, a]=0$ for all $H\in\mathfrak{h}$. Let $\mathfrak{X}$ be the subalgebra of $\mathfrak{J}$ generated by $\mathfrak{h}$. Suppose Q is any positive system of roots. Then, for any $a\in\mathfrak{J}$ of rank 0, there is a unique $\beta_Q(a)$ in $\mathfrak{X}$ such that $a \equiv \beta_Q(a) \bmod \sum_{\alpha\in Q}\mathfrak{J}\mathfrak{G}^\alpha\cdot a\rightarrow\beta_Q(a)$ is a homo-

[1] The present work was done during 1963–1965 when the authors were at the Indian Statistical Institute, Calcutta.

morphism (of the algebra of all zero rank elements) onto $\mathfrak{X}$. We identify $\mathfrak{X}$ with $P(\mathfrak{h}^*)$ in a natural fashion.

Let Ω be the centralizer of $\overline{\mathfrak{G}}$ in $\hat{\mathfrak{J}}$. Given $\omega \in \Omega$ we can find unique elements $\mathfrak{x}(0)$ and $\mathfrak{x}(k)$ of rank 0 in $\mathfrak{J}$ such that

$$\omega \equiv \mathfrak{x}(0)^- + \sum_{k>0} \mathfrak{x}(k)^- \cdot (H_{\alpha_1})^{k_1} \cdots (H_{\alpha_l})^{k_l}$$

modulo $\hat{\mathfrak{J}}\hat{\mathfrak{n}}$, where $\hat{\mathfrak{n}} = \mathbf{C}\cdot(i\cdot\mathfrak{n}_0)$. We define $\boldsymbol{h}_Q(\omega;\ \cdot,\ \cdot)$ as the polynomial on $\mathfrak{h}^* \times \mathfrak{h}^*$ given by

$$\boldsymbol{h}_Q(\omega; \lambda, \nu) = \beta_Q(\mathfrak{x}(0))(\nu) + \sum \beta_Q(\mathfrak{x}(k))(\nu)\lambda(H_{\alpha_1})^{k_1} \cdots \lambda(H_{\alpha_l})^{k_l}.$$

$\omega \rightarrow \boldsymbol{h}_Q(\omega;\ \cdot\ ,\ \cdot\ )$ can then be proved to be a homomorphism of Ω into $P(\mathfrak{h}^* \times \mathfrak{h}^*)$. If V is the space on which $\pi(\lambda, \mu)$ of Theorem 1 acts and if V_0 is the subspace transforming according to $\pi_{\nu 0}^-$ under $\pi(\lambda, \mu)(\overline{\mathfrak{J}})$, then for $\omega \in \Omega$ and $v \in V_0$, $\pi(\lambda, \mu)(\omega)v = \boldsymbol{h}_Q(\omega; \lambda, \nu)v$ where Q is a positive system in which ν is dominant.

Let $2\delta = \sum_{\alpha \in P_0} \alpha$ and for $s \in W$, $\lambda \in \mathfrak{h}^*$, let $s^A\lambda = s\lambda + s\delta - \delta$.

4. The representations $\hat{\pi}_{\lambda,\nu}$. We shall now define a class of integrable irreducible representations of $\hat{\mathfrak{J}}$ (by integrable we mean that they are the infinitesimal forms of representations of the group G; (cf. Harish-Chandra [2(b)], [2(c)]). Let $\nu \in \mathfrak{h}^*$ be integral and $\overline{\mathfrak{M}}_\nu$ the unique maximal left ideal of $\overline{\mathfrak{J}}$ containing all $\overline{X}_\beta$, $\overline{H}_\beta - \nu(H_\beta)\cdot 1$ $(\beta \in Q)$.

