

EXAMPLES OF p -ADIC TRANSFORMATION GROUPS

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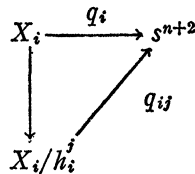
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1. **Introduction.** Our purpose here is to outline the construction of an n -dimensional space X^n , $n \geq 2$, upon which the p -adic group A_p acts so that the orbit space X^n/A_p is of dimension $n+2$. Though such examples are new, it had been known [1; 4], that either they do exist or a certain long standing conjecture on transformation groups must be true. The conjecture states that every compact effective group acting on a (generalized) manifold must be a Lie group; it may well be false.

Another question concerns the amount, k , by which the (cohomology) dimension of a compact space can be raised under the decomposition map $X \rightarrow X/A_p$. By [1; 4], $k \leq 3$. (An example in which $k=1$ is essentially due to Kolmogoroff [2].) No example is known for which $k=3$. The authors expect to have more to say on this subject.

2. **The building blocks.** There exists an $(n+2)$ -dimensional complex X_i , a homeomorphism $h_i: X_i \rightarrow X_i$, maps $q_i: X_i \rightarrow S^{n+2}$ and $r_i: X_i \rightarrow X_i$ such that $(S^{n+2}$ is the standard $n+2$ -simplex)

- (a) h_i is of period p^i .
- (b) $q_i^{-1}(j \text{ skeleton}) = X_i(j)$ is a j -complex, $j=0, 1, \dots, n+2$.
- (c) $q_i h_i = q_i$ so that q_i can be factored through the orbit space: $j=1, p, \dots, p^{i-1}$



- (d) $q_{ij*}: H_{n+2}(X_i/h_i^j, X_i(n+1)/h_i^j; Z_p) \rightarrow H_{n+2}(S^{n+2}, S^{n+2}; Z_p)$ is onto; and
- (e) $r_i: X_i \rightarrow X_i(n)$ is a retraction.

3. (E, π, B, X, q) . Let m be a positive integer and s_0 be the standard m -simplex. We consider 5-tuples (E, π, B, X, q) , where E, B , and X are m -complexes and π, q are simplicial maps such that

- A1. $\pi: E \rightarrow B, q: X \rightarrow s_0$,
- A2. for each m -simplex $s \in B$, there are specified maps

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$$\psi_s: X \rightarrow \pi^{-1}(s) = E_s,$$

$$\phi_s: s_0 \rightarrow s$$

where ϕ_s is a linear homeomorphism;

A3. commutativity holds in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi_s} & E_s \\ q \downarrow & & \downarrow \pi \Big| E_s \\ s_0 & \xrightarrow{\phi_s} & s \end{array}$$

A4. For s, s' two m -simplexes of K ,

$$\psi_s \Big| X_{s \cap s'} = \psi_{s'} \Big| X_{s \cap s'}, \quad \text{and} \quad \phi_s^{-1} \Big| s \cap s' = \phi_{s'}^{-1} \Big| s \cap s',$$

where $X_{s \cap s'} = q^{-1}\phi_s^{-1}(s \cap s')$.

We need two results concerning such 5-tuples.

3.1. *Given X, q , and a complex B which is the barycentric subdivision of a complex, there exists a complex E and a map $\pi: E \rightarrow B$ such that (E, π, B, q, X) satisfies A1–A4 and such that the ψ_s 's are homeomorphisms.*

3.2. *If in addition to A1–A4, we assume*

A5. *$H_m(X, q^{-1}(s); G) \xrightarrow{q_*} H_m(s, s; G)$ is onto, then $H_m(E; G) \rightarrow H_m(B; G)$ is onto.*

4. **The example.** X^n is defined as the inverse limit of a sequence

$$E_0 \leftarrow \begin{array}{c} E_1 \\ \pi_1 \end{array} \leftarrow \begin{array}{c} E_2 \\ \pi_2 \end{array} \leftarrow \dots$$

such that on each E_i (an $(n+2)$ -complex) we have a map \bar{h}_i of period p^i . The E_i are defined inductively. E_0 is taken to be a triangulated $(n+2)$ -sphere, and $\bar{h}_0 = \text{identity}$. Suppose E_i, π_i, \bar{h}_i have been defined. Then set B_{i+1} = the barycentric subdivision of E_i and use 3.1 relative to $B_{i+1}, X_{i+1}, q_{i+1}$ (see §2), to obtain E_{i+1}, π_{i+1} . The homeomorphism \bar{h}_{i+1} is defined on E_{i+1} in terms of h_{i+1}, π_{i+1} and \bar{h}_i , so that we have the equivariance

$$\pi_{i+1} \bar{h}_{i+1} = \bar{h}_i \pi_{i+1}.$$

Thus the map

$$(e_0, e_1, \dots) \rightarrow (\bar{h}_0 e_0, \bar{h}_1 e_1, \dots)$$

defines an effective action of the p -adic group on X^n .

It is next shown that $\bar{E}_i = E_i / \bar{h}_i, i = 1, 2, \dots$ satisfy axioms

A1–A5 of §3, where $\bar{\pi}_i: E_i/\bar{h}_i \rightarrow E_{i-1}/\bar{h}_{i-1}$ is defined via (2c), and the ψ_s 's are essentially like the q_{ij} 's in (2c). By (3.2) all homeomorphisms in the sequence

$$H_{n+2}(\bar{E}_0; Z\mathfrak{p}) \xleftarrow{\bar{\pi}_{1*}} H_{n+2}(\bar{E}_1; Z\mathfrak{p}) \leftarrow \cdots$$

are onto so that the orbit space, X/A_p , which is the inverse limit of the sequence

$$E_0/\bar{h}_0 \xleftarrow{\bar{\pi}_1} E_1/\bar{h}_0 \xleftarrow{\bar{\pi}_2} E_2/\bar{h}_2 \xleftarrow{\bar{\pi}_3} \cdots,$$

is $(n+2)$ -dimensional.

Finally, it follows from (2e) that X^n is n -dimensional.

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