LOCALLY FLAT NONEMBEDDABILITY OF CERTAIN PARALLELIZABLE MANIFOLDS

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1. **Introduction and statement of result.** This note is a supplement to the joint papers of R. H. Szczarba and the author [6], [7]. We proved in [6], [7] that for any positive integer $q>1$, there is a differentiable parallelizable manifold M_q of dimension $(2^{4q+1}-8q-2)$ which can be differentiably immersed in Euclidean space of codimension 1 but can not be differentiably embedded in Euclidean space of codimension $8q$. As a consequence, the dimension difference of the best differentiable immersion and the best differentiable embedding in Euclidean space can be arbitrarily large. One may ask the same type of question for topological or combinatorial immersion and embedding. In this note, we shall modify the argument of [6], [7] to show that M_q $(q>1)$ actually has no locally flat topological (hence no combinatorial) embedding in Euclidean space of codimension $8q$. Since we used the normal bundle of a differentiable embedding and Adams' solution of vector field problem [1] in the original proof of [6], [7], differentiability seemed to be essential. However, we shall replace the normal bundle by the normal fibre space of Nash-Fadell-Spivak [9], [4], [10] and use a corollary of Adams' solution of vector field problem that $[\iota_{2^{4q}-1}, \iota_{2^{4q}-1}]$ is not an $(8q+1)$-fold suspension to show the locally flat nonembeddability of M_q $(q>1)$ in Euclidean space of codimension $8q$. The author is indebted to Professor John Milnor for his comments.

Let us first recall the manifolds M_q $(q\geqq 1)$. If ξ and η are sphere bundles with a common base, we use $\hat{\xi}$ to denote the vector bundle associated with ξ and $\xi * \eta$ the sphere bundle associated with $\hat{\xi}\oplus\hat{\eta}$. Let S^{n-1} be the $(n-1)$-sphere where $n=2^{4q}$, $q\geqq 1$. It follows from results of Eckmann [3] and Adams [1] that S^{n-1} has exactly $8q$ independent vector fields. Thus we can find an $(n-8q-1)$-sphere bundle ξ_q over S^{n-1} with a cross section and with the property that $\xi_q * \theta^{8q-1}=\tau(S^{n-1})$, the tangent sphere bundle of S^{n-1}. (Here θ^r denotes the trivial $(r-1)$-sphere bundle.) Let M_q be the total space of ξ_q.

Let M^n and N^m be two topological manifolds. A topological embedding (immersion) $f\colon M^n\to N^m$ is said to be locally flat [4], [5], if

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for each point $x \in M$, there are neighborhoods $U(x)$ and $V(f(x))$ such that $(V(f(x)), f(U(x)))$ is topologically equivalent to (R^m, R^n). Clearly, a differentiable embedding is locally flat. It follows from Zeeman's unknotting theorem that a combinatorial embedding of codimension greater than 2 is also locally flat [**13**]. Moreover, local flatness is independent of the differential or combinatorial structure.

THEOREM. *For* $q > 1$, M_q *has no locally flat embedding in Euclidean space with codimension* $8q$.

REMARK 1. The 22-dimensional manifold M_1 has no locally flat embedding in R^{28}.

REMARK 2. Let N_q be a combinatorial manifold of the homotopy type of M_q $(q > 1)$. By a recent result of Irwin (Ann. of Math. **82** (1965), 1–14), there is a combinatorial embedding $s: S^{n-1} \subset N_q$ which represents a generator of $H_{n-1}(N_q; Z)$. Let U be a neighborhood of $s(S^{n-1})$ in N_q. One can actually show that U has no locally flat embedding in Euclidean space of codimension $8q$.

COROLLARY 1. *For any integer* k, *there are manifolds which have locally flat immersion in Euclidean space with codimension* 1 *but have no locally flat embedding in Euclidean space with codimension* k.

COROLLARY 2. *For any integer* k, *there are combinatorial manifolds which can be combinatorially immersed in Euclidean space with codimension* 1 *but cannot be combinatorially embedded in Euclidean space with codimension* k.