THEOREM 2. *Let ν, $\lambda \in \mathfrak{h}^*$, ν integral, and let Q be a positive system under which ν is dominant. Then there exists a unique maximal left ideal of $\hat{\mathfrak{J}}$ containing $\overline{\mathfrak{M}}_\nu$ and $\omega - \boldsymbol{h}_Q(\omega; \lambda, \nu)\cdot 1$ for all $\omega \in \Omega$. The representation $\hat{\pi}_{\lambda,\nu}$ of $\hat{\mathfrak{J}}$ defined by the maximal left ideal is integrable. In the restriction of $\hat{\pi}_{\lambda,\nu}$ to $\overline{\mathfrak{J}}$, the representation $\pi_{\nu 0}^-$ occurs exactly once, and for any dominant integral $\rho \in \mathfrak{h}^*$, π_ρ^- cannot occur unless ν is a weight of π_ρ, in which case, π_ρ^- occurs with a multiplicity $\leqq d_\rho^\nu$ where d_ρ^ν is the multiplicity of ν in π_ρ. If λ and $\mu = (\lambda - \nu)^*$ are dominant integral, $\hat{\pi}_{\lambda,\nu}$ is equivalent to $\pi(\lambda, \mu)$. Finally, for fixed λ, ν, all the representations $\hat{\pi}_{s^A\lambda, s\nu}$ are equivalent.*

5. The rings $\mathfrak{R}_\nu$ for dominant integral ν. Let ν be dominant integral. Let $\mathfrak{R}_\nu$ be the range of the homomorphism $\omega \rightarrow \boldsymbol{h}_{P_0}(\omega;\ \cdot\ ,\ \nu\ )$.

$$W_\nu = \{s : s \in W,\ s\cdot\nu = \nu\},$$

$$I_\nu^A = \{p : p \in P(\mathfrak{h}^*),\ s^A \cdot p = p \quad \text{for all } s \in W_\nu\}.$$

THEOREM 3. *$\mathfrak{R}_\nu \subseteq I_\nu^A$. If $\nu = 0$ or if ν is in general position, $\mathfrak{R}_\nu = I_\nu^A$. For general ν, let $\mathfrak{J}_0(\nu)$ be defined by*

$$\mathfrak{J}_0(\nu) = \{a\colon a \in \mathfrak{J},\ a \text{ is of rank 0 and } (\text{ad } X_{\alpha_i})^{\nu_i+1}(a) = 0;\quad \text{for } i = 1, \cdots, l\}$$

(where X_α is a nonzero element of $\mathfrak{G}^\alpha$ and $\nu_i = \nu(H_{\alpha_i})$). Then

$$\mathfrak{R}_\nu = \overset{A}{s_0}\cdot\{\mathfrak{z}_{P_0}(\mathfrak{J}_0(-s_0\nu))\}.$$

By case considerations we can show that $\mathfrak{R}_\nu = I_\nu^A$ for all ν. But we do not possess a uniform proof of this fact.

6. **Representations of class** 0. An integrable representation of $\hat{\mathfrak{J}}$, say π, is said to be of *class* 0 if its restriction to $\bar{\mathfrak{J}}$ contains the trivial representation of $\bar{\mathfrak{J}}$. We can show that every integrable irreducible representation of class 0 of $\hat{\mathfrak{J}}$ is equivalent to some $\hat{\pi}_{\lambda,0}$. We shall say that $\hat{\pi}_{\lambda,0}$ is *complete* if for each ρ such that 0 is a weight of π_ρ, π_ρ^- occurs with multiplicity equal to d_ρ^0 in the restriction of $\hat{\pi}_{\lambda,0}$ to $\bar{\mathfrak{J}}$.

THEOREM 4. *For a $\lambda \in \mathfrak{h}^*$, $\hat{\pi}_{\lambda,0}$ is complete if and only if for each root $\alpha \in P_0$, $(\lambda+\delta)(H_\alpha) \notin Z^*$ where Z^* is the set of nonzero integers.*

Let $\mathfrak{L}$ be the $\mathfrak{Z}$-module ($\mathfrak{Z}$ is the center of $\mathfrak{J}$) of linear maps from the representation space V^ρ of π_ρ into $\mathfrak{J}$ which intertwine π_ρ and the adjoint representation, let $d = d_\rho^0$ and let $\{L_1, \cdots, L_d\}$ be a basis of the free module $\mathfrak{L}$ (cf. Kostant [4]). Let $\{v_1, \cdots, v_d\}$ be a basis of the 0 weight subspace of V^ρ, and let K_ρ' be the $d \times d$ matrix of elements of $\mathfrak{X}$ whose $(i-j)$th element is $\mathfrak{z}_{P_0}(L_i v_j)$.

