A NEW CONSTRUCTION FOR HADAMARD MATRICES[1]

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An *Hadamard matrix* H is a square matrix of *ones* and *minus ones* whose row (and hence column) vectors are orthogonal. The order n of an Hadamard matrix is necessarily 1, 2 or $4t$ with $t=1, 2, 3, \cdots$. It has been conjectured that this condition ($n=1$, 2 or $4t$) also insures the existence of an Hadamard matrix. Constructions have been given for particular values of n and even for various infinite classes of values. While other constructions exist, those given by [1]–[7] exhaust the previously known values of n. This paper gives a new construction which yields, among others, the previously unknown value $n=156$, leaving only two undecided values of $n=4t\leqq 200$ (these are 116 and 188).

An Hadamard matrix is said to be of the *Williamson type* if it has the structure imposed by Williamson [6], that is

$$H = \begin{vmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{vmatrix},$$

where each of A, B, C, D is a symmetric circulant $t\times t$ matrix. Notice that if a Williamson type matrix exists for $n=4t$, then an Hadamard matrix (not obviously Williamson) of order $m=12t$ would exist provided one could find a 12×12 matrix with the following properties. Each row and column must contain precisely three $\pm A$'s, three $\pm B$'s, three $\pm C$'s, three $\pm D$'s and the rows must be formally orthogonal (i.e., A, B, C, D are to be considered as independent quantities). We have discovered such a matrix and display it as Figure 1.

Among the ***known*** orders of Williamson type matrices [1], [6], only 52 yields a new value of n by this construction. This gives an Hadamard matrix of order 156. For definiteness, the first rows of A, B, C, D for one of the Williamson type Hadamard matrices of order 52 are given (here $+$ means $+1$ and $-$ stands for -1).

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$$
\begin{array}{c|ccccccccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
A & + & + & - & - & + & - & + & + & - & + & - & - & + \\
B & + & - & - & - & + & + & + & + & + & + & - & - & - \\
C & + & + & + & - & + & + & - & - & + & + & - & + & + \\
D & + & + & - & + & - & + & + & + & + & - & + & - & +
\end{array}
$$

$$
H = \left|\begin{array}{ccccccccccccc}
A & A & A & B & -B & C & -C & -D & B & C & -D & -D \\
A & -A & B & -A & -B & -D & D & -C & -B & -D & -C & -C \\
A & -B & -A & A & -D & D & -B & B & -C & -D & C & -C \\
B & A & -A & -A & D & D & D & C & C & -B & -B & -C \\
B & -D & D & D & A & A & A & C & -C & B & -C & B \\
B & C & -D & D & A & -A & C & -A & -D & C & B & -B \\
D & -C & B & -B & A & -C & -A & A & B & C & D & -D \\
-C & -D & -C & -D & C & A & -A & -A & -D & B & -B & -B \\
D & -C & -B & -B & -B & C & C & -D & A & A & A & D \\
-D & -B & C & C & C & B & B & -D & A & -A & D & -A \\
C & -B & -C & C & D & -B & -D & -B & A & -D & -A & A \\
-C & -D & -D & C & -C & -B & B & B & D & A & -A & -A
\end{array}\right|
$$

FIGURE 1

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