PROOF. Consider the C^r-triangulation $(r \geqq 1)$ of the differentiable manifolds M_q $(q > 1)$. Following from Zeeman [**13**], every combinatorial embedding with codimension greater than 2 is locally flat. Corollary 2 follows immediately from the theorem.

2. **Spherical normal fibre space.** Let $f: M^n \subset N^m$ be a locally flat embedding. Following [**9**], [**4**], [**10**], we consider the path space

$$E = \{ w \mid w \in (N^m)^I,\ w(t) \in M^n \text{ if and only if } t = 0 \}.$$

The initial projection $p: E \to M^n$ defined by $p(w) = w(0)$ is an $(m-n-1)$-spherical fibre space over M,

$$\nu: S^{m-n-1} \longrightarrow E \longrightarrow M^n.$$

ν is called the spherical normal fibre space of the embedding f. If f is a differentiable embedding with differentiable normal sphere bundle η, then η is fibre homotopically equivalent to the spherical normal fibre space ν when f is considered as a locally flat embedding [**4**].

Lemma 1 (Massey [8], Stallings [11]). *Let* $f\colon S^n \subset S^{n+k}$ $(n \geqq 2, k \geqq 3)$ *be a locally flat embedding. Then, the spherical normal fibre space* ν *of* f *is fibre homotopically trivial.*

Proof. By Stallings [11], $f(S^n) \subset S^{n+k}$ is unknotted. Hence, $S^{n+k} - f(S^n)$ is of the homotopy type of S^{k-1}. Then, ν is fibre homotopically trivial by the argument of [8].

Let $\xi_1\colon S^{k-1} \to E_1 \xrightarrow{p_1} X$, and $\xi_2\colon S^{l-1} \to E_2 \xrightarrow{p_2} X$ be two spherical fibre spaces. Let E be the subset of the join $E_1 * E_2$ of E_1 and E_2, consisting of the points (x_1, t, x_2) with the property that $p_1x_1 = px_2$ where $(x_1, t, x_2) \in E_1 * E_2$. The projection $q\colon E \to X$ defined by

$$q(x_1, t, x_2) = p_1(x_1) = p_2(x_2)$$

is a spherical fibre space ξ called the Whitney join of ν_1 and ν_2, and will be denoted by $\xi_1 * \xi_2$.

Lemma 2 (Spivak [10]). *Let* $f_1\colon M^n \to N_1^m$ *be a differentiable embedding with the normal sphere bundle* ν_1 *and let* $f_2\colon N_1^m \to N_2^r$ *be a locally flat embedding with spherical normal fibre space* ν_2. *Then, the spherical normal fibre space* ν *of the composite embedding*

$$f_2f_1\colon M^n \to N_1^m \to N_2^r$$

(*which is clearly locally flat*) *is fibre homotopically equivalent to* $\nu_1 * (\nu_2 | M^n)$.

Proof. Let D be a closed tubular neighborhood of $f(M^n)$ in N_1^m such that the exponential map (under some proper Riemannian metric of N_1^m) maps the closed disc bundle

$$\tilde{\nu}_1\colon D^{m-n} \to E_1 \xrightarrow{p_1} M^n$$

associated with ν_1 diffeomorphically onto D. We identify D with E_1. Since M^n is a deformation retract of D, the fibre space $p_1^!(\nu_2 | M^n)$ is fibre homotopically equivalent to $\nu_2 | D$. Let $g\colon p_1^!(\nu_2 | M^n) \to \nu_2 | D$ be a fibre homotopical equivalence. Let us construct a fibre map

$$h\colon \nu_1 * (\nu_2 \mid M^n) \to \nu$$

as follows. First recall that (i) every point in D is of the form $\exp(x_1)$ with a vector x_1 of length $\leqq 1$ and normal to M^n (with respect to some proper Riemannian metric of N_1^m), (ii) every point in $(\nu_2 | M^n)$ is a path in N_2^r issuing from M^n and never touching N_1^m again, (iii) every point of $\nu_1 * (\nu_2 | M^n)$ is of the form

$$(\exp(x_1), t, x_2) \quad \text{for } 0 \leqq t \leqq 1.$$

Define

$$\begin{aligned} h(\exp(x_1), t, x_2) &= \exp(ux_1) && \text{for } 0 \leqq u \leqq t, \\ &= g[p_1^! \exp(x_1)(x_2)](u-t) && \text{for } t \leqq u \leqq 1. \end{aligned}$$

Using the differentiability of f_1 and the local flatness of f_2, ir is easy to check that h induces homotopical equivalence on each fibre. By A. Dold's criterion [2], h is a fibre homotopical equivalence.