THEOREM 5. *The multiplicity $m(\rho, \lambda)$ of π_ρ^- in $\hat{\pi}_{\lambda,0}$ is given by*

$$m(\rho, \lambda) = \min\{\text{rank } K_{\rho^*}'(\lambda),\ \text{rank } K_\rho'(\overset{A}{s_0}\lambda)\}$$

where $s_0 \in W$ is such that $s_0 P_0 = -P_0$, $\rho^* = -s_0\rho$.

7. **Representations of** G. Let G be the simply connected group corresponding to $\mathfrak{G}_0$. Let log denote the inverse of the exponential map from A_+ onto $\mathfrak{h}_0$. For any integral $\nu \in \mathfrak{h}^*$ let ψ_ν $(\exp(-1)^{1/2}H_0) = \exp(-1)^{1/2}\cdot\nu(H_0)$ $(H_0 \in \mathfrak{h}_0)$. ψ_ν is a character of M, the connected component of the centraliser of A_+ in K. Let $\mathfrak{H} = \mathfrak{L}^2(K)$,

$$\mathfrak{H}(\nu) = \{f\colon f \in \mathfrak{H},\ R_r(k)f = \psi_{-\nu}(k)f \quad \text{for all } k \in M\}$$

(ν integral, R_r the right regular representation of K). Following Harish-Chandra [2(b), p. 240] we define, for each $\xi \in \mathfrak{h}^*$, the representation $\pi_{\xi,\nu}$ in $\mathfrak{H}(\nu)$ by setting for all $f \in \mathfrak{H}(\nu)$

$$(\pi_{\xi,\nu}(x)f)(k) = \exp[(\xi + 2\delta)(\log a_+(x^{-1}, k))]\cdot f(\sigma_x^{-1}(k))$$

($k\in K$, $x\in G$); here, for $y\in G$, and $k\in K$, $y\cdot k=\sigma_y(k)\cdot a_+(y, k)\cdot n(y, k)$ where $\sigma_y(k)\in K$, $a_+(y, k)\in A_+$, $n(y, k)\in N$. ν^0 is the dominant integral element in $W\cdot\nu$, $\mathfrak{H}(\nu)_{\nu^0}$ is the set of all elements in $\mathfrak{H}(\nu)$ which transform under the left regular representations of K according to $\pi_{\bar{\nu}^0}$.

THEOREM 6. *The representation $\pi_{\xi,\nu}$ of G in $\mathfrak{H}(\nu)$ is homogeneous. Let $\mathfrak{H}(\nu;\ \xi)$ be the smallest closed subspace of $\mathfrak{H}(\nu)$ invariant under $\pi_{\xi,\nu}(G)$ and containing $\mathfrak{H}(\nu)_{\nu^0}$. Then there exists a unique maximal closed subspace $\mathfrak{H}'(\nu;\ \xi)$ of $\mathfrak{H}(\nu;\ \xi)$ invariant under $\pi_{\xi,\nu}(G)$ and not containing $\mathfrak{H}(\nu)_{\nu^0}$. $\mathfrak{H}'(\nu;\xi)$ is orthogonal to $\mathfrak{H}(\nu)_{\nu^0}$ and the representation of G defined by $\pi_{\xi,\nu}$ in $\mathfrak{H}(\nu;\xi)/\mathfrak{H}'(\nu;\xi)$ is irreducible and the associated representation of $\mathfrak{F}$ is equivalent to $\hat{\pi}_{\lambda,\nu}$ where $\lambda=\frac{1}{2}(\nu+\xi)-\delta$.*

The question as to when the $\pi_{\xi,\nu}$ are themselves irreducible is a crucial one; (cf. Bruhat [1]) for $\nu=0$ we have a complete answer to this question.

THEOREM 7. *For $\pi_{\xi,0}$ to be irreducible it is necessary and sufficient that $\frac{1}{2}\xi(H_\alpha)\notin Z^*$ for each root α. In particular, all unitary representations of the principal nondegenerate series, whose restrictions to K contain the trivial representation of K, are irreducible.*

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UNIVERSITY OF SHEFFIELD, SHEFFIELD
UNIVERSITY OF ILLINOIS AND
UNIVERSITY OF CALIFORNIA, LOS ANGELES