Remark. A similar statement of Lemma 2 was proved in [**10**]. Since [**10**] is still unpublished and the case which we need is rather special, we include a somewhat simpler proof for completeness.

3. **Proof of the theorem.** Let G_k be the H-space of degree 1 maps of $S^{k-1} \rightarrow S^{k-1}$. By [2], [**11**], the fibre homotopical equivalence classes of S^{k-1}-fibre spaces over S^p are one-one correspondence to the elements of $\pi_{p-1}(G_k)$. Let $\sigma^{m-k}\colon G_k \rightarrow G_m$ $(m \geqq k)$ be the obvious inclusion induced by the suspensions of the degree 1 maps of S^{k-1}. Let ν_1, ν_2 be S^{k-1}-, S^{l-1}-fibre spaces over S^p corresponding to $\bar{\nu}_1 \in \pi_{p-1}(G_k)$, $\bar{\nu}_2 \in \pi_{p-1}(G_l)$ respectively. By the argument of Lemma 3.1 of [**6**], we see that $\nu_1 * \nu_2$ corresponds to

$$\sigma^l(\bar{\nu}_1) + \sigma^k(\bar{\nu}_2)$$

in $\pi_{p-1}(G_{k+l})$.

Now, suppose that M_q $(q > 1)$ has a locally flat embedding in Euclidean space of codimension $8q$ with ν as the spherical normal fibre space. We choose a differentiable cross section $s\colon S^{n-1} \rightarrow M_q$ (M_q is given the natural differentiable structure) of ξ_q which has a normal sphere bundle ν_s with the property $\nu_s * \theta^1 = \xi_q$. By Lemma 1, the composite embedding

$$S^{n-1} \xrightarrow{s} M_q \rightarrow R^{2n-2}$$

has a fibre homotopically trivial spherical normal fibre space. Hence, $\nu_s * (\nu | S^{n-1})$ is fibre homotopically trivially by Lemma 2. Let $\bar{\nu} \in \pi_{n-2}(G_{8q})$, $\bar{\nu}_s \in \pi_{n-2}(G_{n-8q-1})$, and $\bar{\tau} \in \pi_{n-2}(G_{n-1})$ be the elements corresponding to $(\nu | S^{n-1})$, ν_s, and $\tau(S^{n-1})$ respectively. Since $\nu_s * (\nu | S^{n-1})$ is fibre homotopically trivial,

$$\sigma^{8q}(\bar{\nu}_s) + \sigma^{n-8q-1}(\bar{\nu}) = 0. \tag{1}$$

But $(\nu_s * \theta^1) * \theta^{8q-1} = \xi_q * \theta^{8q-1} = \tau(S^{n-1})$. So

$$\sigma^{8q}(\bar{\nu}_s) = \bar{\tau}. \tag{2}$$

Thus (1) and (2) show that $\bar{\tau}$ is an $(n-8q-1)$-fold suspension. Now let $J: \pi_{n-1}(G_k) \to \pi_{k+n-1}(S^k)$ be defined in the usual way. Then the element

$$[\iota_{n-1}, \iota_{n-1}] = J(\bar{\tau}) \in \pi_{2n-3}(S^{n-1})$$

must be an $(n-8q-1)$-fold suspension. But according to [**1**, Corollary 1.3], this element $[\iota_{n-1}, \iota_{n-1}]$ is not even an $(8q+1)$-fold suspension. Since

$$n - 8q - 1 \geqq 8q + 1$$

for $q > 1$, this gives a contradiction, and completes the proof.